LIFTING CONTINUITY PROPERTIES OF AGGREGATION FUNCTIONS TO THEIR SUPER- AND SUB-ADDITIVE TRANSFORMATIONS

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Abstract. We investigate possible extensions of various types of continuity of aggregation functions to their super- and sub-additive transformations. More specifically, we examine lifts of classical, uniform, Lipschitz and Hölder continuities and differentiability. The classical, uniform, and Lipschitz continuities turn out to be preserved by super- and sub-additive transformations (albeit for uniform continuity and the super-additive case we prove it only in dimension one), while the Hölder continuity and differentiability are not.

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1. Introduction

In the last few years, aggregation functions have been a prolific and rewarding topic of investigation in pure and applied mathematics. A large number of results

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on these functions have been proved and collected in monographs, such as [1] and [2]. An extensively studied sub-topic in aggregation functions appear to be their transformations, among which the super- and sub-additive ones (introduced in [3]) have received particular attention; see e.g. [4, 5, 6, 7, 8, 10, 11, 12].

Preceding a formal definition to be given in the next section, the super- and sub-additive transformations of an aggregation function A may be viewed as the smallest and largest (in the point-wise ordering) functions that are, respectively, super-additive and dominating A, and sub-additive and dominated by A; informally, one may speak about the super-additive and sub-additive 'envelope' of A.

This paper extends some of the ideas initiated by [4] and [9]. In the two papers, the question of 'lifting' continuity of an aggregation function to its super- and sub-additive transformations was stated and solved in the one-dimensional and multi-dimensional case, respectively. As a follow-up we consider the question of lifting different types of continuity of an aggregation function to its super- and sub-additive transformations. In more detail, we are interested in Lipschitz, Hölder and uniform continuity and also in the problem of differentiability inheritance. Interestingly, it turns out that the classical, uniform, and Lipschitz continuities are preserved by super- and sub-additive transformations (however, for uniform continuity and super-additivity we prove this only in dimension one), while differentiability and the Hölder continuity is not.

Our paper is structured as follows. In Section 2 we summarize some basic concepts of the theory of aggregation functions, introduce their super- and sub-additive transformations and list some useful auxiliary results. In Section 3 we provide examples of differentiable aggregation functions whose super- and sub-additive transformations are not differentiable, demonstrating thus the fact that differentiability is not inherited by super- and sub-additive transformations in general. In Section 4 we construct examples of Hölder continuous aggregation functions whose super- nor sub-additive transformations are no longer Hölder continuous, showing that Hölder continuity does not lift, either. In contrast with these negative findings, in Section 5 we prove that Lipschitz continuity is preserved by both the super- and sub-additive transformations. In Section 6 we give a proof of uniform continuity inheritance for sub-additive transformations, leaving the case of super-additive transformation

open except in dimension one. The concluding Section 7 contains a few remarks
and suggestions for further research.

2. Preliminaries

Throughout, for simplicity we will let D denote the interval $[0, \infty[$ and we will
let the domains of all our functions be D^n for some $n \geq 1$. For the purpose of this
paper, an *aggregation function* will be an arbitrary mapping $A : D^n \to D$ that is
increasing in every coordinate and such that $A(0) = A(0, \ldots, 0) = 0$. At this point
we recall that such a function A is *super-additive* and *sub-additive*, respectively, if
$A(u + v) \geq A(u) + A(v)$, resp. $A(u + v) \leq A(u) + A(v)$ for every u, $v \in D^n$ such
that $u + v \in D^n$.

For an aggregation function A, its *super-additive* and *sub-additive transforma-
tions*, A^* and A_*, are defined for every $x \in D^n$ by

$$(1) \qquad A^*(x) = sup \left\{ \sum_{j=1}^{k} A(x^{(j)}) : x^{(j)} \in D^n, \ \sum_{j=1}^{k} x^{(j)} = x \right\}$$

and

$$(2) \qquad A_*(x) = inf \left\{ \sum_{j=1}^{k} A(x^{(j)}) \ ; \ \ x^{(j)} \in D^n, \ \sum_{j=1}^{k} x^{(j)} = x \right\}.$$

As the terms suggest, A^* is super-additive while A_* is sub-additive. We also note
that A^* may not be defined. However, by a result of [4], if there is a point $\bar{\mathbf{x}} \in D^n$
such that $A^*(\bar{\mathbf{x}}) = \infty$ then one necessarily has $A^*(\mathbf{x}) = \infty$ for all $\mathbf{x} \in D^n$ such that
$\mathbf{x} \neq \mathbf{0}$. If this does not happen we will simply say that A^* is well-defined.

