SOME NEW RESULTS ON REMOTAL POINTS IN NORMED SPACES

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Abstract. In this paper, using the best proximity theorems for an extension of Brosowski's theorem. We obtain other results on farthest points. Finally, we define the concept of ϵ- farthest points. We shall prove interesting relationship between the ϵ-best approximation and the ϵ-farthest points in normed linear spaces $(X, \|.\|)$. If $z \in W$ is a ϵ-farthest point from an $x \in X$, then z is also a ϵ-best approximation in W.

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1. Introduction

Let A be a non-empty subset of a normed linear space $(X, \|.\|)$. For $x \in X$, if there exists a point $x_0 \in A$ such that $d(x, A) = \inf\{\|x - y\| : \ y \in A\} = \|x - x_0\|$. The point x_0 is called a best approximation point of A from x. We denote the set

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of best approximation points (nearest point) Of x in A by $P_A(x)$. We can find some results about best approximation points in (see [11]).

Let A be non-empty subset of a normed linear space $(X, \|.\|)$. Consider a map $T: A\cup B \to A\cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$ (T is cyclic map). For $x \in X$, we say that the point x is a best proximity point of map T, if $\|x - Tx\| = d(A, B)$, and we denote the set of all best proximity points of T by $P_T(A, B)$. That is,

$$P_T(A, B) = \{x \in A \cup B : \ \|x - Tx\| = d(A, B)\}.$$

Best proximity points also evolves as a generalization of the concept of fixed point of mappings, because if $A \cap B \neq \phi$, every best proximity point is a fixed point of T (see [8]).

The problem of characterizing remotal points is an interesting problem, though it is much more difficult than the proximinality one. Further, it has its applications in approximation theory and geometry of Banach spaces.

Let A be non-empty bounded subset of normed linear space $(X, \|.\|)$. For $x \in X$, if there exists a point $x_0 \in A$ such that $\delta(x, A) = \sup\{\|x - y\| : \ y \in A\} = \|x - x_0\|$. The point x_0 is called a farthest point of A from x. We denote the set of farthest points of x in A by $F_A(x)$. We can find some results about farthest points in (see [1, 3-7, 9-13]).

Let X be a normed linear space, A, B be non-empty bounded subsets of X and $T : A \cup B \to A \cup B$ is a cyclic map. The point $x \in A \cup B$ is called a remotal point for T, if $\|x - Tx\| = \delta(A, B) = \sup_{x\in B} \delta(x, A)$.

The set of every remotal points for T denoted by $F_T(A, B)$. (see [1])

$$F_T(A, B) = \{x \in A \cup B : \ \|x - Tx\| = \delta(A, B)\}.$$

Let A and B be nonempty subsets of a metric space X and $T : \ A \cup B \to A \cup B$ a map such that $T(A) \subseteq B$, $T(B) \subseteq A$. Put

$$P^{\epsilon}_T(A, B) = \{x \in A \cup B : \ d(x, Tx) \leq d(A, B) + \epsilon \ for\ some\ \epsilon > 0\}.$$

We say that the $x \in A \cup B$ is an approximate best proximity point for T if $P^{\epsilon}_T(A, B) \neq \emptyset$. (see [12])

Let A and B be non-empty bounded subsets of a metric space X and $T : A\cup B \to A\cup B$ be a cyclic map. The point $x \in A\cup B$ is an approximate remotal point for T, if $d(x,Tx) \geq \delta(A,B) - \epsilon$, for some $\epsilon > 0$. We will denote the set of all approximate remotal pair (A,B) by (see [1])

$$F_T^{\epsilon}(A,B) = \{x \in A \cup B : \ d(x,Tx) \geq \delta(A,B) - \epsilon \ for \ some \ \epsilon > 0\}.$$

In the following we shall present a list of known lemmas which are needed in the proof of the main results. The following Lemma is Brosowski's Theorem.

