A TAXICAB VERSION OF A TRIANGLE' S APOLLONIUS CIRCLE

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Abstract. One of the most famous problems of classical geometry is the Apollonius' problem asks construction of a circle which is tangent to three given objects. These objects are usually taken as points, lines, and circles. This well known problem was posed by Apollonius of Perga (about 262 - 190 B.C.) who was a Greek mathematician known as the great geometer of ancient times after Euclid and Archimedes. The Apollonius' problem can be reduced specifically to the question "Is there the circle that touches all three excircles of given triangle and encompasses them? " when all three objects are circles. In literature, altough there are a lot of works on the solution of this question in the Euclidean plane, there is not the work on this question in different metric geometries. In this paper, we give that the conditions of existence of Apollonius taxicab circle for any triangle.

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Keywords: Taxicab distance, Distance Functions, Taxicab geometry, Apollonius circle.

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1. Introduction

Euclidean distance is the most common use of distance. In most cases when people said about distance, they will refer to Euclidean distance. The Euclidean distance between two points in either the plane or $3-$dimensional space is defined as the length of the segment between two points. Although it is the most popular distance function, it is not practical when we measure the distance which we actually move in the real world. We live on a spherical Earth rather than on a Euclidean $3-$space. We must think of the distance as though a car would drive in the urban geography where physical obstacles have to be avoided. So, one had to travel through horizontal and vertical streets to get from one location to another [8]. To compensate disadvantage of the Euclidean distance, the taxicab geometry was first introduced by K. Menger [15] and has developed by E. F. Krause [14] using the taxicab metric d_T of which paths composed of the line segments parallel to coordinate axes. Also, Z. Akca and R. Kaya [1] expand the taxicab distance in $\mathbb{R}^3$.

Later, researchers have wondered whether there are alternative distance functions of which paths are different from path of Euclidean metric. For example, G. Chen [3] developed Chinese checker distance d_{CC} in the $\mathbb{R}^2$ of which paths are similar to the movement made by Chinese checker . Afterwards, Ö. Gelişgen, R. Kaya and M. Özcan [9] defined Chinese checker distance in the $\mathbb{R}^3$.

Another example, S. Tian [19] gave a family of metrics, alpha distance d_α for $\alpha \in [0, \pi/4]$, which includes the taxicab and Chinese checker metrics as special cases. Then, Ö. Gelişgen and R. Kaya extended the d_α to three and n dimensional spaces, respectively [10], [11], [12]. Afterwards, H. B. Çolakoğlu [4] extended for $\alpha \in [0, \pi/2)$.

When we examine the common features of the metrics d_M, d_T, d_{CC} and d_α, we see that these metrics whose paths are parallel to at least one of the coordinate axes. Therefore, it is a logical question " Are there metric or metrics of which paths are not parallel to the coordinate axes. " H. B. Çolakoğlu and R. Kaya [5] give definition of the generalized $m-$distance function which includes the maximum, taxicab, Chinese checker and alpha metrics as follows.

For each real numbers a, b and m such that $a \geq b \geq 0 \neq a$, the distance function d_m between points $P = (x_p, y_p)$ and $Q = (x_q, y_q)$ is defined by

$$d_m\left(P, Q\right) = \left(a\Delta_{PQ} + b\delta_{PQ}\right) \diagup \sqrt{1 + m^2}$$

where $\Delta_{PQ} = \max\{|(x_p - x_q) + m(y_p - y_q)|, |m(x_p - x_q) - (y_p - y_q)|\}$ and $\delta_{PQ} = \min\{|(x_p - x_q) + m(y_p - y_q)|, |m(x_p - x_q) - (y_p - y_q)|\}$. It is the most important property of $m-$metric that its paths are not parallel to the coordinate axes in the real plane (see Figure 1).

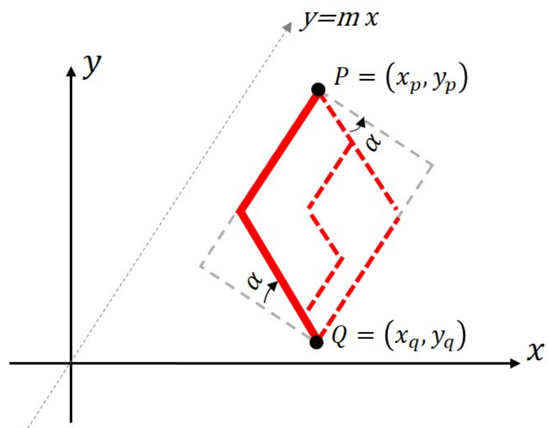

Figure 1. The paths of m-distance d_m

$$d_m = d_T \text{ for } a = b = 1 \text{ and } m = 0$$
$$d_T(P,Q) = |x_p - x_q| + |y_p - y_q|,$$

