NEARLY SINGLE-SOLITON SOLUTION FOR A PERTURBED A HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION

S. ESKANDAR* AND S. M. HOSEINI
DEPARTMENT OF MATHEMATICS, VALI-E-ASR UNIVERSITY OF RAFSANJAN, IRAN
S_ ESKANDAR66@YAHOO.COM
HOSEINI@UOW.EDU.AU

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Abstract. In the present paper we develop the soliton perturbation theory to find nearly soliton solutions for a perturbed higher-order nonlinear Schrödinger (PHNLS) equation. An integral expression for the first-order correction to the wave is found and to avoid the secular terms, the dynamical systems for the soliton parameters are found.

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1. Introduction

Among the Partial Differential Equations (PDEs) soliton equations have vast applications in applied and pure mathematics. A soliton appears as a localized intensity on a stable traveling zero or non-zero wave background. These equations

*CORRESPONDING AUTHOR
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are recognized due to be as the compatibility condition of a so-called Lax pair and are solvable by Inverse Scattering Transformation (IST).

One of the most important model equation in nonlinear sciences especiallay in soliton theory is the nonlinear Schrödinger (NLS) equation,

$$iu_t + u_{xx} + 2|u|^2 u = 0. \tag{1}$$

Physically, the NLS equation (1) describes the modulation of weakly-nonlinear wavetrains in deep water. [2] showed that an uniform wavetrain is unstable to long-wave perturbations. [10] and [19] present a historical overview of fluid mechanics applications of the NLS equation (1) and its physical origins. In the optical context, the NLS equation (1) was derived by [5]. It also describes the evolution of the slowly varying envelope of an optical pulse. Derived asymptotically from Maxwells equations, it assumes slow variation in the carrier frequency and the Kerr dependence (where the nonlinear refractive index $n = n_0 + n_1|u|^2$). The NLS equation (1) is central to understanding soliton propagation in optical fibres, which is of critical importance to the field of fibre-based telecommunications, see [11]. The NLS equation (1) has also key applications in financial, for details *i.e.*, see [12]-[13].

The NLS equation (1) is the first member of an infinity member family of integrable equations via IST, so-called AKNS hierarchy. This family can be easily constructed by a recursive procedure which lets all familiar soliton equations are found as the members, see for more details §2. These members include Korteweg-de Vries (KdV), NLS, modified KdV equations, Hirota equation and Sine-Gordon equations which have several applications in physics and telecommunications. Recently, the general and novel nonlocal AKNS hierarchies were also found [14]-[15]. The hierarchies are shown to possess a Lax pair and infinite number of conservation laws and to be PT-symmetric.

A powerful analytical solution technique is direct soliton perturbation theory. This requires that the complete set of eigenfunctions for the linearized problem, related to the nonlinear wave equation, be determined. [16] constructed this set for a large class of integrable nonlinear wave equations such as the KdV, NLS and modified KdV equations. The same procedure can be exploited to find the eigenstates of the adjoint linearization operator. He found that the eigenfunctions for these hierarchies are the squared Jost solutions. [3] developed direct soliton perturbation theory for

the derivative NLS and the modified NLS equations. Using the similarity between the KdV and derivative NLS hierarchies they showed that the eigenfunctions for the linearized derivative NLS equation (1) are the derivatives of the squared Jost solutions. Suppressing the secular terms, they found the slow evolution of soliton parameters and the perturbation-induced radiation.
We recently investigated the weak interaction for a higher-order nonlinear Schrödinger (HNLS) equation

$$iu_t = u_{xxxx} + 4|u_x|^2u + 8|u|^2u_{xx} + 6u^*u_x^2 + 2u^2u_{xx}^* + 6|u|^4u, \tag{2}$$

by applying the soliton perturbation theory. It is noted that (2) is the next even member of the NLS integrable hierarchy. The HNLS equation (2) is also a member of the integrable three-parameter fifth-order NLS (FONLS) family. Yang *et.al* [18] investigated Rogue waves, rational solitons, and modulational instability in FONLS context. Here in this paper, we develop the soliton perturbation theory for a perturbed higher-order NLS (PHNLS) equation,

