SGLT-MAJORIZATION ON $M_{n,m}$ AND ITS LINEAR PRESERVERS

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Abstract. A matrix R is said to be *g-row substochastic* if $Re \leq e$. For X, $Y \in \mathbf{M}_{n,m}$, it is said that X is *sglt-majorized* by Y, $X \prec_{sglt} Y$, if there exists an n-by-n lower triangular g-row substochastic matrix R such that $X = RY$. This paper characterizes all (strong) linear preservers and strong linear preservers of $\prec_{sglt}$ on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$, respectively.

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1. Introduction

The notion of majorization plays an important role in mathematics, statistics, and economics. Over the years, the theory of majorization as a powerful tool has widely been applied to the related research areas of pure mathematics and applied mathematics (see [10]). A good survey on the theory of majorization was given by Marshall, Olkin, and Arnold [8]. With the development of majorization problem, preserving majorization have attracted much attention of mathematicians as an

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active subject of research in linear algebra. Recently, the concept of generalized stochastic matrices has been attended and many papers have been published in this topic, see [1]-[4], [6], [7], and [9]. The triangular matrices play an important role in the matrix analysis and its application. So, in this work, we pay attention to a new kind of majorization which has been defined on a special type of the triangular matrices.

In [5], the author introduced the notation of sgut-majorization on matrices and characterized all linear preservers and strong linear preservers of $\prec_{sgut}$ on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$, respectively. Here, we introduce the relation $\prec_{sglt}$ on matrices, and we characterize the linear preservers and strong linear preservers of this concept on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$.

The following notations will be fixed throughout the paper.

$\mathbf{M}_{n,m}$ for the set of all n-by-m real matrices;
$\mathbf{M}_n$ for the abbreviation of $\mathbf{M}_{n,n}$;
$\mathbb{R}^n$ for the set of all n-by-1 real vectors;
P_n for the n-by-n backward identity matrix;
e for the vector $(1, 1, \dots, 1)^t$;
$\mathcal{A}(S)$ for the set $\{\sum_{i=1}^m \lambda_i a_i \mid m \in \mathbb{N},\ \sum_{i=1}^m \lambda_i \leq 1,\ a_i \in S,\ \ \forall i \in \mathbb{N}_m\}$, where $S \subseteq \mathbb{R}^n$;
$card(S)$ for the cardinal number of a set S, where S is a finite set;
$\mathbb{N}_k$ for the set $\{1, \dots, k\} \subset \mathbb{N}$;
$[T]$ for the matrix representation of a linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ with respect to the standard basis;
r_i for the sum of entries on the ith row of $[T]$.

Let $\sim$ be a relation on $\mathbf{M}_{n,m}$. A linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ is said to be a linear preserver (or strong linear preserver) of $\sim$, if $TX \sim TY$ whenever $X \sim Y$ (or $TX \sim TY$ if and only if $X \sim Y$).

A real matrix (not necessarily nonnegative) R is called g-row substochastic if $Re \leq e$.

Let $X, Y \in \mathbf{M}_{n,m}$. The matrix X is said to be *sgut-majorized* by Y (in symbol $X \prec_{sgut} Y$) if $X = RY$, for some n-by-n upper triangular g-row substochastic matrix R.

In [5], the author found the (strong) linear preservers of $\prec_{sgut}$ on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$, respectively, as follows.

Theorem 1.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function, and let $[T] = [a_{ij}]$. Then T preserves $\prec_{sgut}$ if and only if one of the following conditions hold.*
(a) $Te_1 = \cdots = Te_{n-1} = 0$. In other words

$$[T] = \begin{pmatrix} 0 & \dots & 0 & a_{1n} \\ 0 & \dots & 0 & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{pmatrix}.$$

(b) There exist $t \in \mathbb{N}_{n-1}$ and $1 \leq i_1 < \cdots < i_m \leq n$ such that
$a_{i_1 t}, a_{i_2 t+1}, \dots, a_{i_m n} \neq 0$,

$$[T] = \begin{pmatrix} 0 & * & & & & & & \\ & a_{i_1 t} & & & * & & & \\ & & \ddots & & & & & \\ & & & a_{i_2 t+1} & & & & \\ & & & & \ddots & & & \\ & 0 & & & & a_{i_{m-1} n-1} & & \\ & & & & & & \ddots & \\ & & & & & & & a_{i_m n} \\ & & & & & & & * \end{pmatrix},$$

