AN APPLICATION OF DAUBECHIES WAVELET IN DRUG RELEASE MODEL

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Abstract. In this paper we investigate wavelet-finite difference method for solving two-dimensional model of drug release in the cardiovascular tissue from the stent. We use a double tensor product to rich a two dimensional wavelet. By using this two dimensional wavelet in space and finite difference method for variable t we convert the drug release model to a system of equation. The Lax-Richtmyer theorem shows that this system is convergent and we obtain a good approximation for a solution of our problem.

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1. Introduction

One of the leading reasons of death in the industrialized world are arterial diseases. Due to the restriction or obstruction of the affected arteries, they may occasion a reduction of the blood flow to main organs and to muscles. A stent is a small tube-like structures that is incorporated constantly into a stenotic artery. Application of stent, may be occur some problems, that Leading to re-narrowing of the treated vessel. To overcome this event drug-eluting stents (DES) have been recently defined. Drug release depends on many factors, such as the geometry and location of the vessel, the geometry of the stent, the coating properties as its chemical composition and porosity, and drug characteristics as for example its diffusivity. A helpful implement to design an proper drug delivery system are mathematical models and prediction of drug release shows an main subject [1, 2]. Some authors considered the convection-diffusion equations. They modeled the spatial and temporal distribution of drug concentration within the vessel wall [3, 4]. Also they demonstrated how numerical simulations are viable tools to study these phenomena. However, to be effective they have to account properly for the expansion of the struts and their interaction with the vascular wall. Indeed, these aspects influence the outcome of the stenting procedure.

The paper is organized as follows. Section 2 is devoted to the description of the model and its initial and boundary conditions. In Section 3 we briefly explain the wavelet and multiresolution analysis (MRA). In Section 4 we present a wavelet-finite difference method for solving the two-dimensional model of drug release from the stent.

2. Description of the model

Let the volume averaged solid concentration of the free drug inside the arterial wall and the dissolved drug inside the coating are given by a and c, respectively. Therefore the convection-diffusion system of equations [5, 6, 7], defined by

$$\begin{aligned} &\frac{\partial a}{\partial t} + \frac{K_{lag}}{k_w} u_w \nabla a - D_w \triangle a = 0, \qquad in\ \Omega_w, \\ &\frac{\partial c}{\partial t} - D_c \nabla c = 0, \qquad in\ \Omega_c, \end{aligned} \tag{1}$$

that we apply to characterize the drug release in the stent coating Ω_c and in the arterial wall Ω_w, where the indexes c, and w, mention to the values inside Ω_c and

Ω_w, respectively. Also, D_w defines the diffusion coefficient of the drug in the tissue and D_c that in the stent coating. The reduction of convective transport because of incidence of the solid bits with the structure of the porous wall, is shown by K_{lag}, that's called the hindrance coefficient ($0 \leq K_{lag} \leq 1$). Moreover, k_w is an additional partition coefficient that defines the ratio between the drug bound to the tissue matrix and that dissolved in the fluid. Finally,

$$u_w = -\frac{k_b}{\mu_b}\nabla p, \tag{2}$$

where $\nabla p = (p_1, p_2)$, k_b and μ_b are the hydraulic permeability of the arterial wall and the viscosity of the blood plasma respectively and p is the pressure. In Fig.1, the geometries of the stent in the arterial wall is shown. The system of partial

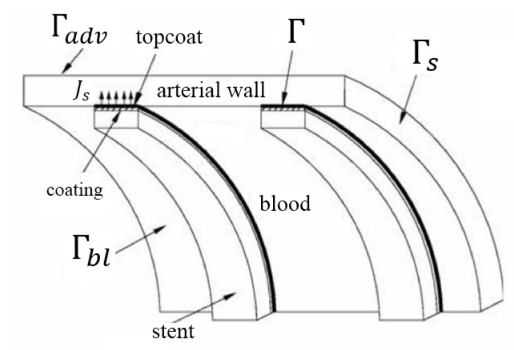

FIGURE 1. The geometries of the stent in the arterial wall.