$$iu_t - u_{xxxx} - 4|u_x|^2u - 8|u|^2u_{xx} - 6u^*u_x^2 - 2u^2u_{xx}^* - 6|u|^4u = \varepsilon F(u), \tag{3}$$

where ε is a small parameter in magnitude and F is a polynomial of u, u^* and their derivatives. It is also noted that the soliton perturbation theory has a central application in phenomena which are formulated by a perturbed soliton equations not an exact soliton equation. The main rigorously part of the theory is to establish the closure relation for continuous and discrete eigenfunctions of the linearized operator corresponding to (2). We shall show that the closure relations for HNLS equation (2) and NLS equation (1) are exactly the same and therefore it is found that the complete set of the eigenfunctions for linearized operator obtained from HNLS equation (2) is the same of that related to NLS equation (1).
This paper contains four sections. In section §2, Lax pair of HNLS equation (2) is introduced and the scattering matrix is obtained. Analytical property of scattering data are also determined. In §3 the closure relation between eigenfunctions is proved. And finally in §4 we consider our HNLS equation (2) under a small perturbation and use direct soliton perturbation theory to find its solution.

2. Lax pair and scattering matrix

It is well known that to establish a general soliton solution of a soliton equation via IST, a pair of ordinary differential equations (ODEs) named Lax pair is needed where their compatibility condition will be the equation. The Lax pair corresponding to the soliton equation can be readily constructed by AKNS procedure, *i.e.*, see [1].

We outline the Lax pair for HNLS equation (2) as

$$Y_x = MY, \quad Y_t = NY, \tag{4}$$

where

$$M = -i\zeta\Lambda + Q, \quad N = -8i\zeta^4\Lambda + 8\zeta^3 Q + 4i\zeta^2 V + R,$$

and $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}$, $V = \begin{pmatrix} |u|^2 & u_x \\ u_x^* & -|u|^2 \end{pmatrix}$,

$$R = \begin{pmatrix} |u_x|^2 - uu_{xx}^* - u_{xx}u^* - 3|u|^2 & -(u_{xxx} + 6|u|^2 u_x) \\ -(u_{xxx}^* + 6|u|^2 u_x^*) & -(|u_x|^2 - uu_{xx}^* - u_{xx}u^* - 3|u|^2) \end{pmatrix},$$

and ζ is scattering parameter, then the compatibility condition for (4), *i.e.*, $Y_{xt} = Y_{tx}$ gives HNLS equation (2).

To study and investigate the relations between the linearized operator's eigenfunctions we need to borrow some facts and relations of Jost solutions of the Lax pair (4) which are essential in IST from [17]. Therefore up to end of this section we will repeatedly apply these relations. The soliton solution u has the spacial boundary condition $|u| \longrightarrow 0$ as $|x| \longrightarrow \infty$. We shall use this property to find the fundamental solution of (4). Therefore, it is clear that $Y \propto \exp\{-i\zeta\Lambda x - 8i\zeta^4\Lambda t\}$ as $x \longrightarrow \pm\infty$. It is convenient to change (4) to

$$J_x = -i\zeta[\Lambda, J] + QJ, \tag{5}$$

$$J_t = -8i\zeta^4[\Lambda, J] + (8\zeta^3 Q + 4i\zeta V + R)J, \tag{6}$$

by

$$Y = Je^{-i\zeta\Lambda x - 8i\zeta^4\Lambda t}, \tag{7}$$

where the Jost solution J is (x,t)-independent at infinity.
As the main step in IST we mainly focus on the first equation of (4) called Zakharov-Shabat (Z-S) system and time evolution of the solitons shall be done by applying the second ODE in (4) when they explicitely determined. HNLS equation (2) shares its Z-S system with NLS and hence the spatial evolution using IST procedure for equations will be analogous. We consider two Jost solutions $J_{\pm}(x,\zeta)$ of the scattering problem (5), with the following asymptotic