and one of the following statement happens.
(i) $card(h_m) \geq 2$, where $h_m = \{r_{i_{m-1}+1}, \dots, r_n\}$.
(ii) there exists $k \in \mathbb{N}_{m-1}$ such that $card(h_k) \geq 2$, $r_{i_k} = r_{i_k+1} = \cdots = r_n$, and for each $i \geq i_k$, and for each $j \in \mathbb{N}_n$, $a_{ij} \geq 0$ or $a_{ij} \leq 0$, where

$$h_1 = \{r_1, r_2, \dots, r_{i_1-1}, r_n\},$$

and

$$h_j = \{r_{i_{j-1}+1}, \dots, r_{i_j-1}, r_n\}$$

for each j $(2 \leq j \leq m-1)$.
(iii) $r_1 = r_2 = \cdots = r_n$, and for each $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ or $a_{ij} \leq 0$.

Theorem 1.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then T strongly preserves $\prec_{sgut}$ if and only if $[T] = \alpha I_n$ for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

Theorem 1.3. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear function. Then T strongly preserves $\prec_{sgut}$ if and only if $TX = XR$ for some invertible matrix $R \in \boldsymbol{M}_m$.*

We wish to find all (strong) linear preservers of sglt-majorization on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$, too.

This paper is organized as follows. In section 2, we state a necessary and sufficient condition for $x \prec_{sglt} y$ on $\mathbb{R}^n$. Then we characterize all (strong) linear preservers of sglt-majorization on $\mathbb{R}^n$. The last section of this paper studies some facts of this concept that are necessary for studying the strong linear preservers of $\prec_{sglt}$ on $\mathbf{M}_{n,m}$. Also, the strong linear preservers of $\prec_{sglt}$ on $\mathbf{M}_{n,m}$ are obtained.

2. Sglt-majorization on $\mathbb{R}^n$

In this section, we focus on the lower triangular g-row substochastic matrices and introduce a new type of majorization. Then we characterize all linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ (strongly) preserving $\prec_{sglt}$.

Definition 2.1. Let $x, y \in \mathbb{R}^n$. We say that x sglt-majorized by y (in symbol $x \prec_{sglt} y$) if $x = Ry$, for some n-by-n lower triangular g-row substochastic matrix R.

Here, we state what is required to continue. For this aim the following function is presented.

Assume that $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Define $\tau : \mathbb{R}^n \to \mathbb{R}^n$ by $\tau(x) = P_n T(P_n x)$.

Proposition 2.2. *Let $x,\ y \in \mathbb{R}^n$. Then $x \prec_{sgut} y$ if and only if $P_n x \prec_{sglt} P_n y$. Also, $P_n x \prec_{sgut} P_n y$ if and only if $x \prec_{sglt} y$.*

Proof. First, suppose that $x \prec_{sgut} y$. So $x = Ry$, for some n-by-n upper triangular g-row substochastic matrix R. It implies that $P_n x = (P_n R P_n)(P_n y)$, and so $P_n x \prec_{sglt} P_n y$.

Next, assume that $P_n x \prec_{sglt} P_n y$. This ensures that there exists some n-by-n lower triangular g-row substochastic matrix R such that $P_n x = R P_n y$, and then $x = (P_n R P_n) y$. Thus, $x \prec_{sgut} y$.

Now, $P_n x \prec_{sgut} P_n y$ if and only if $P_n(P_n x) \prec_{sglt} P_n(P_n y)$ if and only if $x \prec_{sglt} y$. □

This proposition provides a criterion for sglt-majorization on $\mathbb{R}^n$.

Proposition 2.3. *Let $x = (x_1, \ldots, x_n)^t$, $y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$. Then $x \prec_{sglt} y$ if and only if for all $i \in \mathbb{N}_n$*

$$x_i \in \mathcal{A}\{y_1, y_2, \ldots, y_i\}.$$

Now, we assert some prerequisites for introducing the main results of this section.

Proposition 2.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then T preserves $\prec_{sgut}$ if and only if τ preserves $\prec_{sglt}$.*
Also, T preserves $\prec_{sglt}$ if and only if τ preserves $\prec_{sgut}$.

Proof. If T preserves $\prec_{sgut}$; Let x, $y \in \mathbb{R}^n$, and let $x \prec_{sglt} y$. Proposition 2.2 shows that $P_n x \prec_{sgut} P_n y$, and then $T(P_n x) \prec_{sgut} T(P_n y)$. It implies that $P_n T(P_n x) \prec_{sglt} P_n T(P_n y)$. So $\tau(x) \prec_{sglt} \tau(y)$. Therefore, τ preserves $\prec_{sglt}$.
The rest of the proof is omitted for the sake of brevity. □

The following theorem characterizes structure of the linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving sglt-majorization.