$$d_m = d_M \text{ for } a = 1,\ b = 0 \text{ and } m = 0$$
$$d_M(P,Q) = \max\{|x_p - x_q|, |y_p - y_q|\},$$

$$d_m = d_{CC} \text{ for } a = 1,\ b = \left(\sqrt{2} - 1\right) \text{ and } m = 0$$
$$d_{CC}(P,Q) = \max\{|x_p - x_q|, |(y_p - y_q)|\} + b\min\{|x_p - x_q|, |(y_p - y_q)|\},$$

$$d_m = d_\alpha \text{ for } a = 1,\ b = (\sec\alpha - \tan\alpha) \text{ and } m = 0 \quad (\alpha \in [0, \pi/2)),$$
$$d_\alpha(P,Q) = \max\{|x_p - x_q|, |(y_p - y_q)|\} + b\min\{|x_p - x_q|, |(y_p - y_q)|\}.$$

H. G. Park, K. R. Kim, I. S. Ko and B. H. Kim [17] define the polar taxicab distance

$$d_{PT}(P,Q) = \begin{cases} \min\{r_1, r_2\} \times |\theta_1 - \theta_2| + |r_1 - r_2| & , \quad 0 \leq |\theta_1 - \theta_2| \leq 2 \\ \\ r_1 + r_2 & , \quad 2 \leq |\theta_1 - \theta_2| \leq \pi \end{cases}$$

in the $\mathbb{R}^2$ of which paths composed of arc length in circle and line segments, where $P = (r_1, \theta_1)$ and $Q = (r_2, \theta_2)$ are points in a plane expressed by polar coordinates. The polar taxicab metric has very important applications in urban geography beacuse cities formed not only linear streets but also curvilinear streets. Also the distance function d_{PT} have applications in navigation systems [16]. Finally, considering distance of air travel or travel over water in terms of Euclidean distance, these travels are made through the interior of spherical Earth which is impossible. Using the idea given in [17], T. Ermiş and Ö. Gelişgen [8] define a new alternative metric on spherical surfaces due to disadvantage and disharmony of Euclidean distance on earth' s surface. This metric composed of arc length on sphere and length of line segments has been denoted d_{CL} . Also another alternative metric on sphere was defined by A. Bayar and R. Kaya [2].

As mentioned above, the distance geometries have different distance functions. So these geometries have different distance properties. It seems interesting to study the different metric analogues of the topics that include the concept of distance in the Euclidean geometry. In this paper, we study on a taxicab version of Apollonius' s circle.

2. APOLLONIUS CIRCLE IN TAXICAB GEOMETRY

Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions that is different from elliptic and hyperbolic geometry (and from the Minkowskian geometry of space-time). Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. Taxicab plane geometry is one of the geometries of this type. Namely, the taxicab plane $\mathbb{R}^2_T$ is almost the same as the Euclidean analytical plane $\mathbb{R}^2$. The points and the lines are the same, and the angles are measured in the same way. However, the distance function is different [13]. Also, the isometry group of taxicab plane is the semi direct product of $D(4)$ and $T(2)$ where $D(4)$ is the symmetry group of Euclidean square and $T(2)$ is the group of all translations in the plane [18]. So, in the rest of the article, the vertex C of given triangle $\overset{\triangle}{ABC}$ can be taken at origin since all translations of the analytical plane are isometries of the taxicab plane. Notice that this assumption about the position of the triangle does not loose the generality. Also, the other two vertices of triangle $\overset{\triangle}{ABC}$ will be labeled at counterclokwise direction in this paper.

Let l be a line with slope m in the taxicab plane. l is called a gradual line, a steep line, a separator if $|m| < 1$, $|m| > 1$, $|m| = 1$, respectively. In particularly, a gradual line is called horizontal if it is parallel to $x-$axis, and a steep line is called vertical if it is parallel to $y-$axis [6].

