ON STRONGLY H_v-GROUPS

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Abstract. The largest class of hyperstructures is the one which satisfies the weak properties; these are called H_v-structures. In this paper we introduce a special product of elements in H_v-group and define a new class of H_v-groups called strongly H_v-groups. Then we show that in strongly H_v-groups $\beta = \beta^*$. Also we express θ-hyperoperation and investigate some of its properties in connection with strongly H_v-groups.

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1. Introduction

The first definition of hyperoperation and hypergroup was adverted by Frederic Marty in the 8^{th} Congress of Scandinavian Mathematicians in 1934. In 1990, in Greece, T. Vougiouklis introduced the concept of the weak hyperstructures which now are named H_v-structures. Over the last 28 years this class of hyperstructures, which is the largest, has been studied from several aspects as well as in connection with many other topics of mathematics. Basically, the study of H_v-structures has

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been continued in many directions by T. Vougiouklis, B. Davvaz, S. Spartalis, A. Dramalidis, S. Hoskova, and some other mathematicians. We invite the readers for more study about hyperstructure theory and its applications to [1], [2], [3], [4], [7], [13] and [14]. We recall the following definitions from [1].

Definition 1.1. *Let H be a non-empty set and $*: H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, *)$ is called a hypergroupoid. For any two non-empty subset A and B of H and $x \in H$, we define*

$$A * B = \bigcup_{a \in A, b \in B} a * b \ , \quad A * x = A * \{x\}.$$

Definition 1.2. *A hypergroupoid $(H, *)$ is called hypergroup if for all $(x, y, z) \in H^3$, it satisfies the following conditions:*

(1) $(x * y) * z = x * (y * z)$, *which means that*

$$\bigcup_{u \in x*y} u * z = \bigcup_{v \in y*z} x * v,$$

(2) $x * H = H = H * x$.

Definition 1.3. [2]. *A hypergroupoid $(H, *)$ is called a H_v-group if the following axioms hold:*

(1) $x * (y * z) \cap (x * y) * z \neq \emptyset$ *for all* $(x, y, z) \in H^3$; *(weak associativity)*
(2) $x * H = H = H * x$ *for all* x *in* H. *(reproduction)*

In the following for $(x, y) \in (H^2, *)$, we write xy instead of $x * y$.

Example 1.4. *(* [2], *Example 6.1.2). Let $(G, \cdot)$ be a group and R an equivalence relation on G. In $\frac{G}{R}$ consider the hyperaction $\odot$ defined by*

$$\bar{x} \odot \bar{y} = \{\bar{z} \mid z \in \bar{x} \cdot \bar{y}\},$$

where $\bar{x}$ denotes the equivalence class of the element x. Then $(G, \odot)$ is an H_v-group which is not always a hypergroup.

2. Strongly H_v-groups

Consider a special product of elements of an H_v-group H. We introduce a notation as follows. Let $(x_1, x_2, ..., x_n) \in H^n$ and $\mathcal{V}_{x,n} = V(x_1, x_2, ..., x_n)$ be the set of all finite products of $x_1, x_2, ..., x_n$, respectively. Also let $\mathcal{V}_n = \{V(x_1, x_2, ..., x_n) \mid (x_1, x_2, ..., x_n) \in H^n\}$ and $\mathcal{V} = \cup_{n\geq 1}\mathcal{V}_n$. For example

$$\mathcal{V}_{x,2} = \{x_1x_2\}, x = (x_1, x_2)$$

$$\mathcal{V}_{x,3} = \{x_1(x_2x_3), (x_1x_2)x_3\}, x = (x_1, x_2, x_3)$$

$$\mathcal{V}_{x,4} = \{x_1[x_2(x_3x_4)], x_1[(x_2x_3)x_4], [(x_1x_2)x_3]x_4, [x_1(x_2x_3)]x_4, (x_1x_2)(x_3x_4)\},$$