Lemma 1.1. [2] *Let X be a Banach space and $T : X \to X$ a non expansive mapping with a fixed point $\bar{x} \in X$. Let C be a nonempty subset of X such that $T(C) \subseteq C$. Also $P_C(\bar{x})$ is a nonempty compact convex subset of C. Then T has a fixed point in $P_C(\bar{x})$.*

Lemma 1.2. [8] *Let A and B be nonempty closed subsets of a complete metric space X. Suppose that the mapping $T : \ A \cup B \to A \cup B$ satisfy $T(A) \subseteq B$, $T(B) \subseteq A$, and*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma d(A,B),$$

for all $x, y \in A \cup B$, where $\alpha, \gamma, \beta \geq 0$, $\alpha + 2\beta + \gamma \leq 1$. If A (or B) is boundedly compact, then there exists a $x \in A \cup B$ with $d(x,Tx) = d(A,B)$.

Lemma 1.3. [1] *Let A and B be nonempty bounded subsets of a complete metric space X. Suppose that the mapping $T : \ A \cup B \to A \cup B$ satisfy $T(A) \subseteq B$, $T(B) \subseteq A$, and*

$$d(Tx,Ty) \geq \alpha d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma d(A,B),$$

for all $x, y \in A \cup B$, where $\alpha, \beta \geq 0$, $\gamma > 0$, $\alpha + 2\beta < 1$, $\alpha + 2\beta + \gamma \geq 1$. If A (or B) is boundedly compact, then there exists $x \in A \cup B$ with $d(x,Tx) = \delta(A,B)$.

Lemma 1.4. [12] *Let A and B be nonempty closed subsets of a complete metric space X. Suppose that the mapping $T : \ A \cup B \to A \cup B$ satisfy $T(A) \subseteq B$, $T(B) \subseteq A$, and*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma d(A,B)$$

for all $x, y \in A \cup B$, where $\alpha, \beta, \gamma \geq 0$, $\alpha + 2\beta + \gamma \leq 1$. Then T has an approximate best proximity point.

Lemma 1.5. [1] *Let A and B be non-empty bounded subsets of a complete metric space X. Suppose that the map $T : A \cup B \to A \cup B$ is a cyclic map and*

$$d(Tx, Ty) \geq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma\delta(A, B),$$

for all $x, y \in A \cup B$, where $\alpha, \beta \geq 0$, $\gamma > 0$, $\alpha + 2\beta < 1$ and $\alpha + 2\beta + \gamma \geq 1$. Then T has an approximate remotal point.

Lemma 1.6. [1] *Let A and B be non-empty bounded subsets of a metric space X. Suppose that the continuous cyclic mapping $T : A \cup B \to A \cup B$ satisfy $T(A) \subseteq B$, $T(B) \subseteq A$, and*

$$d(Tx, Ty) \geq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma\delta(A, B),$$

for all $x, y \in A \cup B$, where $\alpha, \beta \geq 0$, $\gamma > 0$, $\alpha + 2\beta < 1$ and $\alpha + 2\beta + \gamma \geq 1$. For x_0 is an arbitrary point in A, define $x_{n+1} = Tx_n$ for each $n \geq 1$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A \cup B$ with $d(x, Tx) = \delta(A, B)$.

2. Best proximity points and best approximation points

In this section, we expression conditions where union the set of best proximity points for a map and the set of best approximation points of a set is non-empty. Also we will extend Brosowski's Theorem.

Theorem 2.1. *Let $(X, \|.\|)$ be a Banach space and A and B be non-empty subsets of X. Suppose that the continuous cyclic mapping $T : A \cup B \to A \cup B$ is satisfy*

$$\|Tx - Ty\| \leq \beta[\|x - Tx\| + \|y - Ty\|] + \gamma d(A, B) \ (2.1)$$

for all $x, y \in A \cup B$, where $0 < \beta \leq \frac{1}{2}$, $2\beta + \gamma \leq 1$. Let C be a subset of A such that $T(C) \subseteq C$. Also, there exists a $x_0 \in P_T(A, B) \cap A$, $P_C(x_0)$ and $P_T(A, B)$ are nonempty boundedly compact subset of C. Then $P_T(A, B) \bigcup P_C(Tx_0) \neq \emptyset$.