differential equations (1) needs the suitable boundary and initial conditions. The boundary conditions for equations (1), are given by

$$\begin{aligned}
-D_c\frac{\partial c}{\partial n_c} &= P_c\left(\frac{c}{k_c\varepsilon_c} - \frac{a}{k_w\varepsilon_w}\right), && on\ \Gamma,\\
D_w\frac{\partial a}{\partial n_w} &= -D_c\frac{\partial c}{\partial n_c}, && on\ \Gamma,\\
a &= 0, && on\ \Gamma_{bl}\cap\Gamma_s,\\
-D_w\frac{\partial a}{\partial n_w} &= P_w\frac{a}{k_w\varepsilon_w}, && on\ \Gamma_{adv},\\
-D_c\frac{\partial c}{\partial n_c} &= 0, && in\ \Omega_c\backslash\Gamma,
\end{aligned} \tag{3}$$

The graph of the stent S in contact with the vessel wall V, is shown in Fig.2. Suppose

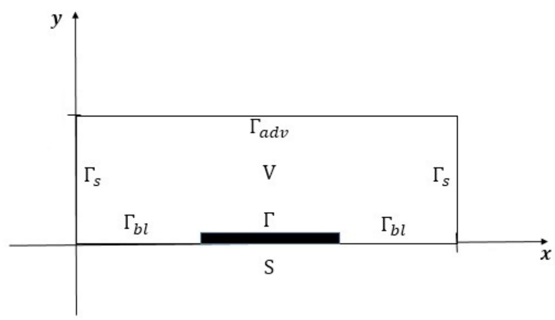

FIGURE 2. Stent S in contact with the vessel wall V.

that the drug concentration is stored into the coating and is zero inside the tissue. Thus the initial conditions are given by

$$\begin{aligned} & c = c_0, \quad in\ \Omega_c, \\ & a = 0, \quad in\ \Omega_w. \end{aligned} \tag{4}$$

The reduced form of the system of equations (1), is illustrated as follows [6]:

$$\begin{aligned} & \frac{\partial a}{\partial t} + \frac{K_{lag}}{k_w} u_w \nabla a - D_w \triangle a = 0, && in\ \Omega_w, \\ & D_w \frac{\partial a}{\partial n_w} + \alpha(t) a = \beta(t) c_0, && on\ \Gamma, \\ & D_w \frac{\partial a}{\partial n_w} + P_w \frac{a}{\epsilon_w k_w} = 0, && on\ \Gamma_{adv}, \\ & a = 0, && on\ \Gamma_{bl} \cup \Gamma_s. \end{aligned} \tag{5}$$

where P_w is the permeability of the tissue and ϵ_w is the its porosity.

3. 2-D MRA AND WAVELET METHOD

A family of orthogonal wavelets defining a discrete wavelet transform and characterized by a maximal number of vanishing moments for some given support, called the Daubechies wavelets [8]. Corresponding to each wavelet type of this class, there is a scaling function φ which generates an orthogonal MRA.

Let φ be the Daubechies wavelet scaling function. Therefore φ is compact support and

$$\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x-k), \tag{6}$$

where $\{a_k\}$ are the filter coefficients and N is an even positive integer. Hence

$$supp\ (\varphi) \subset [0, N-1].$$

Suppose $\varphi(x)$ is normalized such that: $\int_{-\infty}^{\infty} \varphi(x)\ dx = 1$. We introduce [9]

$$\theta(x) := (\varphi * \varphi(-\cdot))(x), \tag{7}$$

where "$*$" denotes the convolution of two functions f and g defined by

$$(f * g) = \int f(x-y)g(y)dy.$$

The function θ is called the autocorrelation function of φ.

Theorem 1. *The function θ, defined as above, have the following properties* [9, 10]*:*

(1) $\theta(x) = \sum_{k=-N+1}^{N-1} c_k \theta(2x-k)$, *that* $c_k = c_{-k} = \frac{1}{2}\sum_{i=0}^{N-1-k} a_i a_{k+i}, \quad k \geq 0,$

(2) $supp\ (\theta) \subseteq [-N+1, N-1]$,

(3) $\theta(k) = \delta_{0,k},\ k \in Z$,

(4) $c_{2k} = \delta_{0,k}$ *and* $c_k = \theta(\frac{k}{2}),\ k \in Z$, *therefore* $\theta(x) = \sum_{k=-N+1}^{N-1} \theta(\frac{k}{2})\theta(2x-k)$,

where N is an even positive integer in definition of the Daubechies wavelet, the sequence $\{c_k\}_{k\in Z}$ is called the scaling filter and $\delta_{0,k}$ is the Kronecker delta function.