$x = (x_1, x_2, x_3, x_4)$

Also let

$$U_{x,2} = U(x_1, x_2) = \{x_1x_2\} = x_1x_2$$

$$U_{x,3} = U(x_1, x_2, x_3) = \{(x_1x_2)x_3\} = (x_1x_2)x_3$$

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$$U_{x,n+1} = U(x_1, x_2, ..., x_{n+1}) = U_{x,n}x_{n+1}.$$

Let $\mathcal{U}_n = \{U_{x,n} = U(x_1, x_2, ..., x_n) \mid (x_1, x_2, ..., x_n) \in H^n\}$ and $\mathcal{U} = \cup_{n\geq 1}\mathcal{U}_n$. It is clear that $\mathcal{U} \subseteq \mathcal{V}$. Now we define the class of strongly H_v-groups.

Definition 2.1. *Let H be a H_v-group. We say that H is a strongly H_v-group if $p \subseteq U_{x,n}$ for all $p \in \mathcal{V}_{x,n}$ and $n \in \mathbb{N}$.*

Remark 2.2. *Due to the above notation if H is a strongly H_v-group then for all $n, m \in \mathbb{N}$ we have $U_nU_m \subseteq U_{n+m}$.*

Proposition 2.3. *The H_v-group H is strongly if and only if for all $(x, y, z) \in H^3$, $x(yz) \subseteq (xy)z$.*

Proof. Let H be strogly H_v-group, then by above notation it is clear that for all $(x, y, z) \in H^3$, $x(yz) \subseteq (xy)z$. We prove the converse by induction on n. Let $x = (x_1, x_2, ..., x_n) \in H$ and $v \in V_{x,n} = V(x_1, x_2, ..., x_n)$. If $n = 3$, then by assumption we have $x_1(x_2x_3) \subseteq (x_1x_2)x_3 = U_3$. Thus $p \subseteq U_{x,3}$, for all $p \in \mathcal{V}_{x,3}$. Now let the the problem be true for all $k < n$. We have $v = wz$ such that $w \in V(x_1, ...x_r)$ and $z \in V(x_{r+1}, ...x_n)$, where $r < n$. So there exist $U_{x,r}, U_{x,n-r}$ such that $w \subseteq$

$U_{x,r}$ and $z \subseteq U_{x,n-r}$. Thus $v = wz \subseteq U_{x,r}U_{x,n-r} = (U_{x,r-1}x_r)(U_{x,n-r-1}x_n) \subseteq [(U_{x,r-1}x_r)U_{x,n-r-1}]x_n \subseteq U_{x,n-1}x_n = U_{x,n}$. This completes the proof. □

Corollary 2.4. *The H_v-group H is strongly if and only if for all subsets A, B, C of H, $A(BC) \subseteq (AB)C$.*

The main tools to study hyperstructures are the fundamental relations β^*, γ^* and ε^*, which are defined, in H_v-groups, H_v-ring and H_v-vector spaces, as the smallest equivalences so that the quotients would be group, ring and vector space, respectively. These relations were introduced by T. Vougiouklis [6, 7]. A way to find the fundamental class is given by theorems as the following [8, 9, 10].
It is defined the relation β in H_v-group H by setting $x\beta y$ if and only if $\{x, y\} \subset p$, for some $p \in \mathcal{V}_{z,n} = V(z_1, ..., z_n)$ and $n \in \mathbb{N}$ and $(z_1, ..., z_n) \in H^n$.

Theorem 2.5. *([12], Theorem 1) Let H be an H_v-group. Then β^* is the transitive closure of β.*

It has been proved that if H is a hypergroup then $\beta = \beta^*$ (see [5]). But in H_v-group so far not proven that $\beta = \beta^*$. In the following we show that by considering special products to wit in strongly H_v-groups we can prove that $\beta = \beta^*$. First we express the below definitions and explain the theorems such that their proofs in H_v-groups, due to the Proposition 2.3, are similar proving in hypergroups, that we avoid presenting their proofs. These theorems are expressed in [1] completely.