Proof. First, we show that $T :\ (P_C(Tx_0) \cap P_T(A, B)) \cup P_T(A, B) \to (P_C(Tx_0) \cap P_T(A, B)) \cup P_T(A, B)$.

case 1: Suppose $y \in P_T(A, B)$. Then $\|Ty - T(Ty)\| \leq \beta[\|y - Ty\| + \|Ty - T^2y\|] +$

$\gamma d(A,B)$. Therefore

$$\begin{aligned} d(A,B) &\leq \|Ty - T^2y\| \\ &\leq \frac{\beta+\gamma}{1-\beta} d(A,B) \\ &\leq d(A,B). \end{aligned}$$

Therefore $d(A,B) = \|Ty - T^2y\|$. implying that $Ty \in P_T(A,B)$.
case 2: If $y \in P_C(Tx_0) \cap P_T(A,B)$, since $Ty \in C$ and $\|Ty - Tx_0\| \geq d(Tx_0, C)$. Therefore

$$\begin{aligned} d(Tx_0, C) &\leq \|Ty - Tx_0\| \\ &\leq \beta[\|x_0 - Tx_0\| + \|y - Ty\|] + \gamma d(A,B) \\ &= \beta[d(A,B) + d(A,B)] + \gamma d(A,B) \\ &\leq (2\beta + \gamma) d(A,B) \\ &\leq d(A,B) \\ &\leq d(Tx_0, C), \end{aligned}$$

implying that $Ty \in P_C(Tx_0)$.
We set $A = P_C(Tx_0) \cap P_T(A,B)$ and $B = P_T(A,B)$. From Lemma 1.2, there exits a $z \in P_C(Tx_0) \cap P_T(A,B) \bigcup P_T(A,B)$. Therefore $z \in P_C(Tx_0) \bigcup P_T(A,B)$

The following Theorem is a extension of **Brosowki's Theorem**.

Theorem 2.2. *Let $(X, \|.\|)$ be a Banach space and $T : X \to X$ satisfy*

$$\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Tx\|,$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$. Let C be a subset of X such that $T(C) \subseteq C$. Assume T has a fixed point $x_0 \in X$ for T and $P_C(x_0)$ is a nonempty boundedly compact subset of C. Then T has a fixed point in $P_C(x_0)$.

Proof. First, we show that $T:\ P_C(x_0)\to P_C(x_0)$. Suppose $y\in P_C(x_0)$. Then

$$\begin{aligned}\|x_0-Ty\| &= \|Tx_0-Ty\|\\ &\le \alpha\|x_0-y\|+\beta\|x_0-Tx_0\|\\ &\le \|x_0-y\|\\ &\le d(x_0,C).\end{aligned}$$

Therefore $Ty\in P_C(x_0)$. We set $A=B=P_C(x_0)$, From Lemma 1.2, there exists $z\in P_C(x_0)$ such that $d(z,Tz)=d(A,B)=0$. That is $Tz=z$.

3. **Remotal Points and farthest points**

In this section, we find conditions where union the set of remotal points and the set of farthest points is non-empty.

Theorem 3.1. *Let $(X,\|.\|)$ be a Banach space and A and B be non-empty bounded subsets of X. Suppose that the continuous cyclic mapping $T: A\cup B\to A\cup B$ is satisfy*

$$\|Tx-Ty\| \quad\ge\quad \beta[\|x-Tx\|+\|y-Ty\|]+\gamma\delta(A,B),$$

for all $x,y\in A\cup B$, where $0<\beta<\frac{1}{2}$, $\gamma>0$ and $2\beta+\gamma\ge 1$. Let C be a bounded subset of A such that $T(C)\subseteq C$. Also, there exists a $x_0\in F_T(A,B)\cap A$, $F_T(A,B)$ and $F_C(Tx_0)$ are nonempty boundedly compact subset of C. Then $F_T(A,B)\bigcup F_C(Tx_0)\neq\emptyset$.