$$J_{\pm}(x,\zeta) \longrightarrow I, \qquad x \longrightarrow \pm\infty, \tag{8}$$

where I is the 2×2 unit matrix. Note that we temporally forget "t" from the notations. Abel's identity shows that

$$\det J_{\pm}(x,\zeta) = 1, \tag{9}$$

for all (x,ζ). As $J_{\pm}E$ ($E = e^{-i\zeta\Lambda x}$) are both solutions of the (linear) Z-S system, they are linearly related as $J_{-}E = J_{+}ES$, where $S = S(\zeta)$ is called scattering matrix and the potential u can be retrieved from the elements of S. Consequently, the property (9) gives

$$\det S(\zeta) = 1, \qquad \zeta \in \mathbb{R}. \tag{10}$$

Up to now, the scattering parameter ζ was considered on real line. Extending the analyticity of Jost solutions $J_{\pm}(x,\zeta)$ and therefore $S(\zeta)$ to the ζ-complex half-planes provides the opportunity of applying Riemann-Hilbert Problem to construct the soliton solutions for HNLS equation (2) from the scattering data stored in S. For example, it can be shown that the first column of J_{-} and the second column of J_{+} can be analytically continued to the upper half plane $\zeta \in \mathbb{C}_{+}$, while the second column of J_{-} and the first column of J_{+} can be analytically continued to the lower half plane $\mathbb{C}_{-}$, for a detailed review of application of Volterra integral equations see [17]. For simplicity, we let $\Phi = J_{-}E$ and $\Psi = J_{+}E$ and express (Φ,Ψ) as a collection of columns and Φ^{-1},Ψ^{-1} as a collection of rows as

$$\begin{aligned} &\Phi = (\phi_1,\phi_2), \qquad \Psi = (\psi_1,\psi_2), \\ &\Phi^{-1} = \begin{pmatrix} \widehat{\phi_1} \\ \widehat{\phi_2} \end{pmatrix}, \qquad \Psi^{-1} = \begin{pmatrix} \widehat{\psi_1} \\ \widehat{\psi_2} \end{pmatrix}. \end{aligned} \tag{11}$$

Finally, collecting the functions with same analyticity areas yields that the Jost solutions

$$P^{+} = (\phi_1, \psi_2)e^{i\zeta\Lambda x} = J_{-}H_1 + J_{+}H_2, \tag{12}$$

where

$$H_1 \equiv diag(1,0), \quad H_2 \equiv diag(0,1), \tag{13}$$

are analytic in $\zeta \in \mathbb{C}_+$, and in a similar consideration the Jost solutions

$$P^{-} = e^{-i\zeta\Lambda x}\begin{pmatrix} \widehat{\phi_1} \\ \widehat{\psi_2} \end{pmatrix} = H_1 J_{-}^{-1} + H_2 J_{+}^{-1} \tag{14}$$

are analytic in $\zeta \in \mathbb{C}_-$, with asymptotics

$$P^{\pm}(x,\zeta) \longrightarrow I, \qquad \zeta \in \mathbb{C}_{\pm} \longrightarrow \infty. \tag{15}$$

Now, the definition of sacttering matrix S

$$S = \Psi^{-1}\Phi = \begin{pmatrix} \widehat{\psi_1} \\ \widehat{\psi_2} \end{pmatrix}(\phi_1, \phi_2), \quad S^{-1} = \Phi^{-1}\Psi = \begin{pmatrix} \widehat{\phi_1} \\ \widehat{\phi_2} \end{pmatrix}(\psi_1, \psi_2), \tag{16}$$

determines the analyticity of the diagonal elements of S. For example, s_{11} and $\widehat{s}_{22}$ are analytical in upper half plane and s_{22} and $\widehat{s}_{11}$ are analytical in lower half plane, where s_{ij} and $\widehat{s}_{ij}$ are elements of S and S^{-1}, respectively.

3. Closure of Zakharov-Shabat Eigenstates

As mentioned earlier, the Z-S relations for NLS equation (1) and HNLS equation (2) are the same, hence the closure of eigenfunctions for the linear operator under investigation related to HNLS equation (2) can be constructed analogous to that related to NLS. There are several alternative methods to do so, but the procedure applied here is based on contour integrations. For continuity the discussion we adopted the notations from [17].