The super- and sub-additive transformations of a given aggregation function A
may also be viewed as the smallest and largest functions (in the ordering $f \leq g$ if
$f(x) \leq g(x)$ for every $x \in D^n$) that are super-additive and sub-additive, respec-
tively, and dominate A, resp. are dominated by A. Informally, A^* is the 'super-
additive envelope' of A while A_* is the 'sub-additive envelope' of A.

The following result, proved in [8], is often useful in determining super- and sub-
additive transformations. If $A \colon D^n \to D$ is an aggregation function, let ∇^A be
the n-dimensional vector with i-th component $(\nabla^A)_i$ equal to $\limsup_{t \to 0^+} A(te_i)/t$,
where e_i is the i-th unit vector, $i \in \{1, ..., n\}$. Similarly, we let ∇_A denote the

n-dimensional vector with i-th component $(\nabla_A)_i = \liminf_{t \to 0^+} A(te_i)/t$ for $i \in \{1, ..., n\}$.

Lemma 1. [8] *If* $A\colon D^n \to D$ *is an arbitrary aggregation function, then* $A^*(x) \geq \nabla^A \cdot x$ *and* $A_*(x) \leq \nabla_A \cdot x$ *for every* $x \in D^n$. $\qquad \square$

We continue with recalling a few concepts related to continuity. For any fixed $\alpha \in\,]0, 1[$, a function $f\colon D^n \to D$ is α-*Hölder continuous* if there exists a constant $c > 0$ such that $|f(x) - f(y)| \leq c \cdot ||x - y||^\alpha$ for every $x, y \in D^n$, where $|| \cdot ||$ denotes the standard Euclidean norm. In the limit case when $\alpha = 1$ the definition still makes sense and leads to the concept of *Lipschitz continuity*.

The previous definitions did not require any relationship between the points (vectors) $x, y \in D^n$. In our later considerations, however, it will be handy to consider the pairs satisfying $x \geq y$, meaning that the inequality is valid for all the corresponding coordinates of the two points. In particular, the fact that $x \in D^n$ is equivalent to writing that $x \geq 0$, the zero vector of dimension n. We will say that a function $f\colon D^n \to D$ is *ordered* α-*Hölder continuous* if there is some $d > 0$ such that $|f(x) - f(y)| \leq d \cdot ||x - y||^\alpha$ for every $x, y \in D^n$ such that $x \geq y$. The concept of *ordered Lipschitz continuity* is defined analogously, letting $\alpha = 1$.

As one would expect, the concepts of Hölder and Lipschitz continuity are equivalent to their ordered versions, which we will show next.

Lemma 2. *A function* $f\colon D^n \to D$ *is* α-*Hölder (or Lipschitz) continuous if and only if it is ordered* α-*Hölder (or Lipschitz) continuous.*

Proof. It is clearly sufficient to prove just the direction that ordered continuity implies the absolute one. Let f be ordered α-Hölder continuous (or Lipschitz continuous for $\alpha = 1$) with multiplicative constant d, and let $x, y \in D^n$ be arbitrary. Since $x, y \geq x \wedge y$ and the latter point is in D^n, applying ordered continuity one obtains

$$|f(x) - f(x \wedge y)| \leq d \cdot ||x - x \wedge y||^\alpha$$

with a similar inequality obtained if y is interchanged with x. But then,

$$
\begin{aligned}
|f(x) - f(y)| \quad &\leq |f(x) - f(x \wedge y)| + |f(y) - f(x \wedge y)| \\
&\leq d \cdot ||x - x \wedge y||^\alpha + d \cdot ||y - x \wedge y||^\alpha \\
&\leq 2d \cdot ||x - y||^\alpha \ .
\end{aligned}
$$

This proves that f is α-Hölder (or Lipschitz) continuous in the 'unordered' version.
$\square$

3. Differentiability

Among the concepts related to continuity we are interested in this article, differentiability is the strongest. We begin by showing that differentiability does not carry over from an aggregation function onto its super- and sub-additive transformations in general. For super-additive transformations we show this on the aggregation function $A \colon D^n \to D$ given by

$$(3) \qquad\qquad A(x) = ||x||^3 - ||x||^2 + ||x||$$

where, for a change $||\cdot||$ will, and for the purpose of this proof only, denote the *sum* of the coordinates of the point x.