Definition 1. [8] *A sequence of closed subspaces $\{Vj\}_{j\in Z}$ in $L^2(R)$ is called a multiresolution analysis for $L^2(R)$ with scaling function φ, if*

1. $V_j \subseteq V_{j+1} \subseteq L^2(R)$,
2. $\bigcap_{j\in Z} V_j = \{0\}$, *and* $\overline{\bigcup_{j\in Z} V_j} = L^2(R)$
3. $f(\cdot) \in V_j \Leftrightarrow f(2^{-j}\cdot) \in V_0$,
4. $f(\cdot) \in V_0 \Leftrightarrow f(\cdot - n) \in V_0$, *for all* $n \in Z$,
5. *There exist a function $\varphi \in V_0$, called scaling function, such that*

$\{\varphi(\cdot - k)\}_{k\in Z}$ *is an orthonormal basis for V_0.*

Let "$\oplus$" denotes the direct sum notation. A simple consequence of condition 1 is

$$V_1 = V_0 \oplus W_0, \tag{8}$$

where W_0 is the orthogonal complement of V_0 in V_1.

If $\{V_j\}_{j\in Z}$ is a multiresolution analysis for $L^2(R)$ with scaling function ϕ and wavelet ψ, then $\{V_j^{'} = V_j \otimes V_j\}_{j\in Z}$ is a multiresolution analysis of $L^2(R^2)$, that "$\otimes$" is tensor product. Let $\{V_j^{(x)}\}_{j\in Z}$ be a multiresolution analysis for $L^2(R)$ at x direction and $\{V_j^{(y)}\}_{j\in Z}$ be a multiresolution analysis for $L^2(R)$ at y direction. Using Eq. (8), one can easily show that

$$\begin{aligned} (9)\quad V_1^{'} &= V_1^{(x)} \otimes V_1^{(y)} = (V_0^{(x)} \oplus W_0^{(x)}) \otimes (V_0^{(y)} \oplus W_0^{(y)}) \\ &= (V_0^{(x)} \otimes V_0^{(y)}) \oplus (V_0^{(x)} \otimes W_0^{(y)}) \oplus (W_0^{(x)} \otimes V_0^{(y)}) \oplus (W_0^{(x)} \otimes W_0^{(y)}) \\ &= V_0^{'} \oplus W_0^{'1} \oplus W_0^{'2} \oplus W_0^{'3}. \end{aligned}$$

This 2-D multiresolution analysis requires one scaling function

$$\Phi(x,y) = \phi(x)\phi(y) \in V_0^{'},$$

and three wavelets

$$\Psi^1(x,y) = \phi(x)\psi(y), \quad \Psi^2(x,y) = \psi(x)\phi(y), \quad \Psi^3(x,y) = \psi(x)\psi(y),$$

where Ψ^i is the wavelet associated to $W^{'i}$ for $i = 1,2,3$, respectively.

Define $V_j = span\{\theta(2^j \cdot -k), k \in Z\}$, that $j \in Z$. Moreover, $\{\theta(\cdot - k), k \in Z\}$ is a Riesz basis for V_0 [10]. Thus by [Theorem 1.6, Sec. 2.1, [11]] and [Theorem 1.7, Sec.2.1, [11]] , $\{V_j\}_{j\in Z}$ generates an MRA with scaling function θ.

Therefore, any $a \in L^2(R^2)$ can be approximated with arbitrary precision by element $a_J \in V_J' = V_J^{(x)} \otimes V_J^{(y)}$, for some suitable $J \in Z$. More precisely, we have the following approximation:

$$(10)\qquad a(x,y,t) \approx a_J(x,y,t) = \sum_{k\in Z}\sum_{l\in Z} a_J(x_k, y_l, t)\theta(2^J x - k)\theta(2^J y - l),$$

where $(x,y) \in [0,l_X] \times [0,l_Y] \subseteq R^2$, $t \in [0,T]$ and $a_J(x_k,y_l,t)$ are suitable coefficients corresponding to collocation points, $x_k = k2^{-J}$ and $y_l = l2^{-J}$.

The first and second derivatives of the function θ defined by

$$\theta(x) = \int \varphi(t)\varphi(t-x)dt \ \text{ are}$$

$$\theta'(l) = \int \varphi(t)\varphi'(t-l)dt, \ \theta''(l) = -\int \varphi'(t)\varphi'(t-l)dt.$$

Define

$$\Gamma_l^1 = \int \varphi(t)\varphi'(t-l)dt, \ \Gamma_l^2 = \int \varphi'(t)\varphi'(t-l)dt.$$

Since φ is compact support on $[0, N-1]$, we have:

$$\Gamma^1_{-l} = -\Gamma^1_l,\ \Gamma^2_{-l} = \Gamma^2_l, \quad \theta'(l) = \Gamma^1_l,\ \theta''(l) = -\Gamma^2_l, \quad |l| \le N-2.$$

MATLAB software is used to compute Γ^1 and Γ^2. Thus we compute derivatives of the function θ at the point $x_l = l2^{-J}$.