Define

$$\begin{aligned} R^{+}(x,y,\zeta) &= \chi^{+}(x,\zeta)diag[\theta(y-x), -\theta(x-y)](\chi^{+})^{-1}(y,\zeta), \\ R^{-}(x,y,\zeta) &= \chi^{-}(x,\zeta)diag[\theta(x-y), -\theta(y-x)](\chi^{-})^{-1}(y,\zeta), \end{aligned} \tag{17}$$

where $\chi^+ = (\phi_1, \psi_2), \chi^- = (\psi_1, \phi_2)$, (see the first relation in (11)) and $\theta(x)$ is the standard step function

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

According to definitions, these functions $R^{\pm}$ are meromorphic in the upper and lower halves of the ζ plane, respectively. It is easy to check that

$$\begin{aligned} \chi^+ &= \Phi(H_1 + S^{-1}H_2), \\ \chi^- &= \Phi(S^{-1}H_1 + H_2). \end{aligned} \tag{18}$$

which yield

$$\det \chi^+ = s_{11} = \hat{s}_{22}, \qquad \det \chi^- = s_{22} = \hat{s}_{11}.$$

Because of the nature of the step functions and also the asymptotics (8), $R^{\pm}$ are satisfied

$$\begin{aligned} R^+(x, y, \zeta) &\longrightarrow diag[\theta(y-x)e^{i\zeta(y-x)}, -\theta(x-y)e^{i\zeta(x-y)}], \\ R^-(x, y, \zeta) &\longrightarrow diag[\theta(x-y)e^{i\zeta(y-x)}, -\theta(y-x)e^{i\zeta(x-y)}], \end{aligned} \tag{19}$$

as $\zeta \longrightarrow \infty$ in the respective half plane of analyticity. These relations show that $R^{\pm}$ are both bounded. Now

$$\begin{aligned} \int_{\gamma_+} R^+(x, y, \zeta) d\zeta &= \int_{\gamma_+} diag[\theta(y-x)e^{i\zeta(y-x)}, -\theta(x-y)e^{i\zeta(x-y)}] d\zeta \\ &= \int_{-\infty}^{\infty} diag[\theta(y-x)e^{i\xi(y-x)}, -\theta(x-y)e^{i\xi(x-y)}] d\xi, \end{aligned} \tag{20}$$

$$\begin{aligned} \int_{\gamma_-} R^-(x, y, \zeta) d\zeta &= \int_{\gamma_+} diag[\theta(x-y)e^{i\zeta(y-x)}, -\theta(y-x)e^{i\zeta(x-y)}] d\zeta \\ &= \int_{-\infty}^{\infty} diag[\theta(x-y)e^{i\xi(y-x)}, -\theta(y-x)e^{i\xi(x-y)}] d\xi, \end{aligned} \tag{21}$$

where γ^+ (γ^-) is a contour starts from $\zeta = -\infty + i0^+$ ($\zeta = -\infty + i0^-$), passes over (under) all zeros of $\hat{s}_{22}(\zeta)$ ($s_{22}(\zeta)$) in $\mathbb{C}_+$ ($\mathbb{C}_-$) and ends at $\zeta = \infty + i0^+$ ($\zeta = \infty + i0^-$). Some calculations show that

$$\int_{\gamma_+} R^+(x, y, \zeta) d\zeta + \int_{\gamma_-} R^-(x, y, \zeta) d\zeta = 2\pi\delta(x-y)\Lambda. \tag{22}$$

Now the residue theorem yields

$$\int_{\gamma_+} R^+(x,y,\zeta)d\zeta + \int_{\gamma_-} R^-(x,y,\zeta)d\zeta = \int_{-\infty}^{\infty} R^+(x,y,\xi) + R^-(x,y,\xi)d\xi$$
$$+2\pi i \sum_j \{Res[R^-(x,y,\zeta),\bar{\zeta}_j] - Res[R^+(x,y,\zeta),\zeta_j]\}, \tag{23}$$

where ζ_j and $\bar{\zeta}_j$ are zeros of $\hat{s_{22}}(\zeta)$ and $s_{22}(\zeta)$, respectively, for left hand side of (22).

