HOPF BIFURCATION CONTROL WITH PD CONTROLLER

M. DARVISHI* AND H.M. MOHAMMADINEJAD
DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICS AND STATISTICS, UNIVERSITY OF BIRJAND, BIRJAND, IRAN
E-MAILS: *M.DARVISHI@BIRJAND.AC.IR,
HMOHAMMADIN@BIRJAND.AC.IR

(Received: 11 January 2017, Accepted: 25 June 2017)

Abstract. In this paper, we investigate the problem of bifurcation control for a delayed logistic growth model. By choosing the timedelay as the bifurcation parameter, we present a Proportional - Derivative (PD) Controller to control Hopf bifurcation. We show that the onset of Hopf bifurcation can be delayed or advanced via a PD Controller by setting proper controlling parameter. Under consideration model as operator Equation, apply orthogonal decomposition, compute the center manifold and normal form we determined the direction and stability of bifurcating periodic solutions. Therefore the Hopf bifurcation of the model became controllable to achieve desirable behaviour which are applicable in certain circumstances.

AMS Classification: 34H20, 34D23.
Keywords: Hopf bifurcation, Bifurcation control, PD Controller, Stability, Time-delay.

1. Introduction

The single-species logistic growth model governed by delay differential equations plays an important role in population dynamics and ecology that has been investigated in-depth involving the stability, persistent, oscillations and chaotic behaviour

* CORRESPONDING AUTHOR

of solutions[6]-[8]. Gopalsamy and weng[2] considered the following system:

$$\begin{cases} \dot{n}(t) = rn(t)[1 - \dfrac{n(t-\tau)}{k} - cv(t)] \\ \dot{v}(t) = -dv(t) + en(t) \end{cases} \tag{1}$$

where r, c, d, e, k $\in (0, +\infty)$ and $\tau \in [0, +\infty)$. the initial conditions for the system (1) take the $n(s) = \phi(s) \geq 0, \phi(0) > 0, \phi \in C([-\tau, 0], R_+)$, $v(0) = v_0$. The solutions of (1) are defined for all $t > 0$ and also satisfy $n(t) > 0, v(t) > 0$ for $t > 0$. And the system (1) has unique positive equilibrium $(n^*, v^*) = (\dfrac{dk}{d+kec}, \dfrac{ek}{d+kec})$. Then by the linear chain trick technique, system (1) can be transformed into the following equivalent system:

$$\begin{cases} \dot{x}(t) = -dx(t) + en^*y(t) \\ \dot{y}(t) = -crx(t) - \dfrac{rn^*}{k}y(t-\tau) - crx(t)y(t) - \dfrac{rn^*}{k}y(t-\tau)y(t) \end{cases} \tag{2}$$

The author obtained when the condition (H) $\dfrac{ec}{d} > \dfrac{1}{k}$ and $d > (1+\sqrt{2})r$ hold, the positive equilibrium (n^*, v^*) of (1) is linearly asymptotically stable irrespective of the size of the delay τ. Xie [3] interested in the effect of delay τ on dynamics of system (1) when the condition (H) is not satisfied. Taking the delay τ as a parameter, they showed that the stability and a Hopf bifurcation occurs when the delay τ passes through a critical value. We summarize these features of the solution via the existence and stability of a positive equilibrium in following:

If $\dfrac{ec}{d} < \dfrac{1}{k}$ then (n^*, v^*) is locally asymptotically stable for $0 \leq \tau < \tau_0$ and unstable for $\tau > \tau_0$ and system (1) undergoes Hopf bifurcation at (n^*, v^*) when $\tau = \tau_n$, n=0,1,2,...

