NONLINEAR OSCILLATION OF CERTAIN THIRD-ORDER NEUTRAL DIFFERENTIAL EQUATION WITH DISTRIBUTED DELAY

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Abstract. The authors obtain necessary and sufficient conditions for the existence of oscillatory solutions with a specified asymptotic behavior of solutions to a nonlinear neutral differential equation with distributed delay of third order. We give new theorems which ensure that every solution to be either oscillatory or converges to zero asymptotically. Examples dwelling upon the importance of applicability of these results.

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1. Introduction

In the present paper, we concerned with the nonlinear oscillation of certain neutral differential equation with continuously distributed delay of third order

$$\left[r_1(t)\left[(r_2(t)y'(t))'\right]^{\gamma}\right]' + \int_c^d q(t,\mu)x^{\gamma}(\sigma(t,\mu))\,d\mu = 0, \tag{E}$$

where $y(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu$ and $a < b$, $c < d$. Throughout this paper we following hypotheses are tacitly supposed to hold:

($\mathbf{H_1}$) $\gamma \geq 1$ is a ratio of two odd positive integers, $r_1(t), r_2(t) \in C^1([t_0, +\infty))$, $r_1(t), r_2(t) > 0$, $r_1'(t) \geq 0$ and

$$\int_{t_0}^{\infty} \frac{1}{r_1^{1/\gamma}(t)}dt = \infty, \quad \int_{t_0}^{\infty} \frac{1}{r_2(t)}dt = \infty. \tag{1.1}$$

($\mathbf{H_2}$) $p(t,\mu) \in C\big([t_0,+\infty) \times [a,b], R^+\big)$,$0 \leq \int_a^b p(t,\mu)d\mu \leq P < 1$, $q(t,\mu) \in C\big([t_0,+\infty)\times[c,d], R^+\big)$ and $q(t,\mu)$ is not identically zero for $[t_*,+\infty)\times[c,d]$, $t_* \geq t$.

($\mathbf{H_3}$) $\tau(t,\mu) \in C\big([t_0,+\infty) \times [a,b], R^+\big)$, $\tau(t,\mu) \leq t$, $\tau(t,\mu)$ is nondecreasing in μ, $\tau(t,\mu) \to \infty$ as $t \to +\infty$ for $\mu \in [a,b]$ and $\sigma(t,\mu) \in C\big([t_0,+\infty)\times[c,d], R^+\big)$, $\sigma(t,\mu) \leq t$, $\sigma(t,\mu)$ is nondecreasing in μ, $\sigma(t,\mu) \to \infty$ as $t \to +\infty$ for $\mu \in [c,d]$.

By a solution of equation (E) we mean a function $x(t) \in C([T_x,\infty))$, $T_x \geq t_0$, which has the property $y'(t)$, $r_2(t)y'(t)$ and $r_1(t)[(r_2(t)y'(t))']^{\gamma}$ are continuous differentiable and satisfies (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We make the standing hypothesis that equation (E) do possess proper solution. A proper solution of (E) called oscillatory if it has a sequence of large zeros lending to ∞; otherwise we call non-oscillatory.

Of late, much attention is being paid in the research activities related to oscillation and asymptotic behavior of various types of differential equations. As a result innumerable research papers [1, 3, 5, 7, 11, 12, 13, 14, 15, 19, 21, 22, 23] as well as several monographs [4, 16] have published and the references quoted therein. The applications of neutral differential equations are manifold. For example the equations are used for the study of distributed networks, automatic control, technology and natural , physical sciences, for instance, Driver [18] and Hale [10].

Very recently, Candan [20], Zhang et al. [24], Bartuek et al. [17], Tian et al. [25], Elabbasy et al. [6], Fu et al. [26], Jiang et al. [2], and Wang et al. [9] are investigated oscillation and asymptotic behavior of solutions of neutral differential equations with distributed delay of third order.

Till necessarily very few result has been initiated with regard to oscillation and asymptotic behavior of equation (E) with distributed delay. By using generalized Riccati transformation and integral averaging technique, this paper presents some sufficient conditions which guarantees that every solution of (E) oscillates or converges to zero.

In the sequel, all inequalities are supposed to hold eventually i.e., for all sufficiently large t.