Let M be a point in the taxicab plane, and r be a positive real number. The set of points $\{X : d_T (M, X) = r\}$ is called taxicab circle, the point M is called center of the taxicab circle, and r is called the length of the radius or simply radius of the taxicab circle. Every taxicab circle in the taxicab plane is an Euclidean square having sides with slopes ± 1. Also, in Euclidean geometry, it is well-known that the number of intersection points of a circle and a line is 0, 1 or 2. In the taxicab plane, this number is 0, 1, 2 or ∞. So far, we have considered that a circle touches (is tangent to) each of the three sides of the triangle at only one point. In this section we will reconsider the concept of a circle tangent to a triangle by using following definition. A steep or a gradual line is tangent to a taxicab circle if the taxicab circle and the steep or gradual line have common only one point. But sides of a taxicab circle always lie on separator lines. If slope of the a side of a triangle is $+1$ or -1 then that side coincides with a side of the taxicab circle along a line segment. Additionally to like Euclidean cases, it has been stated in [7] that we consider the concept of tangent along a line segment if a line segment completely or partially lie on one side of the taxicab circle.

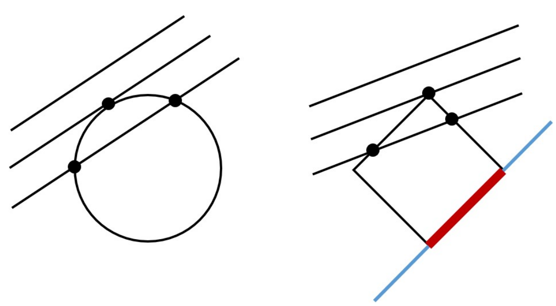

Figure 2. Tangential Cases in the Euclidean Circle and Taxicab Cirle

In plane, the triangles can be classified as seventeen groups according to slopes of the sides of triangles [7]:

- All sides of the triangle lie on gradual (steep) lines.

• Two sides of the triangle lie on gradual (steep) lines, the other side lies on a steep (gradual) line.

• Two sides of the triangle lie on separator lines, the other side lies on a gradual (steep) or a horizontal (vertical) line.

• A side of the triangle lies on a separator line, two sides of the triangle lie on gradual (steep) lines.

• A side of the triangle lies on a separator line, the other side lies on a gradual line and the third side lies on a steep line.

• A side of the triangle lies on a vertical line, the other side lies on a horizontal line and third side lies on a gradual (steep) line or separator line.

• A side of the triangle lies on a vertical (horizontal) line, two sides of the triangle lie on gradual (steep) lines.

• A side of the triangle lies on a vertical (horizontal) line, the other side lies on a gradual line and the third side lies on a steep line.

According to this classification, we give the cases that there is never Apollonius taxicab circle in Teorem 2.1. After that, the conditions for existence of Apollonius taxicab circle in Teorem 2.2 are given. Finally, it is shown the cases that there is always Apollonius taxicab circle in Theorem 2.3. Notice that, m_a, m_b and m_c denote slopes of sides BC, AC, AB of any triangle $\overset{\triangle}{ABC}$, respectively, in the rest of this paper.

Theorem 2.1. *Let $\overset{\triangle}{ABC}$ be any triangle in taxicab plane. Then,*

$$\textit{there is never Apollonius circle iff} \begin{cases} \bullet \quad \begin{array}{l} |m_a| \leq 1,\ |m_b| \leq 1,\ |m_c| < 1 \\ (\ \textit{as subcase}\ |m_a| < 1,\ |m_b| < 1,\ m_c = 0) \end{array} \\ \\ \bullet \quad \begin{array}{l} |m_a| \geq 1,\ |m_b| \geq 1,\ |m_c| > 1 \\ (\ \textit{as subcase}\ |m_a| > 1,\ |m_b| > 1,\ m_c \to \infty\ ) \end{array} \end{cases}$$

Proof. For $|m_a| < 1,\ |m_b| < 1,\ |m_c| < 1$, we know that the taxicab circle compose of the line segments which lie on lines with slopes ± 1. Now, consider the taxicab circle on the longest side the triangle $\overset{\triangle}{ABC}$. This taxicab circle is not excircle of triangle $\overset{\triangle}{ABC}$ because this circle touches at most two sides of the triangle $\overset{\triangle}{ABC}$. So, we can not construct Apollonius taxicab circle (see Figure 3). The other subcases can be similarly explained.

