ASSOCIATED CURVES OF THE SPACELIKE CURVE VIA THE BISHOP FRAME OF TYPE-2 IN $\mathbb{E}_1^3$

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Abstract. The objective of the study in this paper is to define M_1, M_2-direction curves and M_1, M_2-donor curves of the spacelike curve γ via the Bishop frame of type-2 in $\mathbb{E}_1^3$. We obtained the necessary and sufficient conditions when the associated curves to be slant helices and general helices via the Bishop frame of type-2 in $\mathbb{E}_1^3$. After defining the spherical indicatrices of the associated curves, we obtain some relations between associated curves and their spherical indicatrices in terms of the frames used in the present work.

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1. Introduction

There are lots of interesting and important problems in the theory of curves in differential geometry. One of the interesting problems is the problem of characterization of a regular curve in the theory of curves in the Euclidean, Minkowski and different ambient spaces, see, [1], [2], [5], [7], and [8]. Also there are special curves

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which are obtained under some definitions such as Smarandache curves, spherical indicatrices, and curves of constant breadth, and etc.

Special curves are classical differential geometric objects. These curves are obtained by assuming a special property on the original regular curve. Some of them are Smarandache curves, curves of constant breadth, Bertrand curves, and Mannheim curves, etc. Studying curves can be differed according to frame used for curve. In the studies of classical differential geometry of curves, one of the most used frames is parallel transport frame, also called Bishop frame which is an alternative frame needed for non-continously differentiable curves on which Bishop (parallel transport frame) frame is well defined and constructed in Euclidean and its ambient spaces [1]. L. R. Bishop defined Bishop frame, which is known alternative or parallel frame of the curves with the help of parallel vector fields [1]. That is why he defined this frame that curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. Numerous recent research papers related to this concept have been treated, for example, see ([2], [10], [13]).

Choi and Kim introduced the notion of the principal (binormal)-direction curve and principal (binormal)-donor curve of a Frenet curve in $\mathbb{E}^3$ and gave the relationship of curvature and torsion of its mates [3]. Also, Choi et al. introduced the notion of the principal (binormal)-direction curve and principal (binormal)-donor curve of a Frenet curve in $\mathbb{E}^3$ and gave the relationship of curvature and torsion of its mates [4]. New associated curves by using the Bishop frame are obtained by ([6], [12]). S. Yilmaz and M. Turgut examined a new version of the Bishop frame which is called the Bishop frame of type-2 [13].

The objective of the study in this paper is to define M_1, M_2-direction curves and M_1, M_2-donor curves of non-lightlike curve γ via the Bishop frame of type-2 in $\mathbb{E}_1^3$. We obtained the necessary and sufficient conditions when the associated curves to be slant helices or general helices via the Bishop frame of type-2 in $\mathbb{E}_1^3$. After defining the spherical indicatrices of the associated curves, we obtain some relations between associated curves and their spherical indicatrices in terms of the frames used in the present work.

2. Preliminaries

The Minkowski three dimensional space $\mathbb{E}^3_1$ is a real vector space $\mathbb{E}^3$ endowed with the standard flat Lorentzian metric, given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2, \tag{1}$$

where (x_1, x_2, x_3) is rectangular coordinate system of $\mathbb{E}^3_1$, and g is an indefinite metric. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary an vectors in $\mathbb{E}^3_1$, the Lorentzian cross product of u and v defined by

$$u \times v = -\begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Recall that a vector $v \in \mathbb{E}^3_1$ can have one of three Lorentzian characters: it can be spacelike if $g(v, v) > 0$ or $v = 0$; timelike if $g(v, v) < 0$ and null(lightlike) if $g(v, v) = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\gamma = \gamma(s)$ in $\mathbb{E}^3_1$ can locally be spacelike, timelike or null (lightlike) if all of its velocity vector γ' are respectively spacelike, timelike, or null (lightlike), for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in \mathbb{E}^3_1$ is given by

$$\|a\| = \sqrt{|\langle a, a \rangle|}.$$

The curve $\gamma = \gamma(s)$ is called a unit speed curve if velocity vector γ' is unit i.e, $\|\gamma'\| = 1$. For vectors $v, w \in \mathbb{E}^3_1$ it is said to be orthogonal if and only if $g(v, w) = 0$. Denote by $\{T, N, B\}$ the moving Serret-Frenet frame along the curve $\gamma = \gamma(s)$ in the space $\mathbb{E}^3_1$ [9].