2. Hopf Bifurcation in Controlled System

In this section, we focus on designing a controller to control the Hopf bifurcation in model based on the PD strategy[4], [5]. Apply the PD control to system (2), we get

(3)

$$\begin{cases} \dot{x}(t) = -dx(t) + en^*y(t) + k_p x(t) + k_d \dot{x}(t) \\ \dot{y}(t) = -crx(t) - \dfrac{rn^*}{k}y(t-\tau) - crx(t)y(t) - \dfrac{rn^*}{k}y(t-\tau)y(t) + k_p y(t) + k_d \dot{y}(t) \end{cases}$$

where $0 < k_p \leq 1$ and $0 < k_d \leq 1$. whose characteristic linear equation(3) is

(4)

$$\lambda^2 - (a_1 + b_2 + b_3 e^{-\lambda\tau})\lambda + a_1 b_3 e^{-\lambda\tau} + a_1 b_2 - a_2 b_1 = 0$$

That $a_1 = \dfrac{k_p - d}{1 - k_d}, a_2 = \dfrac{en^*}{1 - k_d}, b_1 = \dfrac{-cr}{1 - k_d}, b_2 = \dfrac{k_p}{1 - k_d}$ and $b_3 = \dfrac{-rn^*}{k(1 - k_d)}$. If $\tau > 0$, we assume $\lambda = i\omega$ is a purely imaginary root of (4), then we can obtained

$$\begin{cases} -\omega^2 - b_3\omega sin\omega\tau + a_1 b_3 cos\omega\tau + a_1 b_2 - a_2 b_1 + \\ i[-(a_1 + b_2)\omega - b_3\omega cos\omega\tau - a_1 b_3 sin\omega\tau] = 0 \end{cases} \tag{5}$$

Separating the real and imaginary parts of (5)

$$\begin{cases} \omega^2 + a_2 b_1 - a_1 b_2 = -b_3\omega sin\omega\tau + a_1 b_3 cos\omega\tau \\ -(a_1 + b_2)\omega = b_3\omega cos\omega\tau + a_1 b_3 sin\omega\tau \end{cases} \tag{6}$$

since $sin^2\omega\tau + cos^2\omega\tau = 1$ therefore

$$\omega^4 + [2a_2 b_1 + {a_1}^2 + {b_2}^2 - {b_3}^2]\omega^2 + (a_2 b_1 - a_1 b_2)^2 - (a_1 b_3)^2 = 0 \tag{7}$$

So if the condition ${k_p}^2 - (rn^* + d)k_p + drn^* + en^*cr > 0$ (M) holds, then the equation (7) has a solution $\omega_0 > 0$, since equation of the (6)

(8)

$$\tau_n = \frac{1}{\omega_0} arcsin[\frac{\omega_0 k(1 - k_d)({\omega_0}^2(1 - k_d)^2 - en^*cr + (k_p - d)^2)}{rn^*(1 - k_d)^2{\omega_0}^2 + (k_p - d)^2 rn^*}] + \frac{2n\pi}{\omega_0}, n = 0, 1, 2, ...$$

proposition: In the equation of (4), If the condition (M) holds, then $Re(\dfrac{d\lambda}{d\tau})|_{\lambda = i\omega_0} \neq 0$.

Proof. We compute $\dfrac{d\lambda}{d\tau}$ from equation (4), $\dfrac{d\lambda}{d\tau} = \dfrac{a_1 b_3\lambda - b_3\lambda^2}{2\lambda e^{\lambda\tau} + b_3\tau\lambda - a_1 e^{\lambda\tau} - b_2 e^{\lambda\tau} - b_3 - a_1 b_3\tau}$. The real part of $(\dfrac{d\lambda}{d\tau})|_{\lambda = i\omega_0}$ obtained from

(9)

$$A = \frac{[-2{\omega_0}^3 b_3 - a_1 b_3\omega_0(a_1 + b_2)]sin\omega_0\tau_n + [-b_3{\omega_0}^2(a_1 + b_2) + 2{\omega_0}^2 a_1 b_3]cos\omega_0\tau_n - {b_3}^2{\omega_0}^2}{\Lambda}$$

where

$\Lambda = [-2\omega_0 sin\omega_0\tau_n - (a_1+b_2)cos\omega_0\tau_n - b_3 - a_1b_3\tau_n]^2$
$+ [2\omega_0 cos\omega_0\tau_n + b_3\omega_0\tau_n - (a_1+b_2)sin\omega_0\tau_n]^2$ So $A \neq 0$ completing the proof. □

Consequently, the equilibrium of (3) will occur Hopf bifurcation when $\tau = \tau_n$.