3.1. **Wavelet method.** Suppose $x \in [0, l_X]$ and $y \in [0, l_Y]$. We transferred the physical domain at an interval $[0,1]$ in each direction. Let J be arbitrary. We estimate the solution for equation (5) with corresponding initial and boundary conditions at a fixed time level n using the following approximation:

$$\begin{aligned} a(\xi,\eta) &\approx \sum_{k\in Z}\sum_{l\in Z} a_{kl}\theta(2^J\xi-k)\theta(2^J\eta-l) \approx \sum_{k=1}^{2^J-1}\sum_{l=1}^{2^J-1} a_{kl}\theta(2^J\xi-k)\theta(2^J\eta-l) \\ &+ \sum_{k=-\infty}^{0}\sum_{l=1}^{2^J-1} a_{0l}\theta(2^J\xi-k)\theta(2^J\eta-l) + \sum_{k=2^J}^{\infty}\sum_{l=1}^{2^J-1} a_{2^Jl}\theta(2^J\xi-k)\theta(2^J\eta-l) \\ &+ \sum_{k=1}^{2^J-1}\sum_{l=-\infty}^{0} a_{k0}\theta(2^J\xi-k)\theta(2^J\eta-l) + \sum_{k=1}^{2^J-1}\sum_{l=2^J}^{\infty} a_{k2^J}\theta(2^J\xi-k)\theta(2^J\eta-l), \end{aligned}$$

where $a_{kl} = a(\xi_k, \eta_l)$ and $\xi^J_k = k/2^J$, $\eta^J_l = l/2^J$ are collocation points. For simplicity, let $\xi_k = \xi^J_k$, $\eta_l = \eta^J_l$. Therefore by boundary conditions we have

$$\begin{aligned} a(\xi,\eta) &\approx \sum_{k=1}^{2^J-1}\sum_{l=1}^{2^J-1} a_{kl}\theta(2^J\xi-k)\theta(2^J\eta-l) + \sum_{k=1}^{2^J-1}\sum_{l=-\infty}^{0} a_{k0}\theta(2^J\xi-k)\theta(2^J\eta-l) \\ &+ \sum_{k=1}^{2^J-1}\sum_{l=2^J}^{\infty} a_{k2^J}\theta(2^J\xi-k)\theta(2^J\eta-l). \end{aligned}$$

The first and second order partial derivatives of a in the directions of ξ and η are:

$$\frac{\partial a}{\partial \xi}(\xi_p,\eta_q) = 2^J \sum_{k=1}^{2^J-1} a_{kq}\theta'(p-k) = 2^J \sum_{k=1}^{2^J-1} a_{kq}\Gamma^1_{p-k},$$

$$\frac{\partial^2 a}{\partial \xi^2}(\xi_p,\eta_q) = 2^{2J} \sum_{k=1}^{2^J-1} a_{kq}\theta''(p-k) = -2^{2J} \sum_{k=1}^{2^J-1} a_{kq}\Gamma^2_{p-k},$$

$$\frac{\partial a}{\partial \eta}(\xi_p,\eta_q) = 2^J \left\{ \sum_{l=1}^{2^J-1} a_{pl}\theta'(q-l) + a_{p0} \sum_{l=2-N+q}^{0} \theta'(q-l) + a_{p2^J} \sum_{l=2^J}^{N-2+q} \theta'(q-l) \right\}$$

$$= 2^J \left\{ \sum_{l=1}^{2^J-1} a_{pl}\Gamma^1_{q-l} + a_{p0} \sum_{l=2-N+q}^{0} \Gamma^1_{q-l} + a_{p2^J} \sum_{l=2^J}^{N-2+q} \Gamma^1_{q-l} \right\},$$

$$\frac{\partial^2 a}{\partial \eta^2}(\xi_p,\eta_q) = 2^{2J} \left\{ \sum_{l=1}^{2^J-1} a_{pl}\theta''(q-l) + a_{p0} \sum_{l=2-N+q}^{0} \theta''(q-l) + a_{p2^J} \sum_{l=2^J}^{N-2+q} \theta''(q-l) \right\}$$