Definition 2.6. *Let A be a subset of a H_v-group H. A is called complete part if the following implication is valid:*

$$\forall n \in \mathbb{N}; \quad \forall p \in \mathcal{V}_{x,n}; \quad p \cap A \neq \emptyset \Rightarrow p \subseteq A.$$

Definition 2.7. *Let A be a non-empty subset of H_v-group H, the intersection of the subsets of H which are complete parts and contain A is called the complete closure of A in H; it will be denoted by $C(A)$.*

Theorem 2.8. *Let A be a non-empty subset of H_v-group H, and let $K_1(A) = A$, $K_{n+1}(A) = \{x \in H \mid \exists m \in \mathbb{N}, \quad \exists p \in \mathcal{V}_{y,m}; \quad x \in p, \quad p \cap K_n(A) \neq \emptyset\}$. Let $K(A) = \cup_{n \geq 1} K_n(A)$. Then $K(A) = C(A)$. (Note that $V_{y,m} = V(y_1, ..., y_m)$).*

Proof. It is necessary to prove that i) $K(A)$ is a complete part and ii) if $A \subseteq B$ and B is complete part then $K(A) \subseteq B$.

i) Let $m \in \mathbb{N}$ and $p \in \mathcal{V}_{y,m}$ and $p \cap K(A) \neq \emptyset$. Then there exists $n \in \mathbb{N}$ such that $p \cap K_n(A) \neq \emptyset$, from which follows $p \subseteq K_{n+1}(A) \subseteq K(A)$.
ii) $A \subseteq K_1(A)$. Suppose $K_n(A) \subseteq B$, this implies that $K_{n+1}(A) \subseteq B$, for every $n \in \mathbb{N}$. □

Theorem 2.9. *The relation* $xKy \Leftrightarrow x \in C(y) = \cup_{n \geq 1} K_n(y)$ *is an equivalence.*

Proof. K is clearly reflexive. Now let xKy and yKz. If P is a comlete part and $z \in P$, then $C(z) \subseteq P$ thus $y \in P$ and consequently $x \in C(y) \subseteq P$ and so xKz. □

Theorem 2.10. *For all* x, y *in* H_v*-group* H*, we have* $xKy \Leftrightarrow x\beta^* y$.

Proof. The proof is similar to the proof of Theorem 57 in [1]. □

Definition 2.11. *Let* H *be a* H_v*-group and* $\varphi_H : H \longrightarrow \dfrac{H}{\beta^*}$ *the canonical projection. The kernel of* φ_H *is called heart(or core) of* H *and denoted by* ω_H*, i.e.* $\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$.

Theorem 2.12. ω_H *is the smallest subhypergroup of* H *that is complete part.*

Remark 2.13. *For all* $z \in \omega_H$ *we have* $\omega_H = C(z)$*, since*

$$x \in C(z) \Leftrightarrow xKz \Leftrightarrow x\beta^* z \Leftrightarrow \bar{x} = \bar{y} \Leftrightarrow \varphi_H(x) = \varphi_H(z) = 1 \Leftrightarrow x \in \omega_H.$$

Theorem 2.14. *If* B *is a non-empty subset of* H_v*-group* H*, then* $\varphi_H^{-1}\varphi(B) = \omega_H B = B\omega_H = C(B) = \cup_{b \in B} C(b)$.

Proof. The structure of the proof is like to the proof of Theorem 66 in [1]. □

Corollary 2.15. *By Theorems 2.9, 2.10, 2.14 for all* $x \in H$ *we obtain*

$$\beta^*(x) = \{y \mid y\beta^* x\} = \{y \mid yKx\} = \{y \mid y \in C(x)\} = \{y \mid y \in x\omega_H\}$$

Corollary 2.16. *For all* $(x, y) \in H^2$*, we have* $x\beta^* y \Leftrightarrow xKy \Leftrightarrow y \in C(x) = x\omega_H$.

Now let $P(z)$ be as follows:

$$P(z) = \{U_{x,n} \in \mathcal{U}_n \mid z \in U_{x,n}, n \in \mathbb{N}\}.$$

We set $M(z)$ is the union of $P(z)$ that is $M(z) = \cup_{U_{x,n} \in P(z)} U_{x,n}$.