3. Direction and Stability of Hopf Bifurcation Period Solution

In Section 2, we obtained conditions for the Hopf bifurcation to occur when $\tau = \tau_n$ under the condition (M). In the section we study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions, using techniques from normal form and center manifold theory for delay differential equations[1]. Let $x(t) := x(\tau t), y(t) := y(\tau t)$. Then system (3) can be written as

(10)

$$\begin{cases} \dot{x}(t) = \dfrac{\tau}{1-k_d}[(k_p-d)x(t) + en^* y(t)] \\ \dot{y}(t) = \dfrac{\tau}{1-k_d}[-crx(t) + k_p y(t) - \dfrac{rn^*}{k}y(t-1) - crx(t)y(t) - \dfrac{rn^*}{k}y(t-1)y(t)] \end{cases}$$

Let $\mu = \tau - \tau_n$, functional differential equation (10) in $C = C([-1,0], R^2)$ is

$$\acute{x}(t) = L_\mu(x_t) + f(\mu, x_t) \tag{11}$$

where $x(t) = (x(t), y(t))^T \in R^2$ and $L_\mu : C \longrightarrow R^2$, $f : R \times C \longrightarrow R^2$are given, respectively:

$$\begin{cases} L_\mu(\phi) = \dfrac{(\tau_n+\mu)}{1-k_d}\begin{bmatrix} (k_p-d) & en^* \\ -cr & k_p \end{bmatrix}\begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} \\ +\dfrac{(\tau_n+\mu)}{1-k_d}\begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{rn^*}{k} \end{bmatrix}\begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \end{bmatrix} \end{cases} \tag{12}$$

$$f(\mu,\phi) = \frac{(\tau_n+\mu)}{1-k_d}\begin{bmatrix} 0 \\ Q \end{bmatrix}, Q = -cr\phi_1(0)\phi_2(0) - \frac{rn^*}{k}\phi_2(-1)\phi_2(0) \tag{13}$$

By the Riesz representation theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1,0]$ such that $L_\mu(\theta) = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \phi \in C$. In fact, we can choose :

$\eta(\theta,\mu) = \frac{(\tau_n+\mu)}{1-k_d}\begin{bmatrix}(k_p-d) & en^* \\ -cr & k_p\end{bmatrix}\delta(\theta) - \frac{(\tau_n+\mu)}{1-k_d}\begin{bmatrix}0 & 0 \\ 0 & -\frac{rn^*}{k}\end{bmatrix}\delta(\theta+1)$. That $\delta = \begin{cases}0 & if\theta \neq 0 \\ 1 & if\theta = 0\end{cases}$. For $\phi \in C([-1,0],R^2)$, define:

$A(\mu)\phi = \begin{cases}\frac{d\phi(\theta)}{d\theta} & if\theta \in [-1,0) \\ \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) & if\theta = 0\end{cases}$, $R(\mu)\phi = \begin{cases}0 & if\theta \in [-1,0) \\ f(\mu,\phi) & if\theta = 0\end{cases}$. Then system (11) is equivalent to

$$\acute{x_t} = A(\mu)x_t + R(\mu)x_t \tag{14}$$

where $x_t(\theta) = x(t+\theta)$, $\theta \in [-1,0]$. For $\Gamma \in C([0,1],(R^2)^*)$, define $A^*(\mu)\Gamma = \begin{cases}-\frac{d\Gamma(\theta)}{d\theta} & if\theta \in (0,1] \\ \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) & if\theta = 0\end{cases}$ and a bilinear inner product $<\Gamma,\phi>= \bar{\Gamma}^T(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\Gamma}^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi$ Where $\eta(\theta) = \eta(\theta,0)$. As $<\Gamma, A(0)\phi> = <A^*\Gamma,\phi>$, obviously $A(0)$ and $A^*(0)$ are adjoint operators and $\pm i\omega_0$ are eigenvalues of $A(0)$ and $A^*(0)$. We first need to compute the eigenvector of $A(0)$ and $A^*(0)$ corresponding to $i\omega_0$ and $-i\omega_0$ respectively. Suppose that $q(\theta) = (1,q_1)^T e^{i\omega_0\theta}$ is the eigenvector of $A(0)$ and $q^*(s) = D(1,{q^*}_1)e^{-i\omega_0 s}$ is the eigenvector of $A^*(0)$ corresponding to $i\omega_0$, $-i\omega_0$ respectively. In order to assure $<q^*,q>=1$, we need to determine the value of the D. From inner product we can obtain $D = \frac{1}{1+\bar{q}_1^* q_1 - \frac{rn^*\tau_n e^{i\omega_0}}{k(1-k_d)}\bar{q}_1^* q_1}$.