For a unit speed spacelike curve with first and second curvature(torsion), $\kappa(s)$ and $\tau(s)$ the following Serret-Frenet formulae in $\mathbb{E}^3_1$ are given as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \epsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \cdot \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2}$$

where $\epsilon = \mp 1$ [9]. If $\epsilon = 1$, then $\gamma(s)$ is a spacelike curve with spacelike principal normal N and timelike binormal B and define Serret-Frenet invariants, ([9]),

$$T(s) = \gamma'(s), \quad \kappa(s) = \|T'(s)\|, \quad N(s) = \frac{T'(s)}{\kappa(s)},$$
$$B(s) = T(s) \times N(s) \text{ and } \tau(s) = < N'(s), B(s) > .$$

If $\epsilon = -1$, then $\gamma(s)$ is a spacelike curve with timelike principal normal N and spacelike binormal B then we write

$$T(s) = \gamma'(s), \quad \kappa(s) = \sqrt{-< T'(s), T'(s) >}, N(s) = \frac{T'(s)}{\kappa(s)},$$
$$B(s) = T(s) \times N(s) \text{ and } \tau(s) =< N'(s), B(s) > .$$

The Lorentzian sphere $\mathbb{S}_1^2$ of radius $r > 0$ and with the center in the origin of the space $\mathbb{E}_1^3$ is defined by

$$\mathbb{S}_1^2(r) = \{p = (p_1, p_2, p_3) \in \mathbb{E}_1^3 : \langle p, p \rangle = r^2\}.$$

The pseudo-hyperbolic space $\mathbb{H}_0^2$ of radius $r > 0$ and with the center in the origin of the space $\mathbb{E}_1^3$ is defined by

$$\mathbb{H}_0^2(r) = \{p = (p_1, p_2, p_3) \in \mathbb{E}_1^3 : \langle p, p \rangle = -r^2\}.$$

The Bishop derivative formula of type-2 of a spacelike curve with spacelike principal normal is given

$$\begin{bmatrix} M_1'(s) \\ M_2'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & k_1 \\ 0 & 0 & -k_2 \\ -k_1 & -k_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} M_1(s) \\ M_2(s) \\ B(s) \end{bmatrix} \tag{3}$$

in $\mathbb{E}_1^3$.

Also, the relation between Frenet and Bishop frames of type-2 is given as

$$\begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} \sinh\theta & \cosh\theta & 0 \\ \cosh\theta & \sinh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} M_1(s) \\ M_2(s) \\ B(s) \end{bmatrix}, \tag{4}$$

where the angle

$$\theta = \operatorname{arctanh}\frac{k_2}{k_1}. \tag{5}$$

There are also the following expressions

$$\tau(s) = \sqrt{|k_2^2(s) - k_1^2(s)|}, \quad \kappa(s) = \frac{d\theta(s)}{ds} \tag{6}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion functions of the curve $\alpha(s)$, see [13].

Proposition 2.1. ([5]) Let $\gamma(s)$ be a spacelike curve with curvatures κ and τ. The curve γ lies on the Lorentzian sphere if and only if

$$\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}(\frac{1}{\kappa})\right]=\frac{\tau}{\kappa}.$$

Proposition 2.2. ([5]) Let $\gamma(s)$ be a spacelike curve with curvatures κ and τ. The curve γ is a general helix if and only if

$$\frac{\kappa}{\tau}=\text{constant}. \tag{7}$$

Proposition 2.3. ([5]) Let $\gamma(s)$ be a spacelike curve with curvatures κ and τ. The curve γ is a slant helix if and only if

$$\sigma(s)=\left[\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}(\frac{\tau}{\kappa})'\right]=\text{constant}.$$

Theorem 2.4. Let $\gamma: I \to \mathbb{E}_1^3$ be a unit speed spacelike curve with a spacelike binormal curve with nonzero natural curvatures. Then γ is a slant helix if and only if $\frac{k_1}{k_2}$ is constant [2].

3. Associated curves of the spacelike curve via the Bishop frame of type-2

In this section, we define some associated curves of a spacelike curve γ due to the Bishop frame of type-2 in $\mathbb{E}_1^3$. For a Frenet frame $\gamma: I \to \mathbb{E}_1^3$, consider a vector field V given by the Bishop frame of type-2 as follows:

$$V(s)=u(s)M_1(s)+v(s)M_2(s)+w(s)B(s), \tag{8}$$

where u, v, and w are functions on I satisfying

$$u^2(s)-v^2(s)+w^2(s)=1. \tag{9}$$

Then, an integral curve $\overline{\gamma}(s)$ of V defined on I is a unit speed curve in $\mathbb{E}_1^3$.