Theorem 2.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function, and let $[T] = [a_{ij}]$. Then T preserves $\prec_{sglt}$ if and only if one of the following conditions holds.*
(a)

$$[T] = \begin{pmatrix} a_{11} & \ldots & 0 & 0 \\ a_{21} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \ldots & 0 & 0 \end{pmatrix}.$$

(b) There exist $t \in \mathbb{N}_{n-1}$ *and* $1 \leq l_1 < \cdots < l_{n-t+1} \leq n$ *such that* $a_{l_1 1}, a_{l_2 2}, \ldots, a_{l_{n-t+1} n-t+1} \neq 0$,

$$[T] = \begin{pmatrix} * & 0 & & & & \\ a_{l_1 1} & & & & & \\ & a_{l_2 2} & & & 0 & \\ & & \ddots & & & \\ & * & & a_{l_{n-t} n-t} & & \\ & & & & a_{l_{n-t+1} n-t+1} & \\ & & & & * & 0 \end{pmatrix},$$

and one of the following statement happens.
(i) card $(g_1) \geq 2$, *where* $g_1 = \{r_1, r_2, \ldots, r_{l_2 - 1}\}$.
(ii) there exists $2 \leq k \leq n-t+1$ *such that card* $(g_k) \geq 2$, $r_1 = r_2 = \cdots = r_{l_k}$, *and for each* $i \leq l_k$, *and for each* $j \in \mathbb{N}_n$, $a_{ij} \geq 0$ *or* $a_{ij} \leq 0$, *where*

$$g_{n-t+1} = \{r_{l_{n-t+1}+1}, \ldots, r_n, r_1\},$$

and

$$g_j = \{r_{l_j+1}, \ldots, r_{l_{j+1}-1}, r_1\}$$

for each j $(2 \leq j \leq n-t)$.
(iii) $r_1 = r_2 = \cdots = r_n$, *and for each* $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ *or* $a_{ij} \leq 0$.

Proof. T preserves $\prec_{sglt}$ if and only if τ preserves $\prec_{sgut}$ if and only if, by Theorem 1.1, one of the following conditions holds.
(a) $Te_1 = \cdots = Te_{n-1} = 0$. In other words

$$[T] = \begin{pmatrix} 0 & \ldots & 0 & a_{1n} \\ 0 & \ldots & 0 & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & a_{nn} \end{pmatrix}.$$

(b) There exist $t \in \mathbb{N}_{n-1}$ and $1 \leq i_1 < \cdots < i_m \leq n$ such that $a_{i_1 t}, a_{i_2 t+1}, \ldots, a_{i_m n} \neq 0$,

$$[T] = \begin{pmatrix} 0 & * & & & & & & \\ & a_{i_1 t} & & & * & & & \\ & & \ddots & & & & & \\ & & & a_{i_2 t+1} & & & & \\ & & & & \ddots & & & \\ & 0 & & & & a_{i_{m-1} n-1} & & \\ & & & & & & \ddots & \\ & & & & & & & a_{i_m n} \\ & & & & & & & * \end{pmatrix},$$

and one of the following statement happens.
(i) $card(h_m) \geq 2$, where $h_m = \{r_{i_{m-1}+1}, \ldots, r_n\}$.
(ii) there exists $k \in \mathbb{N}_{m-1}$ such that $card(h_k) \geq 2$, $r_{i_k} = r_{i_k+1} = \cdots = r_n$, and for each $i \geq i_k$, and for each $j \in \mathbb{N}_n$, $a_{ij} \geq 0$ or $a_{ij} \leq 0$, where

$$h_1 = \{r_1, r_2, \ldots, r_{i_1-1}, r_n\},$$

and

$$h_j = \{r_{i_{j-1}+1}, \ldots, r_{i_j-1}, r_n\}$$

for each j $(2 \leq j \leq m-1)$.
(iii) $r_1 = r_2 = \cdots = r_n$, and for each $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ or $a_{ij} \leq 0$.
if and only if, because of $P_n[\tau]P_n = [T]$, one of the following conditions holds.
(a)

$$[T] = \begin{pmatrix} a_{11} & \ldots & 0 & 0 \\ a_{21} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \ldots & 0 & 0 \end{pmatrix}.$$