Proof. First, we show that $T:\ (F_C(Tx_0)\cap F_T(A,B))\cup F_T(A,B)\to (F_C(Tx_0)\cap F_T(A,B))\cup F_T(A,B)$.
case 1: Suppose $y\in F_T(A,B)$. Then $\|Ty-T(Ty)\|\ge\beta[\|y-Ty\|+\|Ty-T^2y\|]+\gamma\delta(A,B)$ Therefore

$$\begin{aligned}\delta(A,B) &\ge \|Ty-T^2y\|\\ &\ge \frac{\beta+\gamma}{1-\beta}\delta(A,B)\\ &\ge \delta(A,B).\end{aligned}$$

Therefore $\delta(A,B)=\|Ty-T^2y\|$. implying that $Ty\in F_T(A,B)$.
case 2: If $y\in F_C(Tx_0)\cap F_T(A,B)$, then $Ty\in C$ and $\|Ty-Tx_0\|\le\delta(Tx_0,C)$.

Therefore $Ty \in F_C(Tx_0)$.

$$\begin{aligned}
\delta(Tx_0, C) &\geq \|Ty - Tx_0\| \\
&\geq \beta[\|x_0 - Tx_0\| + \|y - Ty\|] + \gamma\delta(A, B) \\
&= \beta[\delta(A, B) + \delta(A, B)] + \gamma\delta(A, B) \\
&= (2\beta + \gamma)\delta(A, B) \\
&= \delta(A, B) \\
&\geq \delta(Tx_0, C),
\end{aligned}$$

therefore $\|Tx_0 - Ty\| = \delta(Tx_0, C)$. That is $Ty \in F_C(Tx_0)$. We set $A = F_C(Tx_0) \cap F_T(A, B)$ and $B = P_T(A, B)$. From Lemma 1.2, there exits $z \in (F_C(Tx_0) \cap F_T(A, B)) \bigcup P_T(A, B)$. Therefore $z \in F_C(Tx_0) \bigcup P_T(A, B)$

Theorem 3.2. *Let $(X, \|.\|)$ be a Banach space and A and B be non-empty bounded subsets of X. Suppose that the continuous cyclic mapping $T : A \cup B \to A \cup B$ is satisfy*

$$\|Tx - Ty\| \quad \geq \quad \beta[\|x - Tx\| + \|y - Ty\|] + \gamma\delta(A, B),$$

for all $x, y \in A \cup B$, where $0 < \beta < \frac{1}{2}$, $\gamma > 0$ and $2\beta + \gamma \geq 1$. Let C be a bounded subset of A such that $T(C) \subseteq C$. Also, there exists a $x_0 \in F_T(A, B) \cap A$, define $x_{n+1} = Tx_n$ for each $n \geq 1$. If $\{x_{2n}\}$ has a convergent subsequence in $F_T(A, B) \cap A$. Then $F_T(A, B) \bigcup F_C(Tx_0) \neq \emptyset$.

Proof. With apply lemma 1.6 is similar to theorem 3.1.

4. Approximate remotal points and approximate best proximity points

In this section in first define ϵ-set of farthest points. We expression conditions where union the set of ϵ-best proximity points for a map and the set of approximate best proximity points of a set is non-empty. Also we find conditions where union the set of approximate remotal points and the set of ϵ-farthest points is non-empty

Let $(X, \|.\|)$ be a normed linear space and C be a non-empty bounded subset of X. For $\epsilon > 0$ and $x \in X$, we set

$$F_{C,\epsilon}(x) = \{x_0 \in C : \ \|x - x_0\| \geq \delta(x, C) - \epsilon\}.$$

Also, we can find the definition of the set of ϵ-best approximation in [11] as following definition

$$P_{C,\epsilon}(x) = \{x_0 \in C : \ \|x - x_0\| \leq d(x, C) + \epsilon\}.$$

Theorem 4.1. *Let A and B be non-empty subsets of a Banach space X. Suppose that the map $T : A \cup B \to A \cup B$ is a cyclic map and*

$$\|Tx - Ty\| \leq \beta[\|x - Tx\| + \|y - Ty\|] + \gamma d(A, B),$$

for all $x, y \in A \cup B$, where $\gamma > 0$, $\beta \geq 0$, $2\beta + \gamma \leq 1$ and $\epsilon > 0$. Suppose $x_0 \in P_T^{\epsilon}(A, B) \cap A$, C is a non-empty bounded subset of A, $P_{C,\epsilon}(Tx_0)$ and $P_T^{a}(A, B)$ are non-empty boundedly compact subset of X. Then the exists $z \in P_T^{\epsilon}(A, B) \bigcup P_{C,\epsilon}(Tx_0)$.