Definition 3.1. (M_i-direction curve, $i=1,2$) Let γ be a spacelike curve in $\mathbb{E}_1^3$. An integral curve of M_i is called M_i-direction curve of the spacelike curve γ via the Bishop frame of type-2.

Remark 3.2. An M_1-direction curve is an integral curve of the equation (8) with $u(s)=w(s)=0, v(s)=1$.

Remark 3.3. An M_2-direction curve is an integral curve of the equation (8) with $u(s)=v(s)=0, w(s)=1$.

Theorem 3.4. Let γ be a spacelike curve in $\mathbb{E}_1^3$ with the curvature κ and the torsion τ, and $\overline{\gamma}$ be the M_1-direction curve of the spacelike curve γ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then we have

$$\overline{T} = M_1, \quad \overline{N} = T, \quad \overline{B} = M_2$$
$$\overline{\kappa} = k_1, \quad \overline{\tau} = -k_2.$$

Proof. From Definition 3.1**,** we can write that

$$\overline{\gamma}' = \overline{T} = M_1. \tag{10}$$

Differentiating the expression (10) and then taking its norm, we find

$$\overline{\kappa} = k_1 \tag{11}$$

for $k_1 > 0$.

Differentiation of the expressions (10) with using of (11) gives us

$$\overline{N} = B. \tag{12}$$

The vectorial product of $\overline{T}$ and $\overline{N}$ is as follows:

$$\overline{B} = \overline{T} \times \overline{N}. \tag{13}$$

Using the expressions (10), (12) in (13) we find that

$$\overline{B} = M_2. \tag{14}$$

Finally, differentiating (14) and using (12) in it, we have

$$\overline{\tau} = k_2. \tag{15}$$

Proposition 3.5. Let γ be a spacelike curve in $\mathbb{E}_1^3$ and $\overline{\gamma}$ be the M_1-direction curve of γ. Then the M_1-direction curve of $\overline{\gamma}$ equals to γ up to translation if and only if

$$u(s) = 0, \quad v(s) = \sinh(\textstyle\int k_1(s)ds) \text{ and } \quad w = \cosh(\textstyle\int k_1(s)ds).$$

Proof. Differentiating the expression (9) with respect to s gives

$$uu' - vv' + ww' = 0. \tag{16}$$

Similarly differentiating (8) with respect to s, we obtain

$$V' = (u' - wk_1)M_1 + (v' - wk_2)M_2 + (uk_1 - vk_2 + w\prime)B. \tag{17}$$

Since $V'(s) = \overline{\gamma}''(s) = \overline{T}' = \overline{\kappa}\overline{N}$, $\overline{\gamma}$ is the M_1-direction curve of γ, i.e., $\overline{\gamma}'(s) = \overline{T} = M_1$ if and only if

$$\left\{ \begin{array}{l} u' - wk_1 \neq 0, \\ v' - wk_2 = 0, \\ uk_1 - vk_2 + w\prime = 0. \end{array} \right. \tag{18}$$

Multiplying the third equation in (18) with w and substituting it into (16), we have

$$vwk_2 = uu' + ww'. \tag{19}$$

Similarly multiplying the first equation with u and putting such an obtained equation into (19), we have $u(u' - wk_1) = 0$. Since $u' - wk_2 \neq 0$, it follows that $u = 0$. Hence the solutions of (18) which hold (16) are given by

$$u(s) = 0, \quad v(s) = \sinh(\textstyle\int k_1(s)ds), \text{ and } \quad w = \cosh(\textstyle\int k_1(s)ds).$$

Definition 3.6. An integral curve of

$$\sinh(\textstyle\int k_1(s)ds)M_2(s) + \cosh(\textstyle\int k_1(s)ds)B(s)$$

in the expression (8) is called a M_1-donor curve of γ.

Theorem 3.7. Let γ be a spacelike curve in $\mathbb{E}_1^3$ with the curvature κ and the torsion τ, and $\overline{\gamma}$ be the M_2-direction curve of γ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then we have

$$\begin{array}{l} \overline{T} = M_2, \quad \overline{N} = T, \quad \overline{B} = M_1 \\ \overline{\kappa} = -k_2, \quad \overline{\tau} = k_1. \end{array}$$

Proof. It is similar to the proof of Theorem 3.5.