To compute the coordinates describing center manifold C_0 at $\mu = 0$. Define $z(t) = <q^*, x_t>, W(t,\theta) = x_t - 2Rez(t)q(\theta)$. On the center manifold C_0 we have $W(t,\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + ...$, For the solution x_t of (14) since $\mu = 0$, we have $\acute{z}(t) = i\omega_0 z + \bar{q}^*(0)f(0,z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2}...$, to calculate the coefficients, we have $g_{20} = \frac{2\tau_n\bar{D}{q^*}_1(0)}{1-k_d}[-crq_1 - \frac{rn^*}{k}{q_1}^2 e^{-i\omega_0}]$, $g_{11} = \frac{\tau_n\bar{D}{q^*}_1(0)}{1-k_d}[-cr(\bar{q_1}+q_1) - \frac{rn^*}{k}(q_1\bar{q_1}e^{i\omega_0} + q_1\bar{q_1}e^{-i\omega_0})]$, $g_{02} = \frac{2\tau_n\bar{D}{q^*}_1(0)}{1-k_d}[-cr\bar{q_1} - \frac{rn^*}{k}\bar{q_1}^2 e^{i\omega_0}]$ and $g_{21} = \frac{\tau_n\bar{D}{q^*}_1(0)}{1-k_d}[-cr({W_{20}}^{(1)}(0)q_1 + 2{W_{11}}^{(1)}(0)q_1 + 2{W_{11}}^{(2)}(0) + {W_{20}}^{(2)}(0)) - \frac{rn^*}{k}({W_{20}}^{(2)}(0)\bar{q_1}e^{i\omega_0} + 2{W_{11}}^{(2)}(0)q_1 e^{-i\omega_0} + 2{W_{11}}^{(2)}(-1)q_1 + {W_{20}}^{(2)}(-1)\bar{q_1})]$

that $W_{11} = ({W_{11}}^{(1)}, W_{11}^{(2)})$, $W_{20} = ({W_{20}}^{(1)}, W_{20}^{(2)})$. Let $\Delta = \frac{i}{2\omega_0}\{g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\} + \frac{g_{21}}{2}$. We define $\begin{cases} \mu_2 = -\dfrac{Re\{\Delta\}}{Re\dot{\lambda}(\tau_n)} \\ T_2 = -\dfrac{Im\{\Delta\} + \mu_2 Im\dot{\lambda}(\tau_n)}{\omega_0} \\ \beta_2 = 2Re\{\Delta\} \end{cases}$. According to the case described above, We can summarize the results in the following theorem:

theorem **:** For the controlled system (10), the Hopf bifurcation determined by the parameters μ_2, T_2 and β_2, the conclusions are summarized:(I) Parameter μ_2 determines the direction of the Hopf bifurcation. if $\mu_2 > 0$, the Hopf bifurcation is supercritical, the bifurcating periodic solutions exist for $\tau > \tau_n$, if $\mu_2 < 0$ the Hopf bifurcation is subcritical, the bifurcating periodic solutions exist for $\tau < \tau_n$. (II)Parameter β_2 determines the stability of the bifurcating periodic solutions. If $\beta_2 < 0$, the bifurcating periodic solutions is stable, if $\beta_2 > 0$, the bifurcating periodic solutions is unstable. (III) Parameter T_2 determines the period of the bifurcating periodic solution. If $T_2 > 0$, the period increases, If $T_2 < 0$, the period decreases.