Proof. We show that $T : \ P_T^{\epsilon}(A, B) \cup P_{C,\epsilon}(Tx_0) \cap P_T^{\epsilon}(A, B) \ \to \ P_T^{\epsilon}(A, B) \cup P_{C,\epsilon}(Tx_0) \cap P_T^{\epsilon}(A, B)$.

Case 1. Suppose $y \in P_T^{\epsilon}(A, B)$, then $\|Ty - T^2y\| \leq \beta[\|y - Ty\| + \|Ty - T^2y\|] + \gamma d(A, B)$. Therefore

$$\begin{aligned} \|Ty - T^2y\| &\leq \frac{\beta + \gamma}{1 - \beta}[d(A, B) + \epsilon] \\ &\leq d(A, B) + \epsilon, \end{aligned}$$

it follows that $Ty \in P_T^{\epsilon}(A, B)$.

Case 2. If $y \in P_{C,\epsilon}(Tx_0) \cap P_T^{\epsilon}(A, B)$, therefore

$$\begin{aligned} \|Ty - Tx_0\| &\leq \beta[\|x_0 - Tx_0\| + \|y - Ty\|] + \gamma d(A, B) \\ &\leq \beta[d(A, B) + \epsilon + d(A, B) + \epsilon] + \gamma(d(A, B) + \epsilon) \\ &= (2\beta + \gamma)(d(A, B) + \epsilon) \\ &= d(A, B) + \epsilon \\ &\leq d(Tx_0, C) + \epsilon, \end{aligned}$$

implying that $Ty \in P_{C,\epsilon}(Tx_0)$.

We set $A = P_{C,\epsilon}(Tx_0) \cap P_T^{\epsilon}(A, B)$ and $B = P_T(A, B)$. From Lemma 1.2, there exits

$z \in P_{C,\epsilon}(Tx_0) \cap P_T^\epsilon(A,B) \bigcup P_T(A,B)$. Therefore $z \in P_{C,\epsilon}(Tx_0) \bigcup P_T^\epsilon(A,B)$

Theorem 4.2. *Let A and B be non-empty bounded subsets of a Banach space X. Suppose that the map $T : A \cup B \to A \cup B$ is a cyclic map and*

$$\|Tx - Ty\| \geq \beta[\|x - Tx\| + \|y - Ty\|] + \gamma\delta(A,B),$$

for all $x, y \in A \cup B$, where $\gamma > 0$, $0 < \beta < \frac{1}{2}$ and $2\beta + \gamma \geq 1$ and $\epsilon > 0$.Suppose $x_0 \in F_T^\epsilon(A,B) \cap A$, C is a non-empty bounded subset of A, $F_{C,\epsilon}(Tx_0)$ and $F_T^\epsilon(A,B)$ are non-empty boundedly compact subset of C. Then the exists $z \in F_T^\epsilon(A,B) \bigcup F_{C,\epsilon}(Tx_0)$.

Proof. First we show that $T :\ F_{C,\epsilon}(Tx_0) \cap P_T^\epsilon(A,B) \bigcup F_T^\epsilon(A,B) \to F_{C,\epsilon}(Tx_0) \cap P_T^\epsilon(A,B) \bigcup F_T^\epsilon(A,B)$

case 1. Suppose $y \in F_T^\epsilon(A,B)$, then $\|Ty - T^2y\| \geq \beta[\|y - Ty\| + \|Ty - T^2y\|] + \gamma\delta(A,B)$. Therefore

$$\begin{aligned} \|Ty - T^2y\| &\geq \frac{\beta+\gamma}{1-\beta}[\delta(A,B) - \epsilon] \\ &\leq \delta(A,B) - \epsilon, \end{aligned}$$

it follows that $Ty \in F_T^\epsilon(A,B)$.