Corollary 3.8. Let γ be a spacelike curve in $\mathbb{E}_1^3$ and $\overline{\gamma}$ be the M_i-direction curve of γ, $i = 1, 2$. The Frenet frame of $\overline{\gamma}$ is given in terms of the Bishop frame of type-2 as follows:

$$\begin{array}{l} \overline{T}(s) = \sinh(\int k_i(s)ds)\overline{M}_1(s) + \cosh(\int k_i(s)ds)\overline{M}_2(s), \\ \overline{N}(s) = \cosh(\int k_i(s)ds)\overline{M}_1(s) + \sinh(\int k_i(s)ds)\overline{M}_2(s), \\ \overline{B}(s) = B(s). \end{array} \tag{20}$$

Proof. It is straightforwardly seen by substituting the curvature functions in Theorem 3.7 into (4).

Corollary 3.9. If the curve γ is a M_i-donor curve of the curve $\overline{\gamma}$ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$, then the curvature κ and the torsion τ of the spacelike curve γ are given by

$$\tau = \sqrt{\overline{\kappa}^2 - \overline{\tau}^2}, \quad \kappa = (\frac{\overline{\tau}^2}{\overline{\tau}^2 - \overline{\kappa}^2})(\frac{\overline{\kappa}}{\overline{\tau}})' \tag{21}$$

Proof. It is straightforwardly seen by substituting the expressions in Theorem 3.7 into (6).

Corollary 3.10. Let γ be a spacelike curve with the curvature κ and the torsion τ in $\mathbb{E}_1^3$ and $\overline{\gamma}$ be the M_2-direction curve of γ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then it satisfies

$$\frac{\overline{\tau}}{\overline{\kappa}} = \coth\theta, \quad \frac{\tau}{\kappa} = -\frac{(\overline{\tau}^2 - \overline{\kappa}^2)^{\frac{3}{2}}}{\overline{\tau}^2}(\frac{\overline{\tau}}{\overline{\kappa}})' \tag{22}$$

Proof. It is straightforwardly seen by substituting the expressions in Theorem 3.7 into (5).

Proposition 3.11. Let γ be a spacelike curve in $\mathbb{E}_1^3$ and $\overline{\gamma}$ be the M_2-direction curve of γ. Then the M_2-direction curve of $\overline{\gamma}$ equals to γ up to translation if and only if

$$u(s) = \cosh(-\textstyle\int k_1(s)ds), \quad v(s) = \sinh(-\textstyle\int k_1(s)ds) \text{ and } \quad w = 0.$$

Proof. It is similar to Proposition 3.5.

Definition 3.12. An integral curve of

$$\sin(\textstyle\int k_1(s)ds)M_1(s) + \cos(\textstyle\int k_1(s)ds)B(s)$$

in the expreesion (8) is called a M_2-donor curve of γ.

4. $\overline{M}_i$-Bishop spherical images of a regular M_1-direction curve $\overline{\gamma}$ in $\mathbb{E}_1^3$

Definition 4.1. Let $\overline{\gamma} = \overline{\gamma}(s)$ be an M_i-direction curve of a spacelike curve γ in $\mathbb{E}_1^3$. If we translate the vector field $\overline{M}_i$ $(i = 1, 2)$ of the Bishop frame of type-2 to the center O of the unit Lorentzian sphere S^2, we obtain a spherical image $\varphi = \varphi(s_\varphi)$. This curve is called the $\overline{M}_i$-Bishop spherical image of the regular spacelike curve $\overline{\gamma} = \overline{\gamma}(s)$.

Let $\varphi = \varphi(s_\varphi)$ be the $\overline{M}_1$ Bishop spherical image of a regular spacelike curve $\overline{\gamma} = \overline{\gamma}(s)$. First, we differentiate of the curve φ with respect to s:

$$\varphi' = \frac{d\varphi}{ds_\varphi}\frac{ds_\varphi}{ds} = \overline{k}_1\overline{B}.$$

Taking the norm of the both sides of the above equation, then we have

$$\overline{T}_\varphi = \overline{B} \text{ and } \frac{ds_\varphi}{ds} = \overline{k}_1. \tag{23}$$

Differentiating the expression (23), then we get

$$\overline{T}'_\varphi = \dot{\overline{T}}_\varphi \frac{ds_\varphi}{ds} = -(\overline{k}_1\overline{M}_1 + \overline{k}_2\overline{M}_2)$$

or

$$\dot{\overline{T}}_\varphi = -\left(\overline{M}_1 + \frac{\overline{k}_2}{\overline{k}_1}\overline{M}_2\right),$$

where we denote derivative according to s by a dash, and to s_α by a dot.

