SOME ERGODIC PROPERTIES OF HYPER MV–ALGEBRA DYNAMICAL SYSTEMS

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Abstract. This paper provides a review on major ergodic features of semi-independent hyper MV–algebra dynamical systems. Theorems are presented to make contribution to calculate the entropy. Particularly, it is proved that the total entropy of those semi-independent hyper MV–algebra dynamical systems that have a generator can be calculated with respect to their generator rather than considering all the partitions.

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1. Introduction

The concept of an MV–algebra was introduced by C.C. Chang in 1958 to prove the completeness theorem of infinite valued Lukasiewicz propositional calculus. Hyper structure theory was initiated by F. Marty at 8th congress of Scandinavian

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Mathematicians in 1934. Since then, many researchers have worked in these areas, for example see [2, 4, 7]. For the first time, the notion of hyper MV−algebras was introduced in [5] as a generalization of MV−algebras. In [9], Rasouli and Davvaz studied several properties of hyper MV−algebras. Then in [10], they studied homomorphisms, dual homomorphisms and strong homomorphisms between hyper MV−algebras. Recently, L. Torkzadeh and Sh. Ghorbani found the conditions under which a hyper MV−algebra becomes an MV−algebra, and they characterized hyper MV−algebras of order 2 and order 3 [6]. P. Corsini and V. Leoreanu, in their book "Applications of hyperstructure theory" discussed applications of hyperstructures in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs [2], and B. Davvaz and V. Leoreanu-Fotea presented applications of hyperstructures in chemistry and physics [4], also see [3]. Thus, by extending the entopy function to hyperstructures, one can provide efficient criteria to measure the complexity of the systems in the categories mentioned above.

In this paper, essential ergodic characteristics of HMV−algebra dynamical systems are studied. In the next section that is the main section of this paper, the fundamental properties are studied, and the concept of generator for semi-independent hyper MV−algebra dynamical systems is defined. Then theorems that help calculate the entropy are given. The rest of this section is dedicated to a brief review of hyper MV−algebras and semi-independent systems over them.

Now, we recall the definition of hyper MV−algebra from [5, 8, 9].

Definition. 1.1 ([7])**.** A hyper MV−algebra is a non-empty set, 'M', endowed with a hyperoperation '$\oplus : H \times H \longrightarrow P^*(H)$', a unary operation '$* : H \longrightarrow H$', and a constant '0' satisfying the following axioms for all x, y, $z \in M$:

c1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
c2) $x \oplus y = y \oplus x$;
c3) $(x^*)^* = x$;
c4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$;
c5) $0^* \in x \oplus 0^*$;
c6) $0^* \in x \oplus x^*$;
c7) if $x \ll y$ and $y \ll x$, then $x = y$, where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

Definition. 1.2 ([7])**.** For nonempty subsets A and B of M, we have the following definitions:

d1) $A \ll B$ iff there exist $a \in A$ and $b \in B$ such that $a \ll b$;

d2) $A^* := \{a^* \mid a \in A\}$;

d3) $1 := 0^*$.

Example. 1.3 ([9])**.** There are two methods to obtain a hyper MV–algebra starting from an MV–algebra $(M, \oplus, *, 0, 1)$:

Method 1. Let $(M, \oplus, *, 0, 1)$ be an MV–algebra. Define $x \oplus' y := \{x \oplus y\}$ for each $x, y \in M$. One can see easily that $(M, \oplus', *, 0, 1)$ is a hyper MV–algebra.