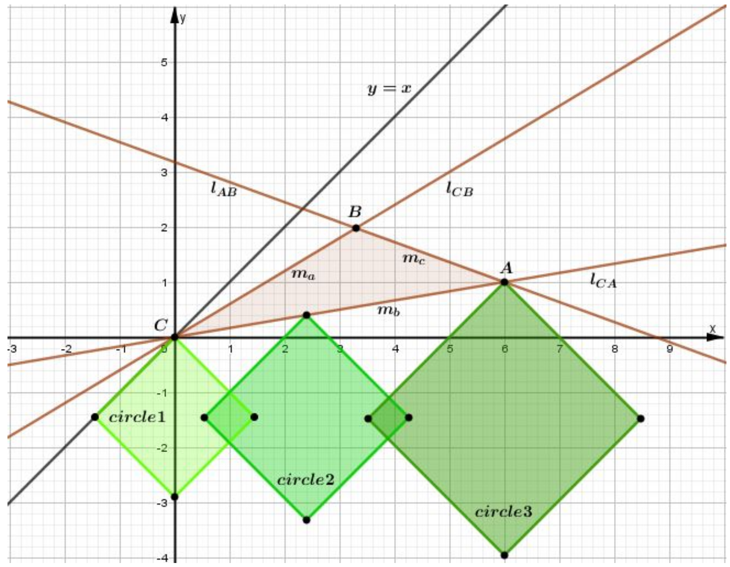

Figure 3.

□

Theorem 2.2. *Let $\overset{\triangle}{ABC}$ be any triangle in taxicab plane. Then, we define the functional relation $\rho(m_a, m_b, m_c) = m_a + m_b - m_c(2 - 3m_a + 3m_b)$;*

i) *For $|m_a| < 1,\ |m_b| < 1,\ |m_c| > 1,$*

$$\text{there is Apollonius taxicab circle} \Leftrightarrow \begin{cases} \rho(m_a, m_b, m_c) < 0 \quad \text{such that} \quad m_c < -1 \\ \\ \rho(m_a, m_b, m_c) > 0 \quad \text{such that} \quad m_c > 1 \end{cases}$$

Specially, there is Apollonius taxicab circle for $|m_a| < 1,\ |m_b| < 1,\ m_c \to \infty,$ iff $2 - 3m_a + 3m_b < 0$.

ii) *For $|m_a| > 1,\ |m_b| > 1,\ |m_c| < 1,$*

$$\text{there is Apollonius taxicab circle} \Leftrightarrow \begin{cases} \rho\left(-m_a^{-1}, -m_b^{-1}, m_c^{-1}\right) > 0,\ m_a m_b < 0 \\ \\ \rho\left(-m_a^{-1}, -m_b^{-1}, m_c^{-1}\right) < 0,\ m_a m_b > 0 \end{cases}$$

Specially, there is Apollonius taxicab circle for $|m_a| > 1,\ |m_b| > 1,\ m_c = 0$ iff $2m_a m_b - 3m_a + 3m_b < 0$.

Proof. **i)** For $|m_a| < 1,\ |m_b| < 1,\ m_c > 1$, let the vertex C of given triangle $\overset{\triangle}{ABC}$ be taken at origin and the other vertices be labeled at counterclokwise direction (see Figure 4).

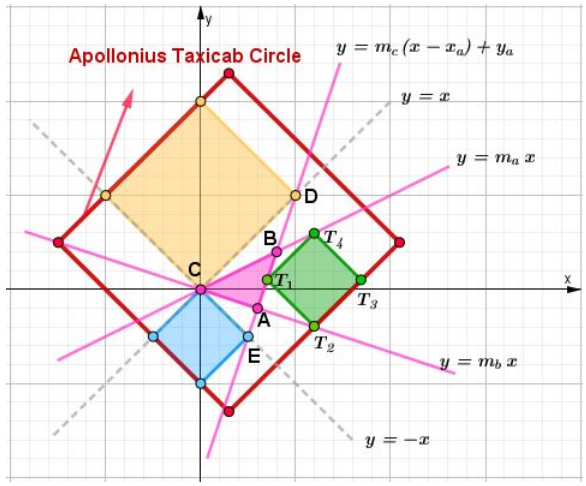

Figure 4.