As $\chi^{\pm}(x,\xi)$ are fundamental matrices of Z-S system on the real axis, then $\chi^+(x,\xi)(\chi^-)^{-1}(x,\xi)$ is x-independent. therefore

$$\int_{-\infty}^{\infty} [R^+(x,y,\xi) + R^-(x,y,\xi)]d\xi = \int_{-\infty}^{\infty} \left\{\chi^+(x,\xi)diag(1,0)(\chi^+)^{-1}(y,\xi)\right.$$
$$\left. -\chi^-(x,\xi)diag(0,1)(\chi^-)^{-1}(y,\xi)\right\} d\xi. \tag{24}$$

The residue terms in (23) are now simplified to

$$Res[R^+(x,y,\zeta),\zeta_j] = Res[\chi^+(x,\zeta)diag(1,0)(\chi^+)^{-1}(y,\zeta),\zeta_j],$$
$$Res[R^-(x,y,\zeta),\bar{\zeta}_j] = -Res[\chi^-(x,\zeta)diag(0,1)(\chi^-)^{-1}(y,\zeta),\bar{\zeta}_j], \tag{25}$$

where the step functions have disappeared.

To end of this section we replace $\chi^{\pm}$ with columns of Jost solutions (Φ,Ψ) and $(\chi^{\pm})^{-1}$ by rows of adjoint Jost solutions (Φ^{-1},Ψ^{-1}) to get the closure relation

$$\delta(x-y)\Lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{1}{s_{11}}(\xi)\phi_1(x,\xi)\widehat{\psi_1}(y,\xi) - \frac{1}{s_{22}}(\xi)\phi_2(x,\xi)\widehat{\psi_2}(y,\xi)\right] d\xi$$
$$-i\sum_j \left[\frac{1}{s_{11}'(\zeta_j)}\phi_1(x,\zeta_j)\widehat{\psi_1}(y,\zeta_j) - \frac{1}{s_{22}'(\bar{\zeta}_j)}\phi_2(x,\bar{\zeta}_j)\widehat{\psi_2}(y,\bar{\zeta}_j)\right]. \tag{26}$$

The closure relation (26) states that

$$\{\phi_1(x,\xi),\phi_2(x,\xi),\ \xi\in\mathbb{R};\quad \phi_1(x,\zeta_j),\phi_2(x,\bar{\zeta}_j),\ 1\le j\le N\} \tag{27}$$

form a complete set as well as

$$\{\widehat{\psi_1}(x,\xi),\widehat{\psi_2}(x,\xi),\ \xi\in\mathbb{R};\quad \widehat{\psi_1}(x,\zeta_j),\widehat{\psi_2}(x,\bar{\zeta}_j),\ 1\le j\le N\}. \tag{28}$$

These results have critical role in our analysis, as any function can be extended as a combination of eigenstates of the Z-S system. The closure relation (26) has a direct implementation in the soliton perturbation theory wich shall be discussed in next section.

4. Soliton Perturbation

It has been shown that the soliton (direct) perturbation theory is a powerful tool to find nearly soliton solutions for a (small) correction to an integrable equation. For a survey on the application of the theory refer to [7] and the references therein. In this section we review in details the soliton perturbation theory for PHNLS equation (3). Using the closure relation established in previous section, the complete set of eigenfunctions for linearized operators is found.
When $\varepsilon = 0$, (2) has a soliton solution as

$$u(x,t) = rsech(r(x - vt - x_0))e^{i(ax+bt+\sigma_0)} \tag{29}$$

where

$$v = 4a(a^2 - r^2), \qquad b = -a^4 + 6a^2r^2 - r^4. \tag{30}$$

There are four free parameters in this solution; r and a are soliton's amplitude and phase parameter and x_0 and σ_0 are also position and phase of the soliton, respectively. This solution can be rewritten as

$$u(x,t) = \Upsilon(\theta)e^{i\phi}, \tag{31}$$

where

$$\Upsilon(\theta) = rsechr\theta, \tag{32}$$

$$\theta = x - \nu, \qquad \phi = a\theta + \sigma,$$

$$\nu = 4a(a^2 - r^2)t + x_0, \qquad \sigma = (3a^4 - r^4 + 2a^2r^2)t + \sigma_0.$$