Method 2. Define $x \oplus'' y := \{t \in M \mid 0 \ll t \ll x \oplus y\} = [0, x \oplus y]$ for each $x, y \in M$. Now, it will be shown that $(M, \oplus'', *, 0, 1)$ is a hyper MV–algebra. Clearly, $\oplus''$ and $*$ are well defined. For each $x, y, z \in M$, $1 \in x \oplus'' 1 = M$, $x \in x \oplus 0 = [0, x]$ and

$$x \oplus'' (y \oplus'' z) = \{t \in M \mid 0 \ll t \ll x \oplus (y \oplus z)\} = \{t \in M | 0 \ll t \ll (x \oplus y) \oplus z\} = (x \oplus'' y) \oplus'' z.$$

Also,

$$\begin{aligned}
(x^* \oplus'' y)^* \oplus'' y &= \bigcup_{\{s \in M | 0 \ll s \ll x^* \oplus y\}} s^* \oplus'' y \\
&= \bigcup_{\{s \in M | 0 \ll s \ll x^* \oplus y\}} \{t \in M \mid 0 \ll t \ll s^* \oplus y\} \\
&= \bigcup \{t \in M \mid 0 \ll t \ll (x^* \oplus y)^* \oplus y\} \\
&= \bigcup \{t \in M \mid 0 \ll t \ll (y^* \oplus x)^* \oplus x\} \\
&= \bigcup_{\{s \in M | 0 \ll s \ll y^* \oplus x\}} \{t \in M \mid 0 \ll t \ll s^* \oplus x\} \\
&= (y^* \oplus'' x)^* \oplus'' x.
\end{aligned}$$

Now, let $x \ll'' y$, thus $1 \in [0, x^* \oplus y]$ and it implies that $1 = x^* \oplus y$, then by MV–algebra's properties, it is obtained that $x \ll y$. Similarly, if $y \ll'' x$, then $y \ll x$. Now, $x \ll'' y$ and $y \ll'' x$ imply that $x \ll y$ and $y \ll x$, respectively which result that $x = y$. Also, it can be shown that $x \odot'' y = [x \odot y, 1]$. This shows that $(M, \oplus'', *, 0, 1)$ is a hyper MV–algebra.

Example. 1.4 ([9])**.** Let $M = \{0, a, b, c, 1\}$. Consider Tables 1(a) and 1(b). Then, $(M, \oplus, *, 0, 1)$ is a hyper MV–algebra.

Proposition. 1.5 ([7])**.** Every hyper MV–algebra satisfies the following statements for every x, y, $z \in M$ and for every subsets A, B and C of M:

TABLE 1. Cayley tables for the hyperoperation and the unary operation

(a) $\oplus$

$\oplus$	0	a	b	c	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,b\}$	$\{0,c\}$	M
a	$\{0,a\}$	$\{0,a\}$	M	$\{0,a,c\}$	M
b	$\{0,a,b\}$	M	$\{0,a,b,1\}$	$\{0,a,b,c\}$	M
c	$\{0,c\}$	$\{0,a,c\}$	$\{0,a,b,c\}$	M	M
1	M	M	M	M	M

(b) $*$

x	x^*
1	0
b	a
c	c
a	b
0	1

e1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$;

e2) $0 \ll x$, $x \ll 1$;

e3) $x \ll x$;

e4) if $x \ll y$, then $y^* \ll x^*$;

e5) if $A \ll B$, then $B^* \ll A^*$;

e6) $A \ll A$;

e7) if $A \subseteq B$, then $A \ll B$, where $A \neq \emptyset$;

e8) $x \ll x \oplus y$, $A \ll A \oplus B$, where $A \neq \emptyset$ and $B \neq \emptyset$;

e9) $(A^*)^* = A$;

e10) $0 \oplus 0 = \{0\}$;

e11) $x \in x \oplus 0$;

e12) if $y \in x \oplus 0$, then $y \ll x$;

e13) if $y \oplus 0 = x \oplus 0$, then $x = y$.

Definition. 1.6 ([8])**.** A partition of unity U of 1 - partition for short - in M is a k–tuple $(u_1, \dots, u_k)$ of elements of M such that $1 \in u_1 \oplus \dots \oplus u_k$. Moreover, the index set of U is denoted by I_U, i.e. $I_U = \{1, \dots, k\}$. Also, $S(U) := \{u_1, \dots, u_k\}$ and $P_M := \{U \mid U \text{ is a partition of unity of } 1\}$.