We find the first curvature of φ as follows:

$$\overline{\kappa}_\varphi = \left\|\dot{\overline{T}}_\beta\right\| = \sqrt{\left|\left(\frac{\overline{k}_2}{\overline{k}_1}\right)^2 - 1\right|} = \frac{\sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}}{\overline{k}_1}. \tag{24}$$

Using the expression (24), then we have the principal normal vector field of φ as

$$\overline{N}_\varphi = -\frac{\overline{k}_1}{\sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}}\overline{M}_1 + \frac{\overline{k}_2}{\sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}}\overline{M}_2.$$

By the cross product of $\overline{T}_\varphi \times \overline{N}_\varphi$, the binormal vector field of φ is obtained as

$$\overline{B}_\varphi = -\frac{\overline{k}_2}{\sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}}\overline{M}_1 + \frac{\overline{k}_1}{\sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}}\overline{M}_2.$$

By means of the obtained equations, we express the torsion of the $\overline{M}_1$ Bishop spherical image of a regular curve $\overline{\gamma} = \overline{\gamma}(s)$ as follows:

$$\overline{\tau}_\varphi = \frac{(\overline{k}_1)^7 \left(\frac{\overline{k}_2}{\overline{k}_1}\right)'}{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}. \tag{25}$$

Consequently, we determined the Frenet-Serret invariants of the $\overline{M}_1$ Bishop spherical image of M_1-direction curve $\overline{\gamma}$ in terms of the Bishop invariants of type-2.

Considering the equations (24) and (25) by Theorem 2.4, then we have the following corollary:

Corollary 4.2. Let $\varphi = \varphi(s_\varphi)$ be a $\overline{M}_1$ Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the spacelike curve $\overline{\gamma} = \overline{\gamma}(s)$ is a slant helix due to the Bishop frameof type 2, then the $\overline{M}_1$ Bishop spherical image φ is a circle in the osculating plane.

In the light of Propositions 2.2, and 2.3, we give the following results without proofs:

Corollary 4.3. Let $\varphi = \varphi(s_\varphi)$ be the $\overline{M}_1$ Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the spacelike curve $\overline{\gamma} = \overline{\gamma}(s)$ is a general helix due to the Bishop frame of type-2, then the Bishop curvatures of φ satisfy

$$\frac{(\overline{k}_1)^5 \left(\dfrac{\overline{k}_2}{\overline{k}_1}\right)'}{(\overline{k}_2^2 - \overline{k}_1^2)^{\frac{3}{2}}} = constant.$$

Corollary 4.4. Let $\varphi = \varphi(s_\varphi)$ be the $\overline{M}_1$ Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the spacelike curve $\overline{\gamma} = \overline{\gamma}(s)$ is a slant helix due to the Bishop frame of type-2, then the Bishop curvatures of φ satisfy

$$\left[\frac{(\overline{k}_1)^5 \left(\dfrac{\overline{k}_2}{\overline{k}_1}\right)'}{(\overline{k}_2^2 - \overline{k}_1^2)^{\frac{3}{2}}}\right] \frac{(\overline{k}_1^2 - \overline{k}_2^2)^4 \overline{k}_1}{\left((\overline{k}_2^2 - \overline{k}_1^2)^3 - (\overline{k}_1)^{16}\left[\left(\dfrac{\overline{k}_2}{\overline{k}_1}\right)'\right]^2\right)^{\frac{3}{2}}} = constant.$$

5. Binormal Bishop spherical images of a regular M_1-direction curve $\overline{\gamma}$ in $\mathbb{E}_1^3$

Definition 5.1. Let $\overline{\gamma} = \overline{\gamma}(s)$ be a M_1-direction curve of a spacelike curve γ in $\mathbb{E}_1^3$. If we translate the binormal vector field of the Bishop frame of type-2 to the center O of the unit Lorentzian sphere S^2, we obtain a spherical image $\alpha = \alpha(s_\alpha)$. This curve is called the binormal Bishop spherical image of the regular spacelike curve $\overline{\gamma} = \overline{\gamma}(s)$.