In the presence of perturbation, the four free parameters of the soliton will slowly change with time, *i.e.*,

$$r = r(T), \qquad a = a(T), \qquad x_0 = x_0(T), \qquad \sigma_0 = \sigma_0(T), \qquad T = \varepsilon t. \tag{33}$$

According to standard multi scale perturbation analysis, we expand the solution $u(x,t)$ into the following perturbation series

$$u(x,t) = e^{i\phi}\left(\Upsilon(\theta) + \varepsilon\Upsilon_1(\theta,t,T) + O(\varepsilon^2)\right) \tag{34}$$

Here Υ, ϕ and θ are given in (32), and

$$\nu = \int_0^t 4a(a^2 - r^2)d\tau + x_0, \tag{35}$$

$$\sigma = \int_0^t (4a^4 - (a^2 - r^2)^2)d\tau + \sigma_0. \tag{36}$$

Substituting (34) in PHNLS equation (3) and collecting $O(\varepsilon)$ terms determines

$$(i\partial_t + L)\, A_1 = W \tag{37}$$

where $A_1 = (\Upsilon, \Upsilon^*), W = (w, -w^*)^T$ and

$$\begin{aligned} w &= F_0 - i\Upsilon_r r_{_T} + ix_{0_T}\Upsilon_\theta - (ax_{0_T} - a_{_T}\theta - \sigma_{0_T})\Upsilon, \\ F_0 &= e^{-i\phi}F(\Upsilon e^{i\phi}). \end{aligned} \tag{38}$$

The operator L is

$$L = \begin{pmatrix} -\partial_{\theta\theta\theta\theta} - 4ia\partial_{\theta\theta\theta} + L_2\partial_{\theta\theta} + L_1\partial_\theta + L_0 & -2\Upsilon^2\partial_{\theta\theta} - 4\Upsilon\Upsilon_\theta\partial_\theta + F_0 \\ 2\Upsilon^2\partial_{\theta\theta} + 4\Upsilon\Upsilon_\theta\partial_\theta - F_0^* & \partial_{\theta\theta\theta\theta} - 4ia\partial_{\theta\theta\theta} - L_2\partial_{\theta\theta} - L_1^*\partial_\theta - L_0^* \end{pmatrix} \tag{39}$$

where

$$\begin{aligned} L_0 &= -4\Upsilon_\theta - 18\Upsilon^4 + r^4 - 24ia\Upsilon\Upsilon_\theta - 6a^2r^2 + 24a^2\Upsilon^2 - 12\Upsilon\Upsilon_{\theta\theta}, \\ L_1 &= i(4ar^2 - 24a\Upsilon^2) - 16\Upsilon\Upsilon_\theta, \quad L_2 = 6a^2 - 8\Upsilon^2, \\ F_0 &= -8\Upsilon\Upsilon_{\theta\theta} - 12\Upsilon^4 - 6\Upsilon_\theta^2 - 24ia\Upsilon\Upsilon_\theta + 12a^2\Upsilon^2. \end{aligned}$$

By straightforward differentiations the adjoin of L, L^A is found as

$$L^A = \sigma_3 L \sigma_3, \qquad \sigma_3 = diag\{1, -1\}, \tag{40}$$

in the sense

$$\langle lV, W\rangle = \langle V, l^A W\rangle, \tag{41}$$

for V and W vanishing at infinity, and where the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f^T(x)g(x)dx \tag{42}$$

is applied through the paper. It is worthy to note that the explicit expressions of L and L^A are not vital for our analysis, however the explicit forms of their eigenfunctions are needed.