Definition. 1.7 ([8])**.** Let $U = (u_1, \dots, u_k)$ and $V = (v_1, \dots, v_n)$ be two partitions of unity. A common refinement - c–refinement for short - of U and V is defined as any matrix '$C = \{c_{ij} \mid i \in I_U \text{ and } j \in I_V\}$' such that $u_i \in c_{i1} \oplus \dots \oplus c_{in}$ for every $i \in I_U$ and $v_j \in c_{1j} \oplus \dots \oplus c_{kj}$ for every $j \in I_V$.

Definition. 1.8 ([8])**.** Let $U, V \in P_M$. The partitions U and V are said to be relatively normal if there exists a c–refinement for U and V. The notation $\{U, V\} \in RN$ is used to show that U and V are relatively normal. Moreover, let X and Y are the sets that are composed of partitions of unity. X and Y are said to be relatively normal if for every $U \in X$ and every $V \in Y$, $\{U, V\} \in RN$. The notation $\{X, Y\} \in RN$ is used to show that X and Y are relatively normal.

Definition. 1.9 ([8])**.** Let $U_i \in P_M$ for $i \in \{1, \ldots, s\}$, and $s \geq 3$. The partitions $U_1, \ldots, U_s$ are said to be relatively normal if every way of computing c–refinements of $U_1, \ldots, U_s$ leads to find at least one c–refinement. The notation $\{U_1, \ldots, U_s\} \in RN$ is used to show that $U_1, \ldots, U_s$ are relatively normal. Furthermore, the notation $\{U_1, \ldots, U_s\} \notin RN$ implies that the partitions $U_1, \ldots, U_s$ are not relatively normal. Moreover, the notation $C \in \vee_{i=1}^{s} U_i$ is applied to show that C is a c–refinement of $U_1, \ldots, U_s$.

Note that $\vee_{i=1}^{1} U_i = U_1$. In this case, by $C \in \vee_{i=1}^{1} U_i$ it is understood that $C = U_1$.

Definition. 1.10 ([8])**.** Let M be a hyper MV–algebra, and $m : M \longrightarrow [0, 1]$ be a mapping. Then, m is called semi-independent if for every $U, V \in P_M$ and every $C \in U \vee V$,

$$\max_{i \in I_U} m(u_i). \max_{j \in I_V} m(v_j) \leq \max_{(i,j) \in I_C} m(c_{ij});$$

where $\{U, V\} \in RN$, and $I_C = \{(i, j) \mid i \in I_U \text{ and } j \in I_V\}$.

Definition. 1.11 ([8])**.** A semi-independent dynamical system on a hyper MV–algebra is a couple of mappings '$m : M \longrightarrow [0, 1]$' and '$T : M \longrightarrow M$' satisfying the following conditions:

f1) $m(t) = m(a) + m(b), \qquad \forall t \in a \oplus b \setminus \{a, b\}$;
f2) $T(a \oplus b) = T(a) \oplus T(b)$;
f3) $m(1) = 1$ and $T(1) = 1$;
f4) $m(T(a)) = m(a)$;
f5) m is semi-independent;

for every $a, b \in M$.

Remark. 1.12 ([8])**.** If $a \ll b$, then $m(a) = m(b)$, where $a \neq 0, a \neq 1$ and $b \neq 1$.

Remark. 1.13 ([8])**.** For every partition of unity $U = (u_1, \ldots, u_k)$, $\max_i m(u_i) \neq 0$.

Definition. 1.14 ([8])**.** Let $U = (u_1, \ldots, u_k)$ be a partition of unity. Its entropy is defined by the formula $H(U) = -\log \max\limits_{i \in I_U} m(u_i)$.