We will often use the symbol $1 \in D^n$ to denote the all-one vector, so that for any $x \in D$ the dot product $1{\cdot}x$ is simply the sum of the coordinates of x.

It is evident that $(\nabla^A) = 1$, the all-one vector of dimension n. Also, a straightforward calculation shows that for every $x \in D^n$ such that $||x|| \leq 1$ one has

$$(4) \qquad\qquad A(x) = ||x||^3 - ||x||^2 + ||x|| \leq ||x|| \ .$$

Thus, (4) shows that $A(x)$ is dominated for $||x|| \leq 1$ by the additive (and hence super-additive function $||x|| = 1 \cdot x$, so that $A^*(x) \leq 1 \cdot x$ for $||x|| \leq 1$. But Lemma 1 implies that $A^*(x) \geq (\nabla^A){\cdot}x = 1 \cdot x$, so that at least in the set $\{x \in D^n \colon ||x|| \leq 1\}$ we conclude that $A^*(x) = 1 \cdot x$.

To prove our claim we will show that A^* does not have a partial derivative at any point $e_i \in D^n$ representing the i-th unit vector. Indeed, for an arbitrary $i \in \{1, 2, \ldots, n\}$ and for every $t \geq 0$ it is obvious that $A^*(te_i) = t$ if $t \in [0, 1]$ and $A^*(te_i) \geq A(te_i) = t^3 - t^2 + t$ for $t \in [1, \infty[$. Noting that $A^*(e_i) = 1$, one obtains

$$\lim_{t \to 1^-} \frac{A^*(te_i) - 1}{t - 1} = 1 \ \ while \ \ \liminf_{t \to 1^+} \frac{A^*(te_i) - 1}{t - 1} \geq \lim_{t \to 1^+} \frac{A(te_i) - 1}{t - 1} = 2$$

which implies that the partial derivative of A^* does not exist at any e_i.

Based on this we can easily construct a differentiability counterexample for sub-additive transformations by observing that the above function A is a composition $f \circ g$ of the function $g(x) = ||x||$ with the monotonous (and hence invertible) function

$f(t) = t^3 - t^2 + t$ in the interval $[0, \infty[$. Taking $B = f^{-1} \circ g$ we can mimic the previous calculations by showing that $B_*(x) = (\nabla_B) \cdot x = 1 \cdot x$; note that, for example, in this setting the inequality (4) is equivalent to $f(t) \leq t$ for $t \in [0, 1]$, which is in turn equivalent to $t \leq f^{-1}(t)$ for $t \in [0, 1]$. The fact that B^* has no derivative and any unit vector point e_i follows by a similar calculation as above.

4. HÖLDER CONTINUITY

In this section we will show that, in general, Hölder continuity does not extend from an aggregation function onto its super- and sub-additive transformation. We will demonstrate this on an example for super-additivity, the case of sub-additivity being analogous. We precede the analysis by a series of auxiliary results.

Lemma 3. *Let a, b, c be positive real numbers such that $a \geq b$ and $c \geq a - b$. Then, $c^\alpha \geq a^\alpha - b^\alpha$ for every $\alpha \in]0, 1]$.*

Proof. We only need to consider $\alpha < 1$, and it is clearly sufficient to prove that $(a - b)^\alpha \geq a^\alpha - b^\alpha$. Letting $a = b + t$ for this is equivalent to showing that the function $h(t) = t^\alpha + b^\alpha - (t + b)^\alpha$ is non-negative for any $t \geq 0$. But it is easy to see that for $0 < \alpha < 1$ and $b > 0$ the derivative $f'(t) = \alpha t^{\alpha - 1} - \alpha(t + b)^{\alpha - 1}$ is non-negative and as $f(0) = 0$ we conclude that f is non-decreasing for $t \geq 0$. This proves our inequality.   □

Lemma 4. *For arbitrary $x, y \in D^n$ such that $x \geq y$ and for any $\alpha \in]0, 1]$ it holds that $(1 + ||x||)^\alpha - (1 + ||y||)^\alpha \leq ||x - y||^\alpha$.*