case 2. Suppose $y \in F_{C,\epsilon}(Tx_0) \cap F_T^\epsilon(A,B)$, then

$$\begin{aligned} \|Ty - Tx_0\| &\geq \beta[\|x_0 - Tx_0\| + \|y - Ty\|] + \gamma\delta(A,B) \\ &\geq \beta[\delta(A,B) - \epsilon + \delta(A,B) - \epsilon] + \gamma(\delta(A,B) - \epsilon) \\ &= (\delta(A,B) - \epsilon)(2\beta + \gamma) \\ &= \delta(A,B) - \epsilon \\ &\geq \delta(Tx_0, C) - \epsilon. \end{aligned}$$

Therefore $Ty \in F_{C,\epsilon}(Tx_0)$. We set $A = F_{C,\epsilon}(Tx_0) \cap F_T^\epsilon(A,B)$ and $B = F_T^\epsilon(A,B)$, From Lemma 1.5, there exits $z \in F_{C,\epsilon}(Tx_0) \cap F_T^\epsilon(A,B) \bigcup F_T^\epsilon(A,B)$. Therefore $z \in F_{C,\epsilon}(Tx_0) \bigcup F_T^\epsilon(A,B)$

5. ϵ-Best Approximation Points and ϵ-Farthest Points

We have the following interesting relationship between the ϵ-best approximation and the ϵ-farthest points in normed linear spaces $(X, \|.\|)$. If $z \in W$ is a ϵ-farthest point from an $x \in X$, then z is also a ϵ-best approximation in W . Indeed, z is a ϵ-best approximation in W from any point which is on the line connecting x and z lies on the opposite side of z to x. So, if there exists no ϵ-best approximation in W, then there exists no ϵ-farthest point in W. This is shown by the following Theorem.

Definition 5.1. [11] Suppose X be a normed linear space, $x, y \in X$ and $\epsilon > 0$. We denote $x \perp_\epsilon y$, if and only if $\|x\| \leq \|x + \alpha y\| + \epsilon$ for every scalar $\alpha \in \mathbb{C}$,
also suppose W is a subspace of X. We define

$$W_\epsilon^\perp = \{x \in X : \ x \perp_\epsilon \omega \ for \ every \ \omega \in W\},$$

we say ϵ-complemented orthogonal set to W.

If $(X, \|.\|)$ is a normed linear space, $x \in X$, $r \in \mathbb{R}$ and $\epsilon > 0$. We put

$$B[x, r + \epsilon] = \{z \in X : \ \|x - z\| \leq r + \epsilon\},$$

and

$$B^c[x, r + \epsilon] = \{z \in X : \ \|x - z\| \geq r + \epsilon\}.$$

We shall denote by $[x, y]$ the line segment joining the points x and y i.e., $[x, y] = \{z \in X : \ \|x - z\| + \|z - y\| = \|x - y\|\}$. The set $[x, y >= \{z \in X : \ \|x - y\| + \|y - z\| = \|x - z\|\}$ denotes a half ray starting from x and passing through y i.e., it is union of the segments $[x, z]$ where $[x, y] \subseteq [x, z]$.

Theorem 5.2. *Let W be a nonempty bounded closed subset of a externally normed space $(X, \|.\|)$ and $\epsilon > 0$. If $z \in F_{W,\epsilon}(x)$, then $z \in P_{W,\epsilon}(x')$ for every $x' \in [x, z > -[x, z]$.*

Proof. Suppose $z \in F_{W,\epsilon}(x)$, then for every $y \in W$

$$\begin{aligned} \|z - x\| &\geq \delta(x, W) - \epsilon \\ &\geq \|x - y\| - \epsilon. \end{aligned}$$

Suppose $x' \in [x, z > -[x, z]$ be arbitrary. For every $y \in W$

$$\begin{aligned} \|x' - z\| &= \|x' - x\| - \|x - z\| \\ &\leq \|x' - y\| + \|y - x\| - \|x - y\| + \epsilon \\ &= \|x' - y\| + \epsilon. \end{aligned}$$

Therefore $\|x' - z\| \leq d(x', W) + \epsilon$. That is $z \in P_{W,\epsilon}(x')$.