Definition. 1.15 ([8])**.** Let $U = (u_1, \ldots, u_k)$ and $V = (v_1, \ldots, v_n)$ be two partitions of unity,
$\{U, V\} \in RN$, and $C \in U \vee V$. The conditional entropy of U given V with respect to C is defined as

$$H_C(U|V) = |\log \frac{\max_{(i,j)\in I_C} m(c_{i,j})}{\max_{j \in I_V} m(v_j)}|.$$

Definition. 1.16 ([8])**.** Let (M, m, T) be a semi-independent hyper MV–algebra dynamical system, and $U \in P_M$. Then, U is said to be a perfect partition for T if for every positive integer $n \geq 2$, $\{U, T(U), \ldots, T^{n-1}(U)\} \in RN$. The collection of all perfect partitions of T is denoted by P_T.

Definition. 1.17 ([8])**.** For any partition $U \in P_T$ and any positive integer n, we define

$$H_n(T, U) = \inf\{H(C) \mid C \in U_n(T)\};$$

where $U_n(T) = \{C \mid C \in \vee_{i=0}^{n-1} T^i(U)\}$. If there is no place for ambiguity, the notation U_n is used rather than $U_n(T)$. If $U \in P_M \smallsetminus P_T$, then we set $H_n(T, U) = 0$ for every positive integer $n \geq N$, where N is the smallest positive integer for which $\{U, T(U), \ldots, T^N(U)\} \notin RN$.

Theorem. 1.18 ([8])**.** $\lim_{n\to\infty} \dfrac{1}{n} H_n(T, U)$ exists.

Definition. 1.19 ([8])**.** Entropy of a semi-independent hyper MV–algebra dynamical system (M, m, T) is defined by the formula

$$h(T) = \sup\{h(T, U) \mid U \in P_M\}, \quad \text{where } h(T, U) = \lim_{n\to\infty} (\frac{1}{n}) H_n(T, U).$$

2. Some ergodic properties

During this section, some characteristics semi-independent hyper MV–algebras dynamical systems and their entropy are studied. a couple of notions and theorems so as to help caculate the entropy are given.

Definition. 2.1. Let U and V be two relatively normal partitions, and C be any c–refinement of U and V. Then, U is said to be C–dominated if for every u_i there exists c_{pj} such that $u_i \ll c_{pj}$, where $u_i \in S(U)$, and $c_{pj} \in S(C)$.

Definition. 2.2. Let $C = \{c_{ij} \mid i \in I_U \text{ and } j \in I_V\}$ be any c–refinement of the relatively normal partitions $U = (u_1, \dots, u_k)$ and $V = (v_1, \dots, v_n)$, and

$$S_i(C) := \{c_{i1}, \dots, c_{in}\} \text{ for } i \in I_U \text{ and } S^j(C) := \{c_{1j}, \dots, c_{kj}\} \text{ for } j \in I_V.$$

U is said to be C_i–dominating if $u_i \notin S_i(C)$. By the statement 'U is C–dominating', it is understood that U is C_i–dominating for every $i \in I_U$. C^j–dominating and V being C–dominating are defined similarly.

For any partition of unity $U = (u_1, \dots, u_k)$ of a semi-independent hyper MV–algebra dynamical system (M, m, T), by u_m we mean the element of U for which $m(u_m) = \max_i m(u_i)$. If C is any c–refinement of the relatively normal partitions U and V, by the statement 'U is C_{im}–dominating', we mean that U is C_i–dominating and $c_m \in S_i(C)$.

Theorem. 2.3. Let (M, m, T) be a semi-independent hyper MV–algebra dynamical system. If $U = (u_1, \dots, u_k)$, $V = (v_1, \dots, v_n)$ and $W = (w_1, \dots, w_q)$ are partitions of unity, then:

g1) $H(C) \leq H(U) + H_C(V|U)$. Moreover, if U is C–dominated, or $u_m \in S(C)$, then $H(C) = H(U) - H_C(V|U)$. If U is C–dominating, or U is C_{im}–dominating, then $H(C) = H(U) + H_C(V|U)$;

g2) if $S(U) \subseteq S(V)$, then $H(U) = H(V)$;