Theorem 2.17. *Let* H *be a strongly* H_v*-group and* $z \in H$*. Then* $M(z)$ *is a complete part.*

Proof. Let $p \in V_{x,n}$ be a finite products of $x_1, ..., x_n$ and $p \cap M(z) \neq \emptyset$. Assume $a \in p \cap M(z)$, so there exists $U_{y,m} \in P(z)$ such that $a \in U_{y,m}, z \in U_{y,m}$. Because H is strongly H_v-group there exists $U_{x,n} \in \mathcal{U}$ such that $p \subseteq U_{x,n}$. By reproductivity property, there exists $(w, b) \in H^2$ such that $x_n \in wz$ and $z \in ab$. Therefore

$$\begin{aligned} p &\subseteq U_{x,n} = U_{x,n-1}x_n \subseteq U_{x,n-1}(wz) \subseteq (U_{x,n-1}w)z \subseteq (U_{x,n-1}w)(ab) \\ &\subseteq (U_{x,n-1}w)(U_{y,m}b) \subseteq [(U_{x,n-1}w)U_{y,m}]b \\ &\subseteq U_{t,n+m}b = U_{t,k}b. \end{aligned}$$

Also

$$\begin{aligned} z &\in ab \subseteq pb \subseteq U_n b \subseteq [U_{x,n-1}x_n]b \subseteq [U_{x,n-1}(wz)]b \\ &\subseteq [(U_{x,n-1}w)z]b \subseteq [(U_{x,n-1}w)U_{y,m}]b \\ &\subseteq U_{t,n+m}b = U_{t,k}b. \end{aligned}$$

So $U_{t,k}b \in P(z)$ and $p \subseteq U_{t,k}b \subseteq M(z)$ and this completes the proof. □

Corollary 2.18. *For all $z \in \omega_H$, we have $M(z) = \omega_H$.*

Proof. Let $U_{x,n} \in P(z)$, so $z \in U_{x,n} \cap \omega_H$. Since ω_H is complete thus $U_{x,n} \subseteq \omega_H$ and $M(z) = \cup_{U_{x,n} \in P(z)} U_{x,n} \subseteq \omega_H$. On the other hand since $M(z)$ is complete, $z \in M(z)$ and $C(z)$ is the smallest complete part that contains z so $C(z) \subseteq M(z)$ and by 2.13 $\omega_H = C(z) \subseteq M(z)$. Therefore $M(z) = \omega_H$ for all $z \in \omega_H$. □

Theorem 2.19. *If H is a strongly H_v-group, then $\beta = \beta^*$.*

Proof. We know that $\beta \subseteq \beta^*$. Let $x, y \in H$ and $x\beta^* y$. Then by Corollary 2.16 we have

$$x\beta^* y \Leftrightarrow xKy \Leftrightarrow y \in C(x) = x\omega_H,$$

Therefore $\exists v, w \in \omega_H$ such that $x \in xv$ and $y \in xw$. According to Corollary 2.18 there exist $U_{t,n} \in P(v)$ such that $w \in U_{t,n}$, where $U_{t,n} = U_{t,n-1}t_n$. Also $v \in U_{t,n}$ and thus $v\beta w$. Therefore $\{x, y\} \subseteq x(U_{t,n-1}x_n)$. Thus $x\beta y$ and $\beta^* \subseteq \beta$. □

3. THE θ-HYPEROPERATION

In this section we investigate a special class of hyperstructures called θ-hyperoperation introduced by Vougiouklis in [11]. By using θ-hyperoperation we obtain a H_v-structure. We investigate θ-hyperoperation by adding strongly H_v-group condition. General definition of θ-hyperoperation is as follows that we can see it in [11, 12].

Definition 3.1. *Let H be a set equipped with n operations (or hyperoperations) $\otimes_1, \otimes_2, ..., \otimes_n$ and a map (or multivalued map) $f\colon H \longrightarrow H$ (or $f\colon H \longrightarrow \mathcal{P}^*(H)$). Then n hyperoperations $\theta_1, \theta_2, ..., \theta_n$ on H can be defined, called θ-hyperoperations by putting*

$$x\theta_i y = \{f(x) \otimes_i y, x \otimes_i f(y)\}, \quad \forall (x, y) \in H^2, i \in \{1, 2, ..., n\}$$

or, in case where $\otimes_i$ is hyperoperation or f is multivalued map, we have

$$x\theta_i y = (f(x) \otimes_i y) \cup (x \otimes_i f(y)), \quad \forall (x, y) \in H^2, i \in \{1, 2, ..., n\}.$$

If $\otimes_i$ is associative then θ_i is weak associative.