2. Main Results

In this section , we present our main result in this paper. For convenience, we use the notations

$$q_*(t) = (1-P)^\gamma \int_c^d q(t,\mu)\, d\mu, \quad \phi'_+(t) = \max\{0, \phi'(t)\},$$

$$\sigma_1(t) = \sigma(t,c), \quad \Phi(t) = \left(\frac{\int_{t_2}^{\sigma_1(t)} \left(\int_{t_1}^{s} r_1^{-1/\gamma}(u)du / r_2(s) \right) ds}{\int_{t_1}^{\sigma_1(t)} r_1^{-1/\gamma}(u)du} \right)^\gamma$$

$$\psi(t) = \phi(s) q_*(s) \Phi(s) - \frac{1}{(1+\gamma)^{1+\gamma}} \frac{r_1(s)(\phi'_+(s))^{1+\gamma}}{\phi^\gamma(s)} \tag{2.1}$$

Theorem 2.1. *Assume* $(H_1)-(H_3)$ *and* (1.1) *holds. If there exists a positive function* $\phi \in C^1([t_0,\infty),\mathbb{R})$*, such that for all sufficiently large* $t_i > t_1 \ge t_0$ $(i = 2,3,4)$*, we have*

$$\limsup_{t\to\infty} \int_{t_3}^{t} \psi(s)ds = \infty, \tag{2.2}$$

and

$$\int_{t_4}^{\infty} \frac{1}{r_2(v)} \int_v^\infty \left(\frac{1}{r_1(u)} \int_u^\infty \int_c^d q(s,\mu)d\mu\, ds \right)^{1/\gamma} du\, dv = \infty, \tag{2.3}$$

where $\psi(t)$ *is defined in* (2.1)*, then every solution* $x(t)$ *of* (E) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume, for sake of contradiction, that equation (E) has an eventually positive solution $x(t)$. That is $x(t) > 0$, $x(\tau(t,\mu)) > 0$ and $x(\sigma(t,\mu)) > 0$ for $t \geq t_1$ some $t_1 \geq t_0$, by definition of $y(t)$. By condition (1.1), there exist two possible cases:

(i) $y(t) > 0$, $y'(t) > 0$, $(r_2(t)y'(t))' > 0$, $\Big(r_1(t)[(r_2(t)y'(t))']^\gamma\Big)' < 0$,

(ii) $y(t) > 0$, $y'(t) < 0$, $(r_2(t)y'(t))' > 0$, $\Big(r_1(t)[(r_2(t)y'(t))']^\gamma\Big)' < 0$, for $t \geq t_1$, t_1 is large enough.

Suppose, Case (i) holds for $t \geq t_2$. From the definition of $y(t)$, $y(t) \geq x(t)$ for $t \geq t_2$ and

$$\begin{aligned} x(t) &= y(t) - \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu \\ &\geq y(t) - \int_a^b p(t,\mu)y(\tau(t,\mu))d\mu \\ &\geq y(t) - y(\tau(t,b))\int_a^b p(t,\mu)d\mu \\ &\geq y(t)\left(1 - \int_a^b p(t,\mu)d\mu\right) = y(t)(1-P). \end{aligned} \tag{2.4}$$

Setting (2.4) into (E), we get

$$\begin{aligned} \Big(r_1(t)\Big[\big(r_2(t)y'(t)\big)'\Big]^\gamma\Big)' &= -\int_c^d q(t,\mu)x^\gamma(\sigma(t,\mu))\,d\mu \\ &\leq -(1-p_0)^\gamma\int_c^d q(t,\mu)y^\gamma(\sigma(t,\mu))\,d\mu \\ &\leq -(1-p_0)^\gamma y^\gamma(\sigma(t,c))\int_c^d q(t,\mu)\,d\mu \\ &= -q_*(t)y^\gamma(\sigma_1(t)). \end{aligned} \tag{2.5}$$

Using the fact that $y'(t) > 0$, we have

$$r_2(t)y'(t) \geq \int_{t_1}^t \frac{r_1^{1/\gamma}(s)(r_2(s)y'(s))'}{r_1^{1/\gamma}(s)}ds \geq r_1^{1/\gamma}(t)(r_2(t)y'(t))'\int_{t_1}^t \frac{1}{r_1^{1/\gamma}(s)}ds.$$