g3) if $S(U) \subseteq S(V)$, then $H_{C'}(U|W) \leq H_{C''}(V|W)$, where U is C'–dominating, V is C''–dominated, and W is C'–dominated;

g4) if $S(U) \subseteq S(V)$, then $H_{C'}(U|W) \geq H_{C''}(V|W)$, where U is C'–dominating, V is C''–dominated, and W is C''–dominating;

g5) if $S(U) \subseteq S(V)$, then $H_{C'}(W|U) \geq H_{C''}(W|V)$, where V is C''–dominated, W is C''–dominating, and W is C'–dominated;

g6) if $S(U) \subseteq S(V)$, then $H_{C'}(W|U) \leq H_{C''}(W|V)$, where U is C'–dominating, W is C''–dominating, and W is C'–dominated;

g7) $H_C(U|V) \leq H(U)$, where V is C–dominating;

g8) $H_{T(C)}(T(U)|T(V)) = H_C(U|V)$;

g9) $H_{W'}(C|W) \leq H_{C'}(U|W) + H_{C''}(V|W)$, where $\{U, V\} \in RN$, $C \in U \vee V$, $\{C, W\} \in RN$, $W' \in C \vee W$, C is W'–dominated, W is W'–dominating, U is C–dominated, and U is C'–dominating;

g10) $H_{W'}(C|W) \leq H_{C'}(U|W) + H_{C''}(V|W)$, where $\{U, V\} \in RN$, $C \in U \vee V$,

$\{C, W\} \in RN$, $W' \in C \vee W$, C is W'–dominating, W is W'–dominated, U is C–dominating, and U is C'–dominated;

g11) $H_{W'}(C|W) \leq H_{C'}(U|W) + H_{D'}(V|D)$, where $\{U, V\} \in RN$, $C \in U \vee V$, $\{C, W\} \in RN$, $W' \in C \vee W$, $D \in U \vee W$, $\{V, D\} \in RN$, $D' \in V \vee D$, C is W'–dominated, W is W'–dominating, U is C–dominated, and U is C'–dominating;

g12) $H_{W'}(C|W) \leq H_{C'}(U|W) + H_{D'}(V|D)$, where $\{U, V\} \in RN$, $C \in U \vee V$, $\{C, W\} \in RN$, $W' \in C \vee W$, $D \in U \vee W$, $\{V, D\} \in RN$, $D' \in V \vee D$, C is W'–dominating, W is W'–dominated, U is C–dominating, and U is C'–dominated.

Proof. g1) We have

$$
\begin{aligned}
H(C) = -\log m(c_m) &= -\log(\frac{m(c_m)}{m(u_m)} m(u_m)) \\
&= -\log \frac{m(c_m)}{m(u_m)} + (-\log m(u_m))
\end{aligned}
$$

$$
\begin{aligned}
&\leq |\log \frac{m(c_m)}{m(u_m)}| + H(U) \\
&= H_C(V|U) + H(U).
\end{aligned}
$$

Notice that if U is C–dominated, or $u_m \in S(C)$, then $m(u_m) \leq m(c_m)$. Thus,

$$
H_C(V|U) = |\log \frac{m(c_m)}{m(u_m)}| = \log \frac{m(c_m)}{m(u_m)}. \tag{1}
$$

Now, using Equation (1), the obtained result is

$$\begin{aligned} H(C) &= -\log m(c_m) = -\log(m(u_m)\frac{m(c_m)}{m(u_m)}) \\ &= (-\log m(u_m)) + (-\log \frac{m(c_m)}{m(u_m)}) \\ &= H(U) - |\log \frac{m(c_m)}{m(u_m)}| = H(U) - H_C(V|U). \end{aligned}$$

The other part is proved similarly.

g2) It is straightforward from the definitions.