Consider taxicab excircle on side BA of the triangle $\overset{\triangle}{ABC}$. Actually, there is always this taxicab excircle, because slopes of sides BC, AC and AB are gradual, gradual and steep, respectively. For $\lambda \in \mathbb{R}^+$, vertices T_2 and T_4 of taxicab excircle on side BA can be coordinated $(\lambda, \lambda m_b)$ and $(\lambda, \lambda m_a)$, respectively. If $T_2 = (\lambda, \lambda m_b)$ and $T_4 = (\lambda, \lambda m_a)$, then radius of the taxicab excircle r is obtained as $\dfrac{\lambda m_a - \lambda m_b}{2}$. So,

$$T_1 = \left(\lambda\left[\frac{2 - m_a + m_b}{2}\right],\ \lambda\left[\frac{m_a + m_b}{2}\right]\right),$$

$$T_3 = \left(\lambda\left[\frac{2 + m_a - m_b}{2}\right],\ \lambda\left[\frac{m_a + m_b}{2}\right]\right)$$

are calculated. Since the point T_1 is on the line $y = m_c(x - x_a) + y_a$, value of λ is founded as $\dfrac{2y_a - 2x_a m_c}{m_a + m_b + m_a m_c - m_b m_c - 2m_c}$. If the system of linear equations consisting of equations of lines $y = x$ and $y = m_c(x - x_a) + y_a$ is solved, then the solution is found as $D = \left(\dfrac{y_a - x_a m_c}{1 - m_c}, \dfrac{y_a - x_a m_c}{1 - m_c}\right)$. Similary, if the system of linear equations consisting of equations of lines $y = -x$ and $y = m_c(x - x_a) + y_a$ is solved, then the solution is found as $E = \left(\dfrac{x_a m_c - y_a}{1 + m_c}, -\dfrac{x_a m_c - y_a}{1 + m_c}\right)$. The line pass through the point D with slope -1 will be denoted by $l_{D_{-1}}$ such that equation of $l_{D_{-1}}$ is $y = -x + \dfrac{2(y_a - x_a m_c)}{1 - m_c}$. Similarly, equation of the line l_{E_1} pass through the point E with slope 1 is $y = x - \dfrac{2(x_a m_c - y_a)}{1 + m_c}$. To construct Apollonius taxicab circle of the triangle $\overset{\triangle}{ABC}$, it is clear that the point T_3 must not be in the region are bounded by lines $x = 0$, l_{E_1} and $l_{D_{-1}}$. To optimize the point T_3 outside this region, the point T_3 must be subject to constraint;

$$\begin{gathered}\frac{\lambda(2 + m_a - m_b)}{2} + \frac{2(y_a - x_a m_c)}{1 - m_c} < \lambda\left[\frac{m_a + m_b}{2}\right] \\ \lambda\left[\frac{m_a + m_b}{2}\right] < \frac{\lambda(2 + m_a - m_b)}{2} - \frac{2(x_a m_c - y_a)}{1 + m_c}.\end{gathered} \tag{2.1}$$

Consequently, the inequality $m_a + m_b - m_c(2 - 3m_a + 3m_b) > 0$ is obtained by inequalities (2.1). The other subcases can be similarly computed.

ii) If m_a, m_b, m_c replace with $-m_a^{-1}$, $-m_b^{-1}$, m_c^{-1} in the functional relation $\rho(m_a, m_b, m_c)$, then this functional relation can be shown as $\rho\left(-m_a^{-1}, -m_b^{-1}, m_c^{-1}\right)$. So, the proof of the case **ii** can be given as analogous to the proof of the previous case. $\square$

The following corollary is a special case of Theorem 2.2. The conditions for existence of Apollonius taxicab circle are given by a simpler relation.

Corollary 2.1. *Let $\overset{\triangle}{ABC}$ be any triangle such that a side of the triangle lies on a vertical line, the other side lies on a horizontal line and third side lies on a gradual or steep line. Also, slope of the side on gradual line is denoted by m_g, and slope of the side on steep line is denoted by m_s.*

$$\text{There is Apollonius taxicab circle iff } \begin{array}{c} -1 < m_g < -1/2 \ \text{ for } \ m_g < 0 \\ \text{or} \\ 1/2 < m_g < 1 \ \text{ for } \ m_g > 0. \end{array}$$

Similarly,

$$\text{There is Apollonius taxicab circle iff } \begin{array}{c} -2 < m_s < -1 \ \text{ for } \ m_s < 0 \\ \text{or} \\ 1 < m_s < 2 \ \text{ for } \ m_s > 0. \end{array}$$

Proof. Let $\overset{\triangle}{ABC}$ be any triangle such that a side of the triangle lies on a vertical line, the other side lies on a horizontal line and third side lies on a gradual line. Also, let the vertex C of given triangle $\overset{\triangle}{ABC}$ be taken at origin and the other vertices be labeled at counterclokwise direction (see Figure 5). For $m_g = m_a > 0$,