Similarly one can use several maps f, instead than only one. We express a special case that way let $(H, \cdot)$ be a hypergroupoid and $f\colon H \longrightarrow H$, be a map on H. It is defined in this case $\bar{\theta}$-hyperoperation as follows:

$$x \ \bar{\theta} \ y = (x \cdot y) \cup (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall (x, y) \in H^2$$

Proposition 3.2. *Let $(H, \cdot)$ be an H_v-group, then $(H, \bar{\theta})$ is an H_v-group.*

Proof. According to [11, 12], $(H, \bar{\theta})$ is weak associative. Since $(H, \cdot)$ is reproductive so for all $x \in H$ we have $x \cdot H = H = H \cdot x$. Therefore

$$x \ \bar{\theta} \ H = \bigcup_{h \in H} x \ \bar{\theta} \ h = \bigcup_{h \in H} \{x \cdot h, f(x) \cdot h, x \cdot f(h)\} = H$$

and $(H, \bar{\theta})$ is reproductive. Thus $(H, \bar{\theta})$ is an H_v-group. □

Proposition 3.3. *Let $(H, \cdot)$ be a strongly H_v-group and f be good homomorphism and projection ($f^2 = f$), then $(H, \bar{\theta})$ is a strongly H_v-group.*

Proof. By proposition 3.2, it is enough to show that H_v-group $(H, \bar{\theta})$ is strongly, i.e. $x \ \bar{\theta} \ (y \ \bar{\theta} \ z) \subseteq (x \ \bar{\theta} \ y) \ \bar{\theta} \ z$, for all $(x, y, z) \in H^3$. We have

$$x \ \bar{\theta} \ (y \ \bar{\theta} \ z) = \cup_{a \in y \bar{\theta} z} x \ \bar{\theta} \ a = \cup_{a \in y \bar{\theta} z} [x \cdot a \cup f(x) \cdot a \cup x \cdot f(a)]$$

such that $a \in (y \cdot z) \cup (f(y) \cdot z) \cup (y \cdot f(z))$. Therefore

$$\begin{aligned} x \,\bar{\theta}\, (y \,\bar{\theta}\, z) &\subseteq [x \cdot (y \cdot z)] \cup [x \cdot (f(y) \cdot z)] \cup [x \cdot (y \cdot f(z))] \cup \\ &\quad [f(x) \cdot (y \cdot z)] \cup [f(x) \cdot (f(y) \cdot z)] \cup [f(x) \cdot (y \cdot f(z))] \cup \\ &\quad [x \cdot f(y \cdot z)] \cup [x \cdot f(f(y) \cdot z)] \cup [x \cdot f(y \cdot f(z))] \\ &\subseteq [(x \cdot y) \cdot z] \cup [(x \cdot f(y)) \cdot z] \cup [(x \cdot y) \cdot f(z)] \cup \\ &\quad [(f(x) \cdot y) \cdot z] \cup [(f(x) \cdot f(y)) \cdot z] \cup [(f(x) \cdot y) \cdot f(z)] \cup \\ &\quad [(x \cdot f(y)) \cdot f(z)] \cup [(x \cdot f(y)) \cdot f(z)] \cup [(x \cdot f(y)) \cdot f(z)] \\ &\subseteq \cup_{b \in x \bar{\theta} y} [b \cdot z, f(b) \cdot z, b \cdot f(z)] \\ &= \cup_{b \in x \bar{\theta} y} b \,\bar{\theta}\, z = (x \,\bar{\theta}\, y) \,\bar{\theta}\, z. \end{aligned}$$

Thus $(H, \bar{\theta})$ is a strongly H_v-group. □

4. aknowledgement

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