g3) Considering $S(U) \subseteq S(V)$, the obtained result is

$$m(u_m) = m(v_m). \tag{2}$$

Since U is C'–dominating, it follows that

$$m(c'_m) \leq m(u_m). \tag{3}$$

Also,

$$m(v_m) \leq m(c''_m); \tag{4}$$

since V is C''–dominated. Now, considering Equations (2), (3) and (4), the obtained result is

$$m(c'_m) \leq m(c''_m). \tag{5}$$

By using Equation (5), we have

$$\log \frac{m(c'_m)}{m(w_m)} \leq \log \frac{m(c''_m)}{m(w_m)} \leq |\log \frac{m(c''_m)}{m(w_m)}| = H_{C''}(V|W). \tag{6}$$

Since W is C'–dominated, it follows that

$$\log \frac{m(c'_m)}{m(w_m)} \geq 0. \tag{7}$$

Considering Equations (6) and (7), then we obtain $H_{C'}(U|W) \leq H_{C''}(V|W)$.

g4) Similarly as the proof of (g3).

g5) Considering W is C''–dominating, the obtained result is $m(c''_m) \leq m(w_m)$. Since W is C'–dominated, then $m(w_m) \leq m(c'_m)$. Thus,

$$m(c''_m) \leq m(c'_m). \tag{8}$$

Since $S(U) \subseteq S(V)$, it follows that

$$\frac{1}{m(v_m)} = \frac{1}{m(u_m)}. \tag{9}$$

By Equations (8) and (9), the obtained result is

$$\log \frac{m(c''_m)}{m(v_m)} \leq \log \frac{m(c'_m)}{m(u_m)} \leq |\log \frac{m(c'_m)}{m(u_m)}| = H_{C'}(W|U). \tag{10}$$

Since V is C''–dominated, it follows that

$$\log \frac{m(c''_m)}{m(v_m)} \geq 0. \tag{11}$$

Now, Equations (10) and (11) imply that $H_{C'}(W|U) \geq H_{C''}(W|V)$.

g6) Similarly as the proof of (g5).

g7) We have $m(u_m) \leq \dfrac{m(c_m)}{m(v_m)}$. Thus,

$$H(U) = -\log m(u_m) \geq -\log \frac{m(c_m)}{m(v_m)}. \tag{12}$$

Since V is C–dominating, it follows that

$$H_C(U|V) = -\log \frac{m(c_m)}{m(v_m)}. \tag{13}$$

Considering Equations (12) and (13), the obtained result is $H_C(U|V) \leq H(U)$.

g8) It is easy to check that $T(C) \in T(U) \vee T(V)$. Then,

$$H_{T(C)}(T(U)|T(V)) = |\log \frac{m(T(c_m))}{m(T(v_m))}| = |\log \frac{m(c_m)}{m(v_m)}| = H_C(U|V).$$

g9) Considering U is C'–dominating, the obtained result is

$$m(c'_m) \leq m(u_m). \tag{14}$$

Since U is C–dominated, it follows that

$$m(u_m) \leq m(c_m). \tag{15}$$

In addition, considering C is W'–dominated, it is obtained that

$$m(c_m) \leq m(w'_m). \tag{16}$$

Equations (14), (15) and (16) occur that $m(c'_m) \leq m(w'_m)$. Then,

$$-\log \frac{m(w'_m)}{m(w_m)} \leq -\log \frac{m(c'_m)}{m(w_m)}. \tag{17}$$

Since W is W'–dominating, it follows that

$$H_{W'}(C|W) = -\log \frac{m(w'_m)}{m(w_m)}. \tag{18}$$

Now, using Equations (17) and (18), the obtained result is

$$\begin{aligned} H_{W'}(C|W) &\leq -\log \frac{m(c'_m)}{m(w_m)} \\ &\leq |\log \frac{m(c'_m)}{m(w_m)}| + H_{C''}(V|W) \\ &= H_{C'}(U|W) + H_{C''}(V|W). \end{aligned}$$