Proof. Let $a = 1 + ||x||$, $b = 1 + ||y||$ and $c = ||x - y||$. From $||x|| = ||x - y + y|| \leq ||x - y|| + ||y||$ we obtain $||x|| - ||y|| \leq ||x - y||$ and so $c \geq a - b$, with $a \geq b \geq 0$. The result now follows from Lemma 3.   □

Lemma 5. *For every $x \in D$ one has $(1 + ||x||)^\alpha - 1 \leq \alpha(1 \cdot x)$.*

Proof. Consider the function $h(x) = 1 + 1 \leq \alpha(1 \cdot x) - (1 + ||x||)^\alpha$, with $h(0) = 0$. For its partial derivatives we obtain $\partial h / \partial x_i = \alpha - \alpha(1 + ||x||)^{\alpha - 1} x_i ||x||^{-1}$. It is easy to check that the inequality $\partial h / \partial x_i > 0$ is equivalent with $(1 + ||x||)^{1 - \alpha} ||x|| > x_i$, which is obviously valid for every $x \in D$ such that $x \neq 0$. Therefore $h(x) \geq 0$, establishing our claim.   □

We are now in position to present the announced examples, and we begin with an example of an α-Hölder continuous aggregation function A such that its super-additive transformation A^* is not β-Hölder continuous for any $\beta \in]0, 1[$. Indeed, for an arbitrary $\alpha \in]0, 1[$ consider the function $A_\alpha \colon D^n \to D$ given by

$$(5) \qquad\qquad A_\alpha(x) = (1 + ||x||)^\alpha - 1 .$$

By Lemmas 4 and 2 the function A_α is α-Hölder continuous; in what follows we determine A^* with the help of our auxiliary results.

For $1 \leq i \leq n$ the i-th coordinate of the vector $(\nabla^{A_\alpha})_i$ introduced above is equal to $\limsup_{t \to 0^+} ((1+t)^\alpha - 1)/t$, which is equal to the one-sided (right) derivative of the function $h(t) = (1+t)^\alpha$ at zero. It follows that $(\nabla^{A_\alpha})_i = \alpha$ for every $i \in \{1, \ldots, n\}$, so that $\nabla^{A_\alpha} = \alpha 1$. Applying Lemma 1 we obtain $A_\alpha^* \geq \alpha 1 \cdot x$. But the function $h \colon D^n \to D \colon x \mapsto \alpha 1 \cdot x$ is linear and hence also super-additive. Further, by Lemma 5, h dominates A_α, and so $A_\alpha^* \leq h$ (by the 'envelope' property of super-additive transformations). We conclude that $A_\alpha^* = h$.

It remains to show that A_α^* is not β-Hölder continuous for any $\beta \in]0, 1[$. Namely, if this was the case, then for every $x, y \in D^n$ one would have $A_\alpha^*(x) - A_\alpha^*(y) = \alpha 1 \cdot (x - y) \leq ||x - y||^\beta$, which is clearly absurd—consider, for example, $x = x e_1$ for arbitrarily large $x > 0$ and $y = 0$.

Now we will construct a similar example to the previous one which will show that Hölder continuity, in general, is not preserved by sub-additive transformations. For $\beta > 1$, let us consider the aggregation function $B^\beta \colon D^n \to D$ given by

$$B^\beta(\mathbf{x}) = A_{1/\beta}(\mathbf{x}) = (1 + ||\mathbf{x}||)^{1/\beta} - 1.$$

By the previous arguments, B^β is α-Hölder continuous, where $\alpha = 1/\beta \in]0, 1[$. Also, it can easily be showed that the vector ∇_{B^β} is equal to $\mathbf{1}/\beta = \alpha \mathbf{1}$. Applying Lemma 1 we obtain that $B_*^\beta \leq \alpha \mathbf{1} \cdot \mathbf{x}$. Observe that $\alpha \mathbf{1} \cdot \mathbf{x}$ is dominated by B^β and thus, in fact, $B_*^\beta = \alpha \mathbf{1} \cdot \mathbf{x}$. Now, if B_*^β was γ-Hölder continuous, the inequality $B_*^\beta(\mathbf{x}) - B_*^\beta(\mathbf{y}) = \alpha \mathbf{1} \cdot (\mathbf{x} - \mathbf{y}) \leq ||\mathbf{x} - \mathbf{y}||^\gamma$ would hold for all $\mathbf{x} \geq \mathbf{y}$. But this is not the case, for the same reasons as given earlier for A_α^*. $\qquad\square$