4.1. **Eigenfunctions of L and its adjoint.** In a similar manner related to NLS the eigenfunctions of L (39) are explicitly determined as

$$Z_1(\theta,k) = \begin{pmatrix} -sech^2 r\theta \\ (ik - tanhr\theta)^2 \end{pmatrix} e^{irk\theta}, \tag{43}$$

$$Z_2(\theta,k) = \begin{pmatrix} (ik + tanhr\theta)^2 \\ -sech^2 r\theta \end{pmatrix} e^{-irk\theta}, \tag{44}$$

which are satisfied in

$$LZ_1(\theta,k) = \lambda(k) Z_1(\theta,k), \tag{45}$$

$$LZ_2(\theta,k) = -\lambda(k) Z_2(\theta,k), \tag{46}$$

where

$$\lambda(k) = r^4(k^4 - 1). \tag{47}$$

Calculations also show that

$$Z_{D,1} = \Upsilon_\theta \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad Z_{D,2} = \Upsilon \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{48}$$

$$Z_{G,1} = \frac{1}{2}\theta\Upsilon \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad Z_{G,2} = \Upsilon_r \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{49}$$

are discrete and generalized discrete eigenfunctions of L with eigenrelations

$$LZ_{D,1} = LZ_{D,2} = 0 \tag{50}$$

$$LZ_{G,1} = -2r^2 Z_{D,1}, \qquad LZ_{G,2} = -4r^3 Z_{D,2}. \tag{51}$$

Similarly we obtain continuous and discrete eigenfunctions of L^A

$$\gamma_1(\theta,k) = \begin{pmatrix} sech^2 r\theta \\ (ik + tanhr\theta)^2 \end{pmatrix} e^{-irk\theta}, \tag{52}$$

$$\gamma_2(\theta,k) = \begin{pmatrix} (ik - tanhr\theta)^2 \\ sech^2 r\theta \end{pmatrix} e^{irk\theta}, \tag{53}$$

$$\gamma_{D,1} = \sigma_3 Z_{D,1},\ \gamma_{D,2} = \sigma_3 Z_{D,2},\ \gamma_{G,1} = \sigma_3 Z_{G,1},\ \gamma_{G,2} = \sigma_3 Z_{G,2}, \tag{54}$$

where

$$L^A\gamma_1(\theta,k)=\lambda(k)\gamma_1(\theta,k),\quad L^A\gamma_2(\theta,k)=-\lambda(k)\gamma_2(\theta,k), \tag{55}$$

$$L^A\gamma_{D,1}=L^A\gamma_{D,2}=0,\quad L^A\gamma_{G,1}=-2r^2\gamma_{D,1},\quad L^A\gamma_{G,2}=-4r^3\gamma_{D,2}. \tag{56}$$

The nonzero inner products between the eigenfunctions of L and L^A are

$$<Z_1(\theta,k),\gamma_1(\theta,k^{'})>=<Z_2(\theta,k),\gamma_2(\theta,k^{'})>=\frac{2\pi}{r}(k^2+1)^2\delta(k-k^{'}), \tag{57}$$

$$<Z_{D,1},\gamma_{G,1}>=<Z_{G,1},\gamma_{D,1}>=-r \tag{58}$$

$$<Z_{D,2},\gamma_{G,2}>=<Z_{G,2},\gamma_{D,2}>=2. \tag{59}$$

4.2. **Solution for the Perturbed Soliton.** After the eigenfunctions and adjoint eigenfunctions of the linearization operator L have been obtained, we can now solve the first-order equation (37) and derive the solution for the soliton under perturbations. To solve (37), we first expand the forcing term W into the complete set of L's eigenfunctions,

$$\begin{aligned} W&=c_1Z_{D,1}(\theta)+c_2Z_{D,2}(\theta)+c_3Z_{G,1}(\theta)+c_4Z_{G,2}(\theta)\\ &+\int_{-\infty}^{\infty}[\alpha_1(k)Z_1(\theta,k)+\alpha_2(k)Z_2(\theta,k)]dk. \end{aligned} \tag{60}$$