g10) Suppose U is C–dominating. This implies that

$$m(c_m) \leq m(u_m). \tag{19}$$

Since U is C'–dominated, it follows that

$$m(u_m) \leq m(c'_m). \tag{20}$$

In addition, suppose that C is W'–dominating. It is obtained that

$$m(w'_m) \leq m(c_m). \tag{21}$$

Equations (19), (20) and (21) imply that $m(w'_m) \leq m(c'_m)$. Then,

$$\log \frac{m(w'_m)}{m(w_m)} \leq \log \frac{m(c'_m)}{m(w_m)}. \tag{22}$$

Since W is W'–dominated, it follows that

$$H_{W'}(C|W) = \log \frac{m(w'_m)}{m(w_m)}. \tag{23}$$

Now, using Equations (22) and (23), it is concluded that

$$\begin{aligned} H_{W'}(C|W) &\leq \log \frac{m(c'_m)}{m(w_m)} \\ &\leq |\log \frac{m(c'_m)}{m(w_m)}| + H_{C''}(V|W) \\ &= H_{C'}(U|W) + H_{C''}(V|W). \end{aligned}$$

g11) The proof is similar to the proof of (g9).

g12) Similarly as the proof of (g10). □

Definition. 2.4. Let (M, m, T) be a semi-independent hyper MV–algebra dynamical system, and $U, V \in P_M$. Then U is said to be a refinement of V modulo m if $m(u_m) \geq m(v_m)$. The notation $V \overset{m}{\leq} U$ is used to show that U is a refinement of V modulo m. U and V are said to be m–equivalent if $V \overset{m}{\leq} U$ and $U \overset{m}{\leq} V$. The notation $U \overset{m}{\sim} V$ is used to show that U and V are m–equivalent.

Moreover, if U' and V' are the sets that are composed of partitions of unity of (M, m, T), by saying U' is a refinement of V' modulo m, $V' \overset{m}{\leq} U'$, it is understood that for every $V \in V'$, $V \overset{m}{\leq} U$ for some $U \in U'$. The notation $U' \overset{m}{\sim} V'$ is used if $V' \overset{m}{\leq} U'$ and $U' \overset{m}{\leq} V'$.

Remark. 2.5. Let (M, m, T) be a semi-independent hyper MV–algebra dynamical system, $U, V \in P_T$, $\{U, V\} \in RN$, and $C \in U \vee V$. It is straightforward to check that

l1) $C \overset{m}{\leq} U$ if and only if $H(C) = H(U) + H_C(V|U)$;
l2) $U \overset{m}{\leq} C$ if and only if $H(C) = H(U) - H_C(V|U)$;
l3) $V \overset{m}{\leq} U$ if and only if $H(V) \geq H(U)$. In particular, $U \overset{m}{\sim} V$ if and only if $H(U) = H(V)$;
l4) $H_C(U|W) = 0$ if and only if $W \overset{m}{\sim} C$.
Moreover, we have
l5) if $U \leq V$, then $U \overset{m}{\leq} V$.

Definition. 2.6. Let $U, V \in P_T$. We say that U is a generator of V of order K if the following conditions are satisfied:

m1) there exists a positive integer N such that for every $n \geq N$, $U_n \overset{m}{\leq} V_n$;
m2) $K = \min\{N \mid N$ satisfies Condition (m1) $\}$.

In this case, the notation $V \ll_{G_K} U$ is applied. If it is not important to emphasize on K, it is just written as $V \ll_G U$. In addition, a perfect partition of unity, U, of T is called a generator of the semi-independent hyper MV–algebra dynamical system (M, m, T) if for every $V \in P_T$, $V \ll_G U$. The notation $U <_G T$ is used to show that U is a generator of T.

Moreover, if U' and V' are the sets that are composed of partitions of unity of (M, m, T), by saying U' is a generator of V', $V' \ll_G U'$, it is understood that for every $V \in V'$, $V \ll_G U$ for some $U \in U'$.