Theorem 5.3. *If W is a bounded subset of a normed linear space $(X, \|.\|)$, $g_0 \in F_{W,\epsilon}(x_0)$ for $x_0 \in X$ and $\epsilon > 0$, then $g_0 \in F_{W,\epsilon}(x_\lambda)$, where $x_\lambda \in [g_0, x_0 >$.*

Proof. Consider $g_0 \in F_{W,\epsilon}(x_0)$, for every $g \in W$

$$\begin{aligned} \|x_\lambda - g_0\| &= \|x_\lambda - x_0\| + \|x_0 - g_0\| \\ &\geq \|x_\lambda - x_0\| + \|x_0 - g\| \\ &\geq \|x_\lambda - g\|. \end{aligned}$$

Therefore $g_0 \in F_W(x_\lambda)$.

Theorem 5.4. *Suppose $(X, \|.\|)$ is a normed linear space, W is a non-empty subset of X, $x \in X$ and $\epsilon > 0$. If $\{r_n\}$ is a sequence and convergent to $d(x, W)$ and $r_n > d(x, W)$. Then*

$$P_{W,\epsilon}(x) = \bigcap_{n=1}^{\infty} B[x, r_n + \epsilon] \cap W.$$

and

$$P_{W,\epsilon}(x) = B[x, d(x, d(x, W) + \epsilon] \cap W.$$

Proof. Suppose $z \in P_{W,\epsilon}(x)$, then $\|x-z\| \leq d(z, W)+\epsilon$. Therefore $\|x-z\| \leq r_n+\epsilon$, that is $z \in B[x, r_n+\epsilon]\cap W$. If $z \in \bigcap_{n=1}^{\infty} B[x, r_n+\epsilon]\cap W$, then $\|x-z\| \leq r_n+\epsilon$, since $r_n \to d(x, W)$ as $n \to \infty$. We have $\|x - z\| \leq d(x, W) + \epsilon$. Therefore $z \in P_{W,\epsilon}(x)$. Also, by definition it is clear that $P_{W,\epsilon}(x) = B[x, d(x, d(x, W) + \epsilon] \cap W$.

Theorem 5.5. *Suppose $(X, \|.\|)$ is a normed linear space, W is a non-empty subset of X, $\omega_0 \in W$ and $\epsilon > 0$. There exits a sequence $\{r_n\}_{n\geq 1}$ such that $r_n > d(z, W)$ and*

$$P^{-1}_{W,\epsilon}(\omega_0) \subseteq \bigcap_{n=1}^{\infty} B[\omega_0, r_n + \epsilon].$$

Proof. We know that $z \in P^{-1}_{W,\epsilon}(\omega_0)$, then $\|z - \omega_0\| \leq d(z, W) + \epsilon$. We set $r_n = d(z, W) + \frac{1}{n}$ for every $n \geq 1$. Then $z \in B[\omega_0, r_n + \epsilon]$ for every $n \geq 1$.

Theorem 5.6. *Suppose $(X, \|.\|)$ is a normed linear space, W is a non-empty bounded subset of X, $x \in X$ and $\epsilon > 0$. Then*

$$F_{W,\epsilon}(x) = B^c[x, \rho(x, W) - \epsilon] \cap W.$$

For $x \in X$ and $\omega_0 \in W$, if $\omega_0 \in F_W(x)$, then

$$F^{-1}_W(\omega_0) \subseteq B^c[\omega_0, \rho(x, W) - \epsilon].$$

Proof. It is clear by definition.

If W be a remotal set of normed linear space $(X, \|.\|)$. We can define the map

$$\psi_{W,\epsilon}: \ X \to 2^W, \ by \ \psi_{W,\epsilon}(x) = F_{W,\epsilon}(x).$$

Let X and Y be metric spaces. A set valued map $f: \ X \to 2^Y$ is called u.s.c. if and only if the set

$$\{x \in X | \ f(x) \cap N \neq \emptyset\},$$

is closed for each subset N of Y .