$$= -2^{2J} \left\{ \sum_{l=1}^{2^J-1} a_{pl}\Gamma^2_{q-l} + a_{p0} \sum_{l=2-N+q}^{0} \Gamma^2_{q-l} + a_{p2^J} \sum_{l=2^J}^{N-2+q} \Gamma^2_{q-l} \right\},$$

for $p = 1, 2, \cdots, 2^J - 1$ and $q = 0, 1, \cdots, 2^J$. Thus the discretization of equation (5) at given collocation points ξ_p and η_q, $p = 1, 2, \cdots, 2^J - 1$, $q = 0, 1, \cdots, 2^J$, is

$$\begin{aligned}
\frac{\partial a}{\partial t}(\xi_p,\eta_q) &= \frac{K_{lag}}{k_w}\frac{k_b}{\mu_b} 2^J \left(\frac{p_1}{l_x} \sum_{k=1}^{2^J-1} a_{kq}\Gamma^1_{p-k} + \frac{p_2}{l_y} \left\{ \sum_{l=1}^{2^J-1} a_{pl}\Gamma^1_{q-l} \right.\right. \\
&+ \left.\left. a_{p0} \sum_{l=2-N+q}^{0} \Gamma^1_{q-l} + a_{p2^J} \sum_{l=2^J}^{N-2+q} \Gamma^1_{q-l} \right\} \right) \\
&- D_w 2^{2J} \left(\frac{1}{(l_x)^2} \sum_{k=1}^{2^J-1} a_{kq}\Gamma^2_{p-k} + \frac{1}{(l_y)^2} \left\{ \sum_{l=1}^{2^J-1} a_{pl}\Gamma^2_{q-l} \right.\right. \\
&+ \left.\left. a_{p0} \sum_{l=2-N+q}^{0} \Gamma^2_{q-l} + a_{p2^J} \sum_{l=2^J}^{N-2+q} \Gamma^2_{q-l} \right\} \right), \qquad (11)
\end{aligned}$$

The first derivative of a with respect to time, are estimated

$$\frac{\partial a}{\partial t} \approx \frac{a^{n+1} - a^n}{\Delta t}. \qquad (12)$$

Therefore

$$
\begin{aligned}
a^{n+1}(\xi_p,\eta_q) &= \frac{K_{lag}}{k_w}\frac{k_b}{\mu_b}2^J\Delta t\left(\frac{p_1}{l_x}\sum_{k=1}^{2^J-1}a^n_{kq}\Gamma^1_{p-k}+\frac{p_2}{l_y}\left\{\sum_{l=1}^{2^J-1}a^n_{pl}\Gamma^1_{q-l}\right.\right.\\
&+ \left.\left. a^n_{p0}\sum_{l=2-N+q}^{0}\Gamma^1_{q-l}+a^n_{p2^J}\sum_{l=2^J}^{N-2+q}\Gamma^1_{q-l}\right\}\right)\\
&- D_w2^{2J}\Delta t\left(\frac{1}{(l_x)^2}\sum_{k=1}^{2^J-1}a^n_{kq}\Gamma^2_{p-k}+\frac{1}{(l_y)^2}\left\{\sum_{l=1}^{2^J-1}a^n_{pl}\Gamma^2_{q-l}\right.\right.\\
&+ \left.\left. a^n_{p0}\sum_{l=2-N+q}^{0}\Gamma^2_{q-l}+a^n_{p2^J}\sum_{l=2^J}^{N-2+q}\Gamma^2_{q-l}\right\}\right)+a^n_{pq}. \qquad (13)
\end{aligned}
$$

Also, for the boundary conditions on Γ and Γ_{adv}, for $p = 1, 2, \cdots, 2^J - 1$ we can write

$$
\begin{aligned}
&\frac{D_w2^J}{l_x}\sum_{k=1}^{2^J-1}a^{n+1}_{k0}\Gamma^1_{p-k}+\alpha(t_{n+1})a^{n+1}_{p0}=\beta(t_{n+1})c_0,\\
&\frac{D_w2^J}{l_x}\sum_{k=1}^{2^J-1}a^{n+1}_{k2^J}\Gamma^1_{p-k}+\frac{P_w}{\epsilon_w k_w}a^{n+1}_{p2^J}=0. \qquad (14)
\end{aligned}
$$

By equations (13) and (14) we have the following system

$$\bar{A}a^{n+1} = \bar{B}a^n + \bar{C}, \qquad (15)$$

or

$$a^{n+1} = Aa^n + B, \qquad (16)$$

where $A = \bar{A}^{-1}\bar{B}$ and $B = \bar{A}^{-1}\bar{C}$.