5. Lipschitz continuity

In this section we show that, in contrast with α-Hölder continuity, the Lipschitz continuity (which is the 'limit' case of Hölder's for $\alpha = 1$ does carry over from an aggregation function to its super- and sub-additive transformaous. We again begin with a lemma that may be of independent interest.

Lemma 6. *For any $k \geq 1$, let $u^{(1)}, u^{(2)}, \ldots, u^{(k)}$ be points in D^n. Then,*

$$(6) \qquad \sum_{j=1}^{k} \left\| u^{(j)} \right\| \leq \sqrt{n} \cdot \left\| \sum_{j=1}^{k} u^{(j)} \right\| .$$

Proof. Clearly, since $u^{(j)} \geq 0$, our vectors satisfy the inequality $\left\| u^{(j)} \right\| \leq 1 \cdot u^{(j)}$, and hence also

$$(7) \qquad \sum_{j=1}^{k} \left\| u^{(j)} \right\| \leq \sum_{j=1}^{k} 1 \cdot u^{(j)} = 1 \cdot v \ , \quad where \ \ v = \sum_{j=1}^{k} u^{(j)} \ .$$

Letting $v = (v_1, v_2, \ldots, v_n)$, so that $1 \cdot v = \sum_{i=1}^{n} v_i$, it now remains to use the well-known inequality

$$(8) \qquad \sum_{i=1}^{n} v_i \leq \sqrt{n} \cdot \sqrt{\sum_{i=1}^{n} v_i^2}$$

in combination with (7) to obtain the result, as the right-hand side of (8) is equal to the right-hand side of (6). $\square$

We would like to point out that the multiplicative constant $\sqrt{n}$ on the right-hand side of the the inequality (6) depends only on the dimension of the points and *not* on their quantity k.

With the help of this result we prove that Lipschitz continuity carries over to super-additive transformations.

Theorem 1. *Let $A \colon D^n \to D$ be an arbitrary aggregation function. If A is Lipschitz continuous and A^* is well-defined, then A^* is also Lipschitz continuous.*

Proof. Let $\mathbf{x}, \mathbf{y} \in D^n$ be any pair of points such that $\mathbf{x} \geq \mathbf{y}$. Take an arbitrary $\varepsilon > 0$. By the definition of a super-additive transformation, there exists some $k \geq 1$

and a sequence $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ of points in D^n summing to $\mathbf{x}$ such that

$$(9) \qquad A^*(\mathbf{x}) - \sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \le \varepsilon \;.$$

Obviously, there also exists a sequence $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(k)}$ of points in D^n summing to $\mathbf{y}$ and such that $\mathbf{x}^{(j)} \ge \mathbf{y}^{(j)}$ for every $j \in \{1, 2, \ldots, n\}$. Moreover, clearly

$$A^*(\mathbf{y}) \ge \sum_{j=1}^{k} A(\mathbf{y}^{(j)}) \;.$$

Also, by the assumed Lipschitz (and hence also ordered Lipschitz) continuity, there exists a constant $c > 0$ such that

$$A(\mathbf{x}^{(j)}) - A(\mathbf{y}^{(j)}) \le c \left\| \mathbf{x}^{(j)} - \mathbf{y}^{(j)} \right\|$$

for every $j \in \{1, 2, \ldots, k\}$. Taking into account all the above facts and using the inequality (6) from Lemma 6 applied to $\mathbf{u}^{(j)} = \mathbf{x}^{(j)} - \mathbf{y}^{(j)}$ we obtain

$$A^*(\mathbf{x}) - A^*(\mathbf{y}) \le \varepsilon + \sum_{j=1}^{k} A(\mathbf{x}^{(j)}) - \sum_{j=1}^{k} A(\mathbf{y}^{(j)}) \le \varepsilon + \sum_{j=1}^{k} c \left\| \mathbf{x}^{(j)} - \mathbf{y}^{(j)} \right\|$$

$$\le \varepsilon + c\sqrt{n} \left\| \sum_{j=1}^{k} \left(\mathbf{x}^{(j)} - \mathbf{y}^{(j)} \right) \right\| = \varepsilon + c\sqrt{n} \left\| \mathbf{x} - \mathbf{y} \right\| \;.$$