Thus

$$\begin{aligned} y(t) &= y(t_2) + \int_{t_2}^{t} \frac{r_2(s)y'(s)}{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du} \frac{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du}{r_2(s)} ds \\ &\geq \frac{r_2(t)y'(t)}{\int_{t_1}^{t} r_1^{-1/\gamma}(u)du} \int_{t_2}^{t} \frac{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du}{r_2(s)} ds. \end{aligned}$$

Then, we get

$$\frac{y(\sigma_1(t))}{r_2(\sigma_1(t))y'(\sigma_1(t))} \geq \frac{\int_{t_2}^{\sigma_1(t)} \left(\frac{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du}{r_2(s)} \right) ds}{\int_{t_1}^{\sigma_1(t)} r_1^{-1/\gamma}(u)du}, \tag{2.6}$$

and

$$\frac{r_2(\sigma_1(t))y'(\sigma_1(t))}{r_2(t)y'(t)} \geq \frac{\int_{t_1}^{\sigma_1(t)} r_1^{-1/\gamma}(u)du}{\int_{t_1}^{t} r_1^{-1/\gamma}(u)du} \tag{2.7}$$

Define

$$W(t) := \phi(t)r_1(t) \left[\frac{(r_2(t)y'(t))'}{r_2(t)y'(t)} \right]^{\gamma}, \tag{2.8}$$

and $W(t) > 0$ for $t \geq t_1$. Differentiating (2.8), we obtain

$$\begin{aligned} W'(t) = \frac{\phi'(t)}{\phi(t)} W(t) + \phi(t) \frac{\left(r_1(t)[(r_2(t)y'(t))']^{\gamma} \right)'}{[r_2(t)y'(t)]^{\gamma}} \\ -\gamma \phi(t) r_1(t) \left[\frac{(r_2(t)y'(t))'}{r_2(t)y'(t)} \right]^{\gamma+1}. \end{aligned} \tag{2.9}$$

By (2.8), we get

$$\left[\frac{W(t)}{\phi(t)r_1(t)} \right]^{(\gamma+1)/\gamma} = \left[\frac{(r_2(t)y'(t))'}{r_2(t)y'(t)} \right]^{\gamma+1}, \tag{2.10}$$

By (2.5), (2.10), and (2.9) that

$$\begin{aligned} W'(t) &\leq \frac{\phi'(t)}{\phi(t)} W(t) - \phi(t) q_*(t) \left(\frac{y(\sigma(t))}{r_2(t)y'(t)} \right)^{\gamma} - \gamma \frac{W^{\frac{(\gamma+1)}{\gamma}}(t)}{[\phi(t)r_1(t)]^{1/\gamma}} \\ &= \frac{\phi'(t)}{\phi(t)} W(t) - \gamma \frac{W^{\frac{(\gamma+1)}{\gamma}}(t)}{[\phi(t)r_1(t)]^{1/\gamma}} \\ &\quad -\phi(t) q_*(t) \left(\frac{y(\sigma_1(t))}{r_2(\sigma_1(t))y'(\sigma_1(t))} \frac{r_2(\sigma_1(t))y'(\sigma_1(t))}{r_2(t)y'(t)} \right)^{\gamma} \end{aligned} \tag{2.11}$$

Taking (2.6), (2.7) and (2.11), into account

$$\begin{aligned} W'(t) &\le \frac{\phi'_+(t)}{\phi(t)}W(t) - \gamma\frac{W^{\frac{(\gamma+1)}{\gamma}}(t)}{[\phi(t)r_1(t)]^{1/\gamma}} \\ &\quad -\phi(t)q_*(t)\left(\frac{\int_{t_2}^{\sigma_1(t)}\left(\frac{\int_{t_1}^{s} r_1^{-1/\gamma}(u)du}{r_2(s)}\right)ds}{\int_{t_1}^{\sigma_1(t)} r_1^{-1/\gamma}(u)du}\right)^{\gamma} \\ &\le -\phi(t)q_*(t)\Phi(t) + \frac{\phi'_+(t)}{\phi(t)}W(t) - \gamma\frac{W^{\frac{\gamma+1}{\gamma}}(t)}{[\phi(t)r_1(t)]^{1/\gamma}}. \qquad (2.12) \end{aligned}$$