Utilizing the orthogonality relations (58)-(59), we find that:

$$c_1=-\frac{1}{r}<W,\gamma_{G,1}>,\qquad c_2=-\frac{1}{2}<W,\gamma_{G,2}>, \tag{61}$$

$$c_1=-\frac{1}{r}<W,\gamma_{D,1}>,\qquad c_2=-\frac{1}{2}<W,\gamma_{D,2}>, \tag{62}$$

$$\alpha_j(k)=\frac{r}{2\pi(k^2+1)^2}<\mathcal{F},\gamma_j>,\qquad j=1,2, \tag{63}$$

where

$$\mathcal{F}=(F_0,-F_0^*)^T \tag{64}$$

Next, we expand the solution A_1 into the complete set of L's eigenfunctions as well,

$$\begin{aligned} A_1&=h_1(t)Z_{D,1}(\theta)+h_2(t)Z_{D,2}(\theta)+h_3(t)Z_{G,1}(\theta)+h_4(t)Z_{G,2}(\theta)\\ &+\int_{-\infty}^{\infty}[g_1(t,k)Z_1(\theta,k)+g_2(t,k)Z_2(\theta,k)]dk. \end{aligned} \tag{65}$$

Finally If we follow the same way for NLS (1), try to (34) be valid lead us to the following dynamical equations for the soliton parameters:

$$\frac{dr}{dT} = \int_{-\infty}^{\infty} Im(F_0) r sech(r\theta) d\theta, \tag{66}$$

$$\frac{da}{dT} = -\int_{-\infty}^{\infty} Re(F_0) r sech(r\theta) tanh(r\theta) d\theta, \tag{67}$$

$$\frac{dx_0}{dT} = \int_{-\infty}^{\infty} Im(F_0) \theta sech(r\theta) d\theta, \tag{68}$$

$$\frac{d\sigma_0}{dT} = a x_{0_T} - \int_{-\infty}^{\infty} Re(F_0) r sech(r\theta)(1 - r\theta tanh(r\theta)) d\theta. \tag{69}$$

Evolution equations for the position ν and phase σ then can be found from these equations and (35) and (36) as:

$$\frac{d\nu}{dt} = 4a(a^2 - r^2) + \varepsilon \int_{-\infty}^{\infty} Im(F_0) \theta sech(r\theta) d\theta, \tag{70}$$

$$\begin{aligned} \frac{d\sigma}{dt} &= -a^4 + 6a^2r^2 - r^4 + a \int_{-\infty}^{\infty} Im(F_0) \theta sech(r\theta) d\theta \\ &- \int_{-\infty}^{\infty} Re(F_0) r sech(r\theta)(1 - r\theta tanh(r\theta)) d\theta. \end{aligned} \tag{71}$$

With straightforward calculations we finally obtain the perturbed soliton solution (up to $O(\varepsilon)$) as:

$$\begin{aligned} u(x,t) = e^{i\phi}\Big\{ & r sech(r\theta) - \varepsilon sech^2(r\theta) \int_{-\infty}^{\infty} \frac{1 - e^{ir^4(k^4+1)t}}{2\pi r(k^2+1)^3(k^2-1)} e^{irk\theta} < \mathcal{F}, \gamma_1 > dk \\ & -\varepsilon \int_{-\infty}^{\infty} \frac{1 - e^{-ir^4(k^4+1)t}}{2\pi r(k^2+1)^3(k^2-1)} (ik + tanh(r\theta))^2 e^{-irk\theta} < \mathcal{F}, \gamma_2 > dk \Big\}. \end{aligned} \tag{72}$$

5. Conclusion

Using relationship between scattering data and the potential, the squared eigenfunctions for the Z-S system corresponding to the higher-order nonlinear Schrödinger equation are constructed. These eigenfunctions have direct implementation in soliton perturbation theory where the perturbed solutions for the nearly integrable equation is explored. Applying the explicit forms of eigenfunctions and avoiding secular terms, the dynamical systems for the soliton's parameters are found.

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