In the (2) let $r = 1, k = 1, c = \frac{1}{2}, d = e = 2$. we have:

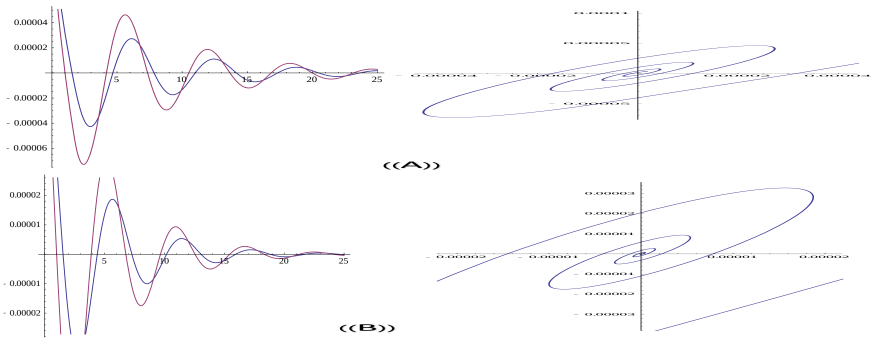

FIGURE 1. The numerical solution of uncontrol modle with corresponding to ((A)): $\tau = 1.74281$ and ((B)): $\tau = 1.5$.

CONCLUSION

In this paper, the problem of Hopf bifurcation control for an logistic growth model with time delay was studied. In order to control the Hopf bifurcation, a

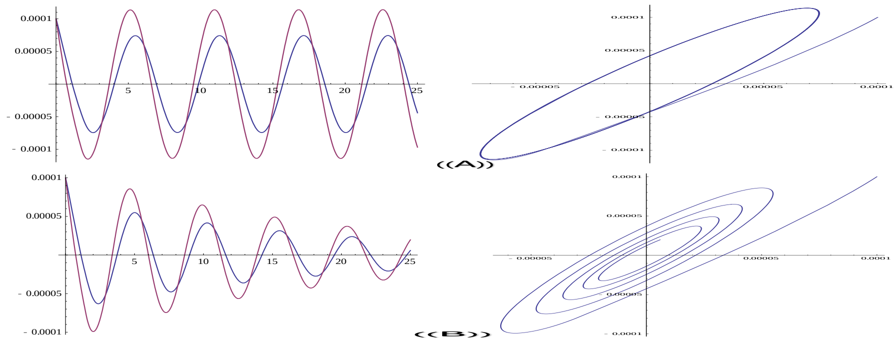

FIGURE 2. The numerical solution of control modle with corresponding to $k_d = 0.3, k_p = 0.1$ ((A)): $\tau = 1.74281$ and ((B)): $\tau = 1.5$.

PD controller is applied to the model. This PD controller can successfully delay or advance the onset of an inherent bifurcation. The end theorem helped to improve model.

REFERENCES

[1] Balakumar Balachandran, Tamas Kalmar Nagy, David E. Gilsinn, *Delay Differential Equations*, Recent Advances and New Directions (Springer, New York), (2009).

[2] K. Gopalsamy, P. X. Weng, Feedback regulation of logistic growth, *Internat. J. Math.* **16**, (1993), 177-192.

[3] Xiaojie Gong, Xianddong Xie, Rongyu Han, Liya Yang, Hopf bifurcation in a delayed logistic growth with feedback control, *Commun. Math. Biol. Neurosci*, (2015), 2052-2541.

[4] Dawei Ding, Xiaoyun Zhang, Jinde Cao, Nian Wang, Dong Liang, Bifurcation control of complex networks model via PD controller, *Neurocomputing* (2015).

[5] Mohammad Darvishi, Haji Mohammad Mohammadinejad, Control of Bifurcation in Internet Model with Time-Delay, *Applied and Computational Mathematics,* **5**,(2016), 186-192.

[6] K. Gopalsamy, *Stability and oscillations in delay differential equations of population dynamics*, London:Kluwer Academic Press, (1992).

[7] M. H. Jiang, Y. Shen, J. G. Jian, X. X. Liao, Stability, bifurcation and a new chaos in the logistic differential equation with delay, *Phys. Lett. A*, 350 (2006), 221-227.

[8] Y. Kuang, *Delay differential equations with applications in population dynamics*, New York: Academic Press,(1993).