Theorem. 2.7. Let (M, m, T) be a hyper MV–algebra dynamical system, and $P_T \neq \emptyset$. If $G = P_T \setminus U_0$, then $h(T) = \sup\{h(T, W) \mid W \in G\}$, where

$$U_0 = \{U \in P_T \mid U \ll_G V \text{ for some } V \in P_T\}.$$

Proof. Suppose that $U \ll_{G_K} V$ for some $V \in P_T$. Let $n \geq K$; then, for every $C \in V_n$, $C \overset{m}{\leq} D$ for some $D \in U_n$. Thus, $H_n(T, U) \leq H_n(T, V)$ for $n \geq K$. It occurs that $h(T, U) \leq h(T, V)$. □

Theorem. 2.8. Let (M, m, T) be a semi-independent hyper MV–algebra dynamical system, and k be any positive integer. If $P_T \neq \emptyset$, and for every $U \in P_T$, the following statements are satisfied:

n1) there exists $N > 0$ such that for every $n \geq N$, and every $C \in U_k(T)$,

$$C \in P_{T^k} \text{ and } C_n(T^k) \overset{m}{\sim} U_{nk}(T);$$

n2) $U \ll_G W$ for some $W \in G_k$, where $G_k = \bigcup_{V \in P_T} V_k(T)$;

then, $h(T^k) = kh(T)$. If $P_T = \emptyset$, then $h(T^k) \geq kh(T)$.

Proof. Evidently, we have

$$kh(T, U) = h(T^k, C). \tag{24}$$

Now, considering (n2), there exists $V \in P_T$ such that $H_n(T^k, U) \leq H_n(T^k, W)$ for some $W \in V_k(T)$; thus, $h(T^k, U) \leq h(T^k, W)$. It follows that

$$\sup_{W \in G_k} h(T^k, W) = \sup_{U \in P_M} h(T^k, U). \tag{25}$$

Considering, Equations (24) and (25), it is obtained that

$$kh(T) = k \sup_{U \in P_T} h(T, U) = \sup_{C \in G_k} h(T^k, C) = \sup_{W \in G_k} h(T^k, W) = \sup_{U \in P_M} h(T^k, U) = h(T^k).$$

If $P_T = \emptyset$, then $h(T) = 0$. Therefore, $h(T^k) \geq 0 = kh(T)$. □

Theorem. 2.9. Let (M, m, T) be a semi-independent hyper MV–algebra dynamical system, and $U \in P_T$ for which there exists $N > 0$ such that for every $n \geq N$, $\{U\} \overset{m}{\leq} U_n$. Then $h(T, U) = 0$.

Proof. Let $n \geq N$. Since $\{U\} \overset{m}{\leq} U_n$, then $H(C) \leq H(U)$ for some $C \in U_n$. Thus, $H_n(T, C) \leq H(U)$; then,

$$h(T, U) = \lim_{n\to\infty} \frac{1}{n} H_n(T, C) \leq \lim_{n\to\infty} \frac{1}{n} H(U) = 0.$$

□

Corollary. 2.10. Let (M, m, T) be a semi-independent hyper MV–algebra dynamical system. If for every $U \in P_T$, there exists $N > 0$ such that for every $n \geq N$, $\{U\} \overset{m}{\leq} U_n$, then $h(T) = 0$.

Proof. The proof is clear by using Theorem 2.9. □

Theorem. 2.11. Let (M, m, T) be a semi-independent hyper MV–algebra dynamical system, and $U <_G T$. Then $h(T) = h(T, U)$.

Proof. Let $V \in P_T$ be given. Since $U <_G T$, it follows that $V \ll_G U$. Considering the proof of Theorem 2.7, the obtained result is $h(T, V) \leq h(T, U)$. Thus, $\sup_{V\in P_T} h(T, V) \leq h(T, U)$. □

3. Conclusion

In this paper, the essential ergodic properties of semi-independent systems over hyper MV–algebras are discussed. Specifically, it is proved that the total entropy of a system that has a generator is calculated with respect to its generator. For the purpose of calculating the entropy, it is paramount to make use of sequences rather than tuples as partitions to define the idea of the entropy of a semi-independent hyper MV–algebra dynamical system for future research.

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