Then, using (2.12) and inequality

$$Bu - Au^{(\gamma+1)/\gamma} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^\gamma}, \qquad (2.13)$$

where $u = W(t)$, $A = \frac{\gamma}{[\phi(t)r_1(t)]^{1/\gamma}}$, $B = \frac{\phi'_+(t)}{\phi(t)}$. We find that

$$W'(t) \le -\phi(t)q_*(t)\Phi(t) + \frac{1}{(1+\gamma)^{1+\gamma}}\frac{r_1(t)(\phi'_+(t))^{1+\gamma}}{\phi^\gamma(t)}. \qquad (2.14)$$

Integrating (2.14) from t_3 ($> t_2$) to t gives

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left(\phi(s)q_*(s)\Phi(s) - \frac{1}{(1+\gamma)^{1+\gamma}}\frac{r_1(s)(\phi'_+(s))^{1+\gamma}}{\phi^\gamma(s)}\right)ds \le W(t_3), \qquad (2.15)$$

which contradicts (2.2).

Suppose Case (ii) holds. Since $y(t) > 0$ and $y'(t) < 0$, we have $y(t) \to L \ge 0$. If $L > 0$, then for $\epsilon = \frac{L(1-P)}{2P} > 0$, there exists $t_4 \ge t_1$ such that $L < y(t) < L + \epsilon$ for $t \ge t_4$. Then for $t \ge t_4$, we have

$$\begin{aligned} x(t) &= y(t) - \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu > L - \int_a^b p(t,\mu)y(\tau(t,\mu))d\mu \\ &\ge L - Py(\tau(t,a)) \ge L - P(L+\epsilon) > ky(t) \qquad (2.16) \end{aligned}$$

where $k = \frac{L-P(L+\epsilon)}{L+\epsilon}$ and $\sigma_0(t) = \sigma(t,d)$. Using (2.16), we obtained from (E) , we have

$$\Big(r_1(t)\big(r_2(t)y'(t)\big)'\Big)' = -k^\gamma\int_c^d q(t,\mu)y(\sigma(t,\mu))\,d\mu = -q^*(t)y(\sigma_0(t)),$$

where $q^*(t) = k^\gamma\int_c^d q(t,\mu)d\mu$. Integrating from $t(\ge t_4)$ to ∞ and $r_1(t)\big(r_2(t)y'(t)\big)' \ge 0$ is decreasing, we get

$$\big(r_2(t)y'(t)\big)' \ge \left(\frac{1}{r_1(t)}\int_t^\infty q^*(s)y^\gamma(\sigma_0(s))ds\right)^{1/\gamma}.$$

Using $y(\sigma_0(t)) \geq L$, we obtain

$$\big(r_2(t)y'(t)\big)' \geq L\left(\frac{1}{r_1(t)}\int_t^\infty q^*(s)ds\right)^{1/\gamma}.$$

Again integrating

$$r_2(t)y'(t) \geq -L\int_v^\infty \left(\frac{1}{r_1(u)}\int_u^\infty q^*(s)\,ds\right)^{1/\gamma} du,$$

and finally integration from t_4 to ∞, we get

$$y(t_4) \geq L\int_{t_4}^\infty \frac{1}{r_2(v)}\int_v^\infty \left(\frac{1}{r_1(u)}\int_u^\infty q^*(s)ds\right)^{1/\gamma} du\,dv.$$

This contradicts to (2.3) and hence $L = 0$. □

Remark 2.2. Suitable choice of $\phi(t)$, by Theorem 2.1 gives a various asymptotic criteria for (E).

Corollary 2.3. *If* $(H_1)-(H_3)$ *and* (2.3) *holds. If* $\phi = 1$, *such that for all sufficiently large* $t_3 > t_1 \geq t_0$, *we have*

$$\limsup_{t\to\infty}\int_{t_3}^t q_*(s)\Phi(s)ds = \infty, \tag{2.17}$$

then all solution $x(t)$ *of* (E) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Next, examine the oscillation results of solutions of (E) by Philos-type.