Let $\alpha = \alpha(s_\alpha)$ be the binormal Bishop spherical image of a regular spacelike curve $\overline{\gamma} = \overline{\gamma}(s)$. We can write that

$$\alpha' = \frac{d\alpha}{ds_\alpha}\frac{ds_\alpha}{ds} = -(\overline{k}_1\overline{M}_1 + \overline{k}_2\overline{M}_2).$$

Here, we denote derivative according to s by a dash, and to s_α by a dot.

In terms of type-2 Bishop frame vector field in (3), we obtain the tangent vector of the binormal Bishop spherical image as follows:

$$\overline{T}_\alpha = \frac{-(\overline{k}_1\overline{M}_1 + \overline{k}_2\overline{M}_2)}{\sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}} = -\frac{\overline{k}_1}{k_2}\overline{M}_1 - \frac{\overline{k}_2}{k_2}\overline{M}_2, \tag{26}$$

where

$$\frac{ds_\alpha}{ds} = \sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|} = \overline{\tau}(s) = k_2(s).$$

In order to determine the first curvature of the binormal Bishop spherical image α, we can write

$$\dot{\overline{T}}_\alpha = -(P'(s)\overline{M}_1 + Q'(s)\overline{M}_2 + (P(s)\overline{k}_1 - Q(s)\overline{k}_2)\overline{B}),$$

where

$$P(s) = \frac{\overline{k}_1}{\sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}}, \text{ and } Q(s) = \frac{\overline{k}_2}{\sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}}.$$

Immediately, we arrive at

$$\overline{\kappa}_\alpha = \left\|\dot{\overline{T}}_\alpha\right\| = \sqrt{\left|(P'(s))^2 + (Q'(s))^2 - (P(s)\overline{k}_1 - Q(s)\overline{k}_2)^2\right|}.$$

So, the principal normal vector field of the binormal Bishop spherical image α is obtained as

$$\overline{N}_\alpha = \frac{-1}{\overline{\kappa}_\alpha}\left\{(P'(s)\overline{M}_1 + Q'(s)\overline{M}_2 + (P(s)\overline{k}_1 - Q(s)\overline{k}_2)\overline{B})\right\}$$

By the cross product of $\overline{T}_\alpha \times \overline{N}_\alpha$, we have the binormal vector field of the binormal Bishop spherical image α as

$$\overline{B}_\alpha = \frac{1}{\overline{\kappa}_\alpha\sqrt{\left|\overline{k}_2^2 - \overline{k}_1^2\right|}}\left\{\left[Q(s)\overline{k}_2 - P(s)\overline{k}_1\right]\overline{M}_1 + \left[P(s)\overline{k}_1 - Q(s)\overline{k}_2\right]\overline{M}_2 - \left[Q'(s)\overline{k}_1 + P'(s)\overline{k}_2\right]\overline{B}\right\}.$$

By means of the obtained equations, we express the torsion of the binormal Bishop spherical image α as follows:

(27)

$$\tau = \frac{1}{(\overline{\kappa}_\alpha)^2}\Big\{\overline{k}_1\left[\overline{k}_1\overline{k}_1'\overline{k}_2' + \overline{k}_2\overline{k}_2'^2 + \overline{k}_2'(\overline{k}_1^2 + \overline{k}_2^2)' - (\overline{k}_1^2 + \overline{k}_2^2)(\overline{k}_2^2 + (\overline{k}_1^2 + \overline{k}_2^2)\overline{k}_2\right] + \overline{k}_2\left[(\overline{k}_1^2 + \overline{k}_2^2)(\overline{k}_1^2 + (\overline{k}_1^2 + \overline{k}_2^2)\overline{k}_1 - \overline{k}_1\overline{k}_1'^2 - \overline{k}_1'\overline{k}_2\overline{k}_2' - \overline{k}_1'(\overline{k}_1^2 + \overline{k}_2^2)\right]\Big\}$$

Consequently, we determined the Frenet-Serret invariants of the binormal Bishop spherical image of M_1-direction curve $\overline{\gamma}$ in terms of the Bishop invariants of type-2.

Corollary 5.2. Let $\alpha = \alpha(s_\alpha)$ be a binormal Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the spacelike curve $\overline{\gamma} = \overline{\gamma}(s)$ is a slant helix due to the Bishop frame of type-2, then the binormal spherical image α is a circle in the osculating plane.

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