Since for the given $\mathbf{x} \ge \mathbf{y}$ our $\varepsilon > 0$ was chosen arbitrarily, the above chain of inequalities implies that $A^*(\mathbf{x}) - A^*(\mathbf{y}) \le c\sqrt{n} \left\| \mathbf{x} - \mathbf{y} \right\|$. This means that A^* is ordered Lipschitz continuous and hence also Lipsichtz continuous, by Lemma 2. □

A similar method applies to proving an analogous result for sub-additive transformations.

Theorem 2. *Let $A \colon D^n \to D$ be an arbitrary aggregation function. If A is Lipschitz continuous, then so is A_*.*

Proof. The argument is analogous to what has been used in the proof of the previous theorem. Let $\mathbf{x} \ge \mathbf{y}$. By the definition of infimum, for every $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ and a sequence $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(k)}$ of points in D^n summing to $\mathbf{y}$ such that

$$\sum_{j=1}^{k} A(\mathbf{y}^{(j)}) - A_*(\mathbf{y}) \le \varepsilon \;.$$

It is also easy to show that there is a sequence $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ of points in D^n summing to $\mathbf{x}$ and satisfying $\mathbf{x}^{(j)} \geq \mathbf{y}^{(j)}$ for $j \in \{1, 2, \ldots, k\}$, such that

$$A_*(\mathbf{x}) \leq \sum_{j=1}^{k} A(\mathbf{x}^{(j)}).$$

By the assumption that A is Lipschitz continuous, there exists some $c > 0$ such that

$$A(\mathbf{x}^{(j)}) - A(\mathbf{y}^{(j)}) \leq c \left\| \mathbf{x}^{(j)} - \mathbf{y}^{(j)} \right\|.$$

The above inequalities imply:

$$A_*(\mathbf{x}) - A_*(\mathbf{y}) \leq \varepsilon + \sum_{j=1}^{k} A(\mathbf{x}^{(j)}) - \sum_{j=1}^{k} A(\mathbf{y}^{(j)}) \leq \varepsilon + \sum_{j=1}^{k} c \left\| \mathbf{x}^{(j)} - \mathbf{y}^{(j)} \right\|$$

$$\leq \varepsilon + c\sqrt{n} \left\| \sum_{j=1}^{k} \left(\mathbf{x}^{(j)} - \mathbf{y}^{(j)} \right) \right\| = \varepsilon + c\sqrt{n} \left\| \mathbf{x} - \mathbf{y} \right\|.$$

By the same reasoning as in the previous proof we conclude that the function A_* is Lipschitz continuous. $\square$

6. Uniform continuity

In [9] it was proved that continuity of an aggregation function extends to its super- and sub-additive transformations. The question that remains to be answered in this context is if a similar result holds also for extension of *uniform* continuity. We begin with sub-additive transformations, as the corresponding result assumes only continuity of the aggregation function at the origin (and its proof simplifies the approach used in [9]).

Theorem 3. *Let $A \colon D^n \to D$ be an aggregation function. If A is continuous at the origin, then A_* is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be arbitrary; we need to show that there is some $\delta > 0$ such that for any $x, y \in D^n$ with $x \geq y$, the inequality $\|x - y\| < \delta$ implies $A_*(x) - A_*(y) < \varepsilon$.

Using continuity of A at the origin it follows that for $\varepsilon/2$ there exists a $\delta > 0$ with the property that, given any $z \in D^n$ with $\|z\| < \delta$, one has $A(z) < \varepsilon/2$.