Let $\mathbb{S}_0 = \{(t,s) : a \leq s < t < +\infty\}$, $\mathbb{S} = \{(t,s) : a \leq s \leq t < +\infty\}$ the continuous function $E(t,s)$, $E : \mathbb{S} \to \mathbb{R}$ belongs to the class function $\Re$

(i) $E(t,t) = 0$ for $t \geq t_0$ and $E(t,s) > 0$ for $(t,s) \in \mathbb{S}_0$,

(ii) $\frac{\partial E(t,s)}{\partial s} \leq 0$, $(t,s) \in \mathbb{S}_0$ and some locally integrable function e such that

$$\frac{\partial E(t,s)}{\partial s} + \frac{\phi'(t)}{\phi(t)}E(t,s) = -e(t,s)(E(t,s))^{\frac{\gamma}{1+\gamma}} \quad \text{for all } (t,s) \in \mathbb{S}_0.$$

Theorem 2.4. *Assume that* (2.3) *holds. If there exists a positive function* $E \in \Re$ *and* $\phi \in C^1([t_0,\infty),\mathbb{R})$, *such that for all sufficiently large* $t_5 > t_1 \geq t_0$, *we have*

$$\limsup_{t\to\infty}\frac{1}{E(t,t_5)}\int_{t_5}^t\left[E(t,s)\phi(s)q_*(s)\Phi(s) - \frac{r_1(s)\phi(s)|e(t,s)|^{\gamma+1}}{(\gamma+1)^{\gamma+1}E^\gamma(t,s)}\right]ds = \infty, \tag{2.18}$$

then all solution $x(t)$ *of* (E) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a positive solution of (E). Then by the proof of Theorem 2.1, we have Cases (i) and (ii). Let Case (i) hold; $W(t)$ is defined as in (2.8). Then, we get

$$W'(t) \leq -\phi(t)q_*(t)\Phi(t) + \frac{\phi'(t)}{\phi(t)}W(t) - \gamma\frac{W^{\frac{\gamma+1}{\gamma}}(t)}{[\phi(t)r_1(t)]}. \tag{2.19}$$

Take $X(t) = \frac{\phi'(t)}{\phi(t)}$, $Y(t) = \gamma\left(\frac{1}{\phi(t)r_1(t)}\right)^{1/\gamma}$, we have

$$\phi(t)q_*(t)\Phi(t) \leq -W'(t) + X(t)W(t) - Y(t)W^{\frac{\gamma+1}{\gamma}}. \tag{2.20}$$

Multiplying (2.20) by $E(t,s)$ and integrating from t_5 to t, with $T \geq t_1$, we have

$$\begin{aligned}
&\int_{t_5}^{t} E(t,s)\phi(s)q_*(s)\Phi(s)ds \\
&\quad\leq \int_{t_5}^{t} E(t,s)\left[-W'(s) + X(s)w(s) - B(s)W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&\quad= E(t,t_5)W(t_5) + \int_{t_5}^{t}\left[\frac{\partial E(t,s)}{\partial s} + E(t,s)X(s)\right]W(s)ds \\
&\qquad - \int_{t_5}^{t}\left[E(t,s)Y(s)W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&\quad\leq E(t,t_5)W(t_5) + \int_{t_5}^{t}\left[|e(t,s)|W(s) - E(t,s)Y(s)W^{\frac{\gamma+1}{\gamma}}(s)\right]ds.
\end{aligned}$$

Then, using (2.12) and inequality, we obtain

$$\begin{aligned}
&\int_{t_5}^{t} E(t,s)\phi(s)q_*(s)\Phi(s)ds \leq \\
&\qquad E(t,t_5)W(t_5) + \int_{t_5}^{t}\frac{r_1(s)\phi(s)|e(t,s)|^{\gamma+1}}{(\gamma+1)^{\gamma+1}E^{\gamma}(t,s)}ds.
\end{aligned} \tag{2.21}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{E(t,t_5)}\int_{t_5}^{t}\Bigg[E(t,s)\phi(s)q_*(s)\Phi(s) \\
&\qquad -\frac{r_1(s)\phi(s)|e(t,s)|^{\gamma+1}}{(\gamma+1)^{\gamma+1}E^{\gamma}(t,s)}\Bigg]ds \leq W(t_5),
\end{aligned} \tag{2.22}$$

which contradicts (2.18).

Next, Assume that Case (ii) holds, we get $\lim_{t\to\infty} x(t) = 0$. □

Next, Based on Theorem 2.1, we present a Kamenev-type criterion for (E).