(b) There exist $t \in \mathbb{N}_{n-1}$ and $1 \leq l_1 < \cdots < l_{n-t+1} \leq n$ such that
$a_{l_1 1}, a_{l_2 2}, \ldots, a_{l_{n-t+1} n-t+1} \neq 0$,

$$[T] = \begin{pmatrix} * & 0 & & & & \\ a_{l_1 1} & & & & & \\ & a_{l_2 2} & & & 0 & \\ & & \ddots & & & \\ & * & & a_{l_{n-t} n-t} & & \\ & & & & a_{l_{n-t+1} n-t+1} & \\ & & & & * & 0 \end{pmatrix},$$

and one of the following statement happens.
(i) *card* $(g_1) \geq 2$, where $g_1 = \{r_1, r_2, \ldots, r_{l_2 - 1}\}$.
(ii) there exists $2 \leq k \leq n-t+1$ such that *card* $(g_k) \geq 2$, $r_1 = r_2 = \cdots = r_{l_k}$, and for each $i \leq l_k$, and for each $j \in \mathbb{N}_n$, $a_{ij} \geq 0$ or $a_{ij} \leq 0$, where

$$g_{n-t+1} = \{r_{l_{n-t+1}+1}, \ldots, r_n, r_1\},$$

and

$$g_j = \{r_{l_j+1}, \ldots, r_{l_{j+1}-1}, r_1\}$$

for each j $(2 \leq j \leq n-t)$.
(iii) $r_1 = r_2 = \cdots = r_n$, and for each $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ or $a_{ij} \leq 0$. □

In the following theorem the structure of linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ strongly preserving sglt-majorization will be characterized.

Lemma 2.6. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function that strongly preserves $\prec_{sglt}$. Then T is invertible.*

Proof. Let $x \in \mathbb{R}^n$, and let $Tx = 0$. Since $Tx = T0$ and T strongly preserves $\prec_{sglt}$, we have $x \prec_{sglt} 0$. So $x = 0$. Therefore, T is invertible. □

Theorem 2.7. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then T strongly preserves $\prec_{sglt}$ if and only if $[T] = \alpha I$, for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. We only prove the necessity of the condition. Suppose that T strongly preserves $\prec_{sglt}$. This ensures that T and T^{-1} preserve $\prec_{sglt}$. So τ is invertible. Proposition 2.4 implies that τ and τ^{-1} preserve $\prec_{sgut}$, and then τ strongly preserves

$\prec_{sgut}$. By Theorem 1.2, $[\tau] = \alpha I$, for some $\alpha \in \mathbb{R} \setminus \{0\}$. It follows that $[T] = \alpha I$, and the statement holds. □

3. Sglt-majorization on $\mathbf{M}_{n,m}$

In this section, we talk about sglt-majorization on $\mathbf{M}_{n,m}$.

Definition 3.1. Let $X, Y \in \mathbf{M}_{n,m}$. We say that X sglt-majorized by Y (in symbol $X \prec_{sglt} Y$) if there exists an n-by-n lower triangular g-row substochastic matrix R such that $X = RY$.

Suppose that $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear function. Define $\tau : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ by $\tau(X) = P_n T(P_n X)$.

Proposition 3.2. *Let X, $Y \in \mathbf{M}_{n,m}$. Then $X \prec_{sgut} Y$ if and only if $P_n X \prec_{sglt} P_n Y$.*
Also, $P_n X \prec_{sgut} P_n Y$ if and only if $X \prec_{sglt} Y$.

Proposition 3.3. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear function. Then T preserves $\prec_{sgut}$ if and only if τ preserves $\prec_{sglt}$.*
Also, T preserves $\prec_{sglt}$ if and only if τ preserves $\prec_{sgut}$.

The following theorem characterizes the strong linear preservers of $\prec_{sglt}$ on $\mathbf{M}_{n,m}$.

Theorem 3.4. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear function. Then T strongly preserves $\prec_{sglt}$ if and only if $TX = XR$, for some invertible matrix $R \in \boldsymbol{M}_m$.*

Proof. First, assume that T strongly preserves $\prec_{sglt}$. This ensures that τ strongly preserves $\prec_{sgut}$, and, by Theorem 1.3, $\tau X = XR$, for some invertible matrix $R \in \mathbf{M}_m$. It implies that $P_n T(P_n X) = XR$, and then $T(P_n X) = P_n XR$. So $TX = XR$, where $R \in \mathbf{M}_m$, and is invertible. Observe that the statement holds.

Next, assume that $TX = XR$, for some invertible matrix $R \in \mathbf{M}_m$. It follows that $T(P_n X) = P_n XR$, and then $P_n T(P_n X) = XR$. Then $\tau X = XR$. It implies that τ strongly preserves $\prec_{sgut}$, and so T strongly preserves $\prec_{sglt}$. □

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