Let $x, y \in D^n$ be any pair of points such that $x \geq y$ and $\|x - y\| < \delta$. By the definition of A_*, for $\varepsilon/2$ there exists a sequence $(y^{(j)})$ of points in D^n, $1 \leq j \leq k$ for some $k \geq 1$, such that $A_*(y) + \varepsilon/2 > \sum_{j=1}^{k} A(y^{(j)})$, or equivalently,

$$(10) \qquad -A_*(y) < \varepsilon/2 - \sum_{j=1}^{k} A(y^{(j)}) \ .$$

Define now $x^{(j)} = y^{(j)}$ for $j \in \{1, \ldots, k\}$ and $x^{(0)} = x - y \in D^n$. Clearly, $\sum_{j=0}^{k} x^{(j)} = x$, and

$$(11) \qquad A_*(x) \leq \sum_{j=0}^{k} A(x^{(j)}) \ .$$

Letting $z = x^{(0)}$ we have $\|x^{(0)}\| < \delta$, and by the first part of the proof we also have $A(x^{(0)}) < \varepsilon/2$. This together with (10), (11) and $A(x^{(j)}) = A(y^{(j)})$ for $1 \leq j \leq k$ implies

$$A_*(x) - A_*(y) < \varepsilon/2 + \sum_{j=0}^{k} A(x^{(j)}) - \sum_{j=1}^{k} A(y^{(j)}) = \varepsilon/2 + A(x^{(0)}) < \varepsilon \ ,$$

completing the proof. $\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \square$

The situation for super-additive transformations appears to be more complex. We first offer an auxiliary result, in which the norm $\|x\|$ of a point $x \in D^n$ is the sum of its coordinates.

Proposition 1. *Let $A \colon D^n \to D$ be a uniformly continuous aggregation function such that A^* is well-defined. Then, $\sup\{A(x)/\|x\| : x \in D^n, \ x \neq 0\}$ is finite.*

Proof. The assumption that A^* is well-defined implies that, for any positive constant c, the quantity $\sup\{A(x)/\|x\| \, ; \ x \in D^n, \ 0 < \|x\| \leq c\}$ is a positive real number. All that remains to be shown is that this extends to points $x \in D^n$ of arbitrarily large norm.

Take an arbitrary but fixed $\varepsilon > 0$. By uniform continuity of A there exists a $\delta > 0$ such that for arbitrary $x \in D^n \backslash \{0\}$ and for any $y \in D^n$, $y \leq x$ with $\|x - y\| \leq \delta$ one has $A(x) - A(y) < \varepsilon$. For such an x let t be the unique positive integer such that $(t-1)\delta < \|x\| \leq t\delta$. By the observation made above we may assume that, say, $t \geq 3$.

For $j = 1, \ldots, t$ define $y^{(j)} = b_j x$ for $b_j = (j-1)\delta/\|x\|$; note that $\|y^{(j)}\| = (j-1)\delta$ and, in particular, $y^{(1)} = 0$. Since $\|y^{(j+1)} - y^{(j)}\| = \delta$ for $1 \le j \le t-1$, by uniform continuity of A we have $A(y^{(j+1)}) - A(y^{(j)}) < \varepsilon$, and, by the same token, also $A(x) - A(y^{(t)}) < \varepsilon$. Summing up these t inequalities gives $A(x) < t\varepsilon$. Further, from $(t-1)\delta < \|x\|$ one obtains $t < 1 + \|x\|/\delta$, which together with $A(x) < t\varepsilon$ gives $A(x) < (\varepsilon/\delta)\|x\| + \varepsilon$. But then, $A(x)/\|x\| < (\varepsilon/\delta) + \varepsilon/\|x\|$. As ε (and hence δ) were constants in the above arguments while x was arbitrary, the last inequality shows that the ratio $A(x)/\|x\|$ is bounded for any $x \in D^n$ with a sufficiently large norm. $\square$

For lifting uniform continuity to super-additive transformations we just consider the one-dimensional case.

Theorem 4. *Let* $A\colon D \to D$ *be a uniformly continuous aggregation function. If* A^* *exists, then it is uniformly continuous.*

Proof. Since we are going to deal with functions of one variable we will use lower-case subscripts to denote sequences of real numbers. Suppose that A is uniformly continuous on D. To show that A^* is also uniformly continuous on D we need to establish, for every $\varepsilon > 0$, the existence of a $\delta > 0$ with the property that whenever $x, y \in D$ are such that $x > y$ and $x - y < \delta$, one has $A^*(x) - A^*(y) < \varepsilon$.