Theorem 2.5. *Assume that* (2.3) *holds. If there exists a positive function* $\phi \in C^1([t_0,\infty),\mathbb{R})$*, such that for sufficiently large* $t_6 > t_0$*, we have*

$$\limsup_{t\to\infty} \frac{1}{t^n}\int_{t_6}^{t}(t-s)^n\psi(s)ds = \infty, \tag{2.23}$$

where $\psi(t)$ *is defined in* (2.1)*, then all solution* $x(t)$ *of* (E) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume, for sake of contradiction, that equation (E) has an eventually positive solution $x(t)$. Then by the proof of Theorem 2.1, we have Cases (i) and (ii). Let Case (i) hold. By using the same arguments as in the proof of Theorem 2.1, we obtain (2.14), then

$$W'(t) \le -\phi(t)q_*(t)\Phi(t) + \frac{1}{(1+\gamma)^{1+\gamma}}\frac{r_1(t)(\phi'_+(t))^{1+\gamma}}{\phi^\gamma(t)}. \tag{2.24}$$

Multiplying by $(t-s)^n$ and integrating (2.24) from t_6 to t gives

$$\int_{t_6}^{t}(t-s)^n\,\psi(s)ds \;=\; -\int_{t_1}^{t}(t-s)^nW'(s)ds. \tag{2.25}$$

We get

$$\begin{aligned}\frac{1}{t^n}\int_{t_6}^{t}(t-s)^n\,\psi(s)ds &= -\frac{1}{t^n}\int_{t_6}^{t}(t-s)^nW'(s)ds\\ &\le -\frac{n}{t^n}\int_{t_6}^{t}(t-s)^{n-1}W(s)ds + \big(1-\frac{t_6}{t}\big)^nW(t_1)\\ &< \big(1-\frac{t_6}{t}\big)^nW(t_1) < \infty,\end{aligned} \tag{2.26}$$

which contradicts (2.24).
Next, Assume that Case (ii) holds, we get $\lim_{t\to\infty} x(t) = 0$. □

We give the following examples illustrate applications of theoretical results presented in this paper.

Example 2.6. For $t \ge 1$, consider the 3^{rd} order differential equation

$$\Big[t\Big[t^{-1}\Big[x(t) + \frac{1}{2}\int_{-1}^{0}x\big(\frac{t+\xi}{3}\big)d\xi\Big]''\Big]^3\Big]' + \int_0^1\frac{\xi}{t^3}x^3\big(\frac{t+\xi}{2}\big)d\xi = 0, \tag{2.27}$$

where $\gamma = 3$, $r_1(t) = t$, $r_2(t) = t^{-1}$, $\tau(t,\mu) = (t+\mu)/3$, $\sigma(t,\mu) = (t+\mu)/2$, $a = -1$, $b = 0$, $c = 0$, $d = 1$, then we obtain $\int_c^d q(t,\mu)\,d\mu = 1/2 = P$, $\sigma_1(t) = \sigma(t,c) = t/2$. Choose $\phi(t) = t$, easily verified that the conditions (2.2), (2.3) of Theorem 2.1 are satisfied. Since all solutions of (2.27) is oscillatory or $\lim_{t\to\infty} x(t) = 0$..

Example 2.7. Consider the 3^{rd} order differential equation

$$\Big[x(t) + \int_{1/2}^{1} \frac{1}{2} x\big(\frac{t+\xi}{3}\big)\Big]''' + \int_{1}^{2} \frac{\xi}{t^2} x\big(\frac{t+\xi}{2}\big) d\xi = 0, \quad t \geq 1, \tag{2.28}$$

where $\gamma = 1$, $r_1(t) = r_2(t) = 1$, $\tau(t, \mu) = (t + \mu)/3$, $\sigma(t, \mu) = (t + \mu)/2$, $a = 1/2$, $b = 1$, $c = 1$, $d = 2$, then we obtain $\int_c^d q(t, \mu)\, d\mu = 1/2 = P$, $\sigma_1(t) = \sigma(t, c) = (t + 1)/2$. Pick $\phi(t) = 1$, it is not difficult to verified that the conditions of Theorem 2.5. Since all solutions of (2.28) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.

3. Conclusions

In this paper, we using generalized Riccati transformation, Philos and Kamenev-type to established three new oscillation and asymptotic theorems for (E) in the case of (1.1) . Our result improves and complements results in the cited papers. This results easily extended to the corresponding dynamic equations on time scales. The details are left to the reader.

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