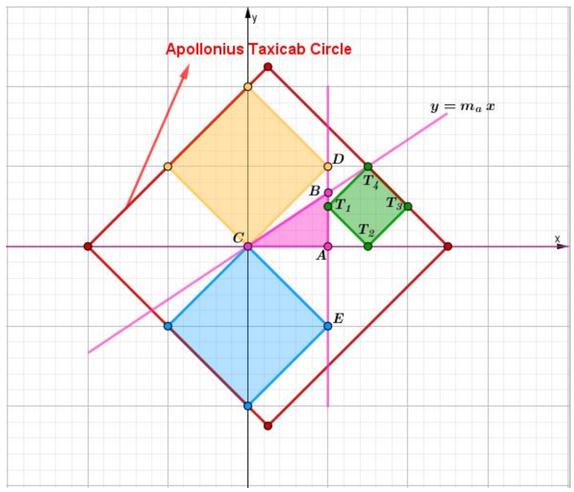

Figure 5.

consider taxicab excircle on side BA of the triangle $\overset{\triangle}{ABC}$. For λ and $x_a \in \mathbb{R}^+$, vertex T_1 of taxicab excircle on side BA can be coordinated by (x_a, λ). Therefore, radius of the taxicab excircle r is λ. So, $T_2 = (\lambda + x_a, 0)$, $T_3 = (2\lambda + x_a, \lambda)$ and $T_4 = (\lambda + x_a, 2\lambda)$. Since the point T_4 is on the line $y = m_a x$, value of λ is founded as $\frac{x_a m_a}{2 - m_a}$. Also, $D = (x_a, x_a)$ and $E = (x_a, -x_a)$. The line pass through the point D with slope -1 will be denoted by $l_{D_{-1}}$ such that equation of $l_{D_{-1}}$ is $y = -x + 2x_a$. Similarly, equation of the line l_{E_1} pass through the point E with slope 1 is $y = x - 2x_a$. To construct Apollonius taxicab circle of the triangle $\overset{\triangle}{ABC}$, it is clear that the point T_3 must not be in the region are bounded by lines $x = x_a$, l_{E_1} and $l_{D_{-1}}$. To optimize the point T_3 outside this region, the point T_3 must be subject to constraint;

$$
\begin{aligned}
&-(x_a + 2\lambda) + 2x_a < \lambda \\
&\lambda < (x_a + 2\lambda) - 2x_a
\end{aligned}
\tag{2.2}
$$

Consequently, the inequality $1/2 < m_g$ is obtained by inequalities (2.2). □

Theorem 2.3. *Let $\overset{\triangle}{ABC}$ be any triangle in taxicab plane. Then, there is always Apollonius taxicab circle iff for $|m_a| = 1,\ |m_b| < 1,\ |m_c| > 1$.*

Proof. For $|m_a| = 1,\ |m_b| < 1,\ |m_c| > 1$, (see Figure 6), consider three taxicab excircles on the side with slope 1 (or -1) the triangle $\overset{\triangle}{ABC}$. Because any taxicab circle compose of the line segments which lie on lines with slopes ± 1, there are always these taxicab excirles lying outside the triangle, tangent to the side with slope ∓ 1 and tangent to the extensions of the other two sides. Due to similar reason, it is clear that there will be a taxicab circle enclosing three taxicab excircles.

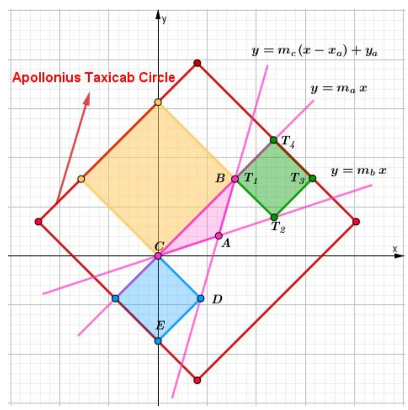

Figure 6.

□

3. Conclusion

There are several different alternative definitions of the circle of Apollonius in the literature [20]. For a given triangle, the Apollonius circle is the circle tangent internally to each of the three excircles. Using this definition of Apollonius circle based on any triangle, we have shown that conditions of existence of Apollonius circle in taxicab plane geometry. Moreover, these conditions are given in terms of the slopes of the sides of the triangle. Using a simple geometric approach based on slopes of the sides of any triangle, conditions of existence of Apollonius circle can be given in different metric plane geometry (e.g. plane geometries equipped with d_M, d_{CC}, d_α, d_m and d_{PT}). So, this is an open problem in different distance geometries.

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