Thus, let $\varepsilon > 0$ be given. Uniform continuity of A on D implies that to the value $\varepsilon/2$ there exists a $\delta' > 0$ such that for any $x, y \in D$ with $x > y$ one has $A(x) - A(y) < \varepsilon/2$. Let $\alpha > 0$ be a real number for which $A(x) \le \alpha x$ for every $x \in D$; its existence is guaranteed by Proposition 1. Define now $\delta = min\{\delta', \varepsilon/(4\alpha)\}$.

Let now x, y be arbitrary points of D satisfying $0 < x - y < \delta$. By definition of A^*, to $\varepsilon/2$ there exists a sequence $\{x_j\}_{j=1}^k$ of points in D such that $\sum_{j=1}^k x_j = x$ and

$$(12) \qquad\qquad A^*(x) < \varepsilon/2 + \sum_{j=1}^k A(x_j) \,.$$

Without loss of generality assume that $x_1 \le \ldots \le x_k$. We will consider two cases.

Case 1: $x_k \geq x - y$. We introduce the sequence $\{y_j\}_{j=1}^k$ of points in D by letting $y_j = x_j$ for $j < k$ and $y_k = x_k - (x - y)$. Observe that $\sum_{j=1}^k y_j = y$ and so

$$(13) \qquad A^*(y) \geq \sum_{j=1}^k A(y_j) \ .$$

Of course we have $A(x_j) = A(y_j)$ for $j < k$, and as $x_k - y_k = x - y < \delta$ by the beginning of the proof one sees that $A(x_k) - A(y_k) < \varepsilon/2$. Combining this with (12) and (13) gives

$$A^*(x) - A^*(y) < \varepsilon/2 + \sum_{j=1}^k A(x_j) - \sum_{j=1}^k A(y_j) = \varepsilon/2 + A(x_k) - A(y_k) < \varepsilon \ .$$

Case 2: $x_j \leq x - y$ for every $j \leq k$. Let ℓ be the smallest positive integer for which $\sum_{j \leq \ell} x_j \geq x - y$; such an ℓ clearly exists. Since $\sum_{j < \ell} x_j < x - y$, we have

$$(14) \qquad \sum_{j \leq \ell} x_j = \sum_{j < \ell} x_j + x_\ell < 2(x - y) < 2\delta = \varepsilon/(2\alpha) \ .$$

With the help of $A(z) \leq \alpha z$ for every $z \in D$ we then obtain from (14) the estimate

$$(15) \qquad \sum_{j \leq \ell} A(x_j) \leq \alpha \sum_{j \leq \ell} x_j < \alpha\varepsilon/(2\alpha) = \varepsilon/2 \ .$$

Define now the sequence $\{y_j\}_{j=\ell}^k$ of points in D by letting $y_\ell = \sum_{j \leq \ell} x_j - (x - y)$ and $y_j = x_j$ for $j > \ell$. It can be checked that $\sum_{j \geq \ell} y_j = y$, so that, as before in (13), we again have

$$(16) \qquad A^*(y) \geq \sum_{j=\ell}^k A(y_j) \ .$$

Collecting the information contained in the inequalities (12), (16) and (15) gives

$$A^*(x) - A^*(y) < \varepsilon/2 + \sum_{j=1}^k A(x_j) - \sum_{j=\ell}^k A(y_j) \leq \varepsilon/2 + \sum_{j \leq \ell} A(x_j) < \varepsilon$$

and this completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

7. Concluding remarks

In this paper we have considered lifting of various continuity types of aggregation functions to their super- and sub-additive transformations. We have shown that differentiability and Hölder continuity are not preserved by super- and sub-additive transformations in general by providing examples illustrating this fact. We have also showed that the Lipschitz continuity is preserved by both super- and sub-additive transformations and that the uniform continuity is preserved by sub-additive transformations. Lifting of uniform continuity to super-additive transformations has been proved for one-dimensional aggregation functions only, leaving the multi-dimensional case open.

We have focused on aggregation functions defined on $[0, \infty[^n$ for $n \geq 1$. However, aggregation functions have also been considered on compact domains, with $[0, 1]^n$ being a representative case. Questions analogous to the ones considered in this paper can also be posed for this other type of aggregation functions. However, our negative results on lifting differentiability and Hölder continuity and the positive results on preservation of Lipschotz continuity can easily be adapted to the case of a compact domain, and since continuous functions on a compact domain are automatically uniformly continuous, lifting of uniform continuity to both super- and sub-additive transformations for a compact domain follows from [9].

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