Lemma 5.7. *Let $(X, \|.\|)$ be a normed linear space, W is bounded subset of X and $\epsilon > 0$. Then $\psi_{W,\epsilon}$ is u.s.c if and only if for each closed subset N of W, the subset*

$$\bigcup_{y \in N} \{x \in X | \ y \in \psi_{W,\epsilon}(x)\},$$

is closed.

Proof. It is clear, by definition.

Theorem 5.8. *If W is a subspace of the normed linear space $(X, \|.\|)$, then $\psi_{W,\epsilon}$ is u.s.c. if and only if for each closed subset N of W, $N + \psi^{-1}_{W,\epsilon}(0)$ is closed.*

Proof. It is enough, we prove that

$$N + \psi^{-1}_{W,\epsilon}(0) = \bigcup_{y \in N} \{x \in X | \ y \in \psi_{W,\epsilon}(x)\}.$$

$$\begin{aligned}
z \in N + \psi^{-1}_{W,\epsilon}(0) \quad &\Leftrightarrow \quad z = u + y, \ for \ some \ u \in F^{-1}_{W,\epsilon}(x), \ y \in N \\
&\Leftrightarrow \quad \|z - y\| \geq \rho(z - y, W) - \epsilon, \ y \in N \\
&\Leftrightarrow \quad \|z - y\| \geq \rho(z, W) - \epsilon, \ y \in N, \ W \ is \ subspace \\
&\Leftrightarrow \quad y \in N, \ y \in F_{W,\epsilon}(z) \\
&\Leftrightarrow \quad y \in N, \ z \in \{x \in X: \ y \in F_{W,\epsilon}(x) \\
&\Leftrightarrow \quad z \in \bigcup_{y \in N} \{x \in X | \ y \in \psi_{W,\epsilon}(x)\}.
\end{aligned}$$

Theorem 5.9. *Let $(X, \|.\|)$ be a normed linear space, W is subspace of X, $\omega_0 \in W$ and $\epsilon > 0$. Then*

$$X = W + P^{-1}_{W,\epsilon}(\omega_0).$$

Proof. If $x \in X$, the set $P_{W,\epsilon}(x) \neq \emptyset$, therefore there exists a $y \in W$ such that $y \in P_{W,\epsilon}(x)$. Then $x - y + \omega_0 \in P^{-1}_{W,\epsilon}(\omega_0)$. We set $u = x - y + \omega_0$, then $x = y - \omega_0 + u$. Therefore $X = W + F^{-1}_{W,\epsilon}(\omega_0)$.

Theorem 5.10. *Let $(X, \|.\|)$ be a normed linear space, W is a bounded subset of X and $\epsilon > 0$. Then*

$$X = W + F^{-1}_{W,\epsilon}(0).$$

Proof. If $x \in X$, the set $F_{W,\epsilon}(x) \neq \emptyset$, therefore there exists a $y \in W$ such that $y \in F_{W,\epsilon}(x)$. Then $u = x - y \in F^{-1}_{W,\epsilon}(0)$. Therefore $x = y + u$, and $X = W + F^{-1}_{W,\epsilon}(0)$.

Theorem 5.11. *If W is a linear subspace of a normed linear space X. Then $P^{-1}_{W,\epsilon}(0) = W^{\perp}_{\epsilon}$, therefore for $\omega_0 \in W$, we have $P^{-1}_{W,\epsilon}(\omega_0) = \omega_0 + W^{\perp}_{\epsilon}$.*

Proof. It is clear that $P^{-1}_{W,\epsilon}(\omega_0) = \omega_0 + P^{-1}_{W,\epsilon}(0)$. We must prove $P^{-1}_{W,\epsilon}(0) = W^{\perp}_{\epsilon}$.

$$\begin{aligned} x \in P^{-1}_{W,\epsilon}(0) \quad &\Leftrightarrow \quad \|x\| \leq d(x, W) + \epsilon \\ &\Leftrightarrow \quad \|x\| \leq \|x + \alpha y\| + \epsilon \ \forall y \in W \\ &\Leftrightarrow \quad x \in W^{\perp}_{\epsilon}. \end{aligned}$$

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