4. The convergence of the proposed method

First, we recall the Lax-Richtmyer theorem [12, 13].

Theorem 2. *A consistent finite-difference scheme for a partial differential equation for which the initial-value problem is well posed is convergent if and only if it is stable.*

We show that the scheme is stable and consistent. Suppose $P(a_{pq})$ represent the PDE operator of Equation (11) at fixed point $(\xi_p,\ \eta_q)$, and $P^n(a_{pq})$ indicate the approximating wavelet-finite difference operator for a fixed time-level, that the exact solution is a_{pq}. Let V_{pq} be a continuous function of t with a sufficient number of continuous derivatives to enable $P(V_{pq})$ to be evaluated at point $n\Delta t$. Then the truncation error $E^n(V_{pq})$ at the point $n\Delta t$ for all $p = 0, 1, ..., 2^J$, $q = 0, 1, ..., 2^J$, is defined by

$$E^n(V_{pq}) = P^n(V_{pq}) - P(V_{pq}^n), \tag{17}$$

where $V_{pq}^n = V_{pq}(n\Delta t)$.

The Equation (11) is said to be consistent with the PDE (13), if the truncation error, $E^n(V_{pq})$, tend to zero as Δt tend to zero [12].

The following theorem, shows that our method presented in section 3.1 yields a consistent approximation that is first order in time and spectral accuracy in space.

Theorem 3. *Let $a(x, y, t)$ be as in (10) and its derivatives of order two, with respect to x, y and t exist. Then the recursive formula defined in (13) is convergent.*

Proof. Let us to expand a_{pq}^{n+1} and a_{pq}^n appearing in Equation (13) around the point $(\xi_p,\ \eta_q, n\Delta t) \in \Omega \times [0, T]$, then we have

$$\begin{aligned} a_{pq}^n &+ \Delta t(\frac{\partial a}{\partial t})_{pq}^n + O(\Delta t) \\ &= \frac{K_{lag}}{k_w}\frac{k_b}{\mu_b}2^J\Delta t\left(\frac{p_1}{l_x}\sum_{k=1}^{2^J-1} a_{kq}^n\Gamma_{p-k}^1 + \frac{p_2}{l_y}\left\{\sum_{l=1}^{2^J-1} a_{pl}^n\Gamma_{q-l}^1\right.\right. \\ &+ \left.\left. a_{p0}^n\sum_{l=2-N+q}^{0}\Gamma_{q-l}^1 + a_{p2^J}^n\sum_{l=2^J}^{N-2+q}\Gamma_{q-l}^1\right\}\right) \\ &- D_w 2^{2J}\Delta t\left(\frac{1}{(l_x)^2}\sum_{k=1}^{2^J-1} a_{kq}^n\Gamma_{p-k}^2 + \frac{1}{(l_y)^2}\left\{\sum_{l=1}^{2^J-1} a_{pl}^n\Gamma_{q-l}^2\right.\right. \\ &+ \left.\left. a_{p0}^n\sum_{l=2-N+q}^{0}\Gamma_{q-l}^2 + a_{p2^J}^n\sum_{l=2^J}^{N-2+q}\Gamma_{q-l}^2\right\}\right) + a_{pq}^n. \end{aligned} \tag{18}$$

for $n = 0, 1, 2, ..., \ p = 1, 2, ..., 2^J - 1, \ q = 0, 1, ..., 2^J$. In light of Equation (11) we can write

$$
\begin{aligned}
(\frac{\partial a}{\partial t})^n(\xi_p, \eta_q) &= \frac{K_{lag}}{k_w}\frac{k_b}{\mu_b}2^J\left(\frac{p_1}{l_x}\sum_{k=1}^{2^J-1}a^n_{kq}\Gamma^1_{p-k} + \frac{p_2}{l_y}\left\{\sum_{l=1}^{2^J-1}a^n_{pl}\Gamma^1_{q-l}\right.\right.\\
&+ \left.\left. a^n_{p0}\sum_{l=2-N+q}^{0}\Gamma^1_{q-l} + a^n_{p2^J}\sum_{l=2^J}^{N-2+q}\Gamma^1_{q-l}\right\}\right)\\
&- D_w 2^{2J}\left(\frac{1}{(l_x)^2}\sum_{k=1}^{2^J-1}a^n_{kq}\Gamma^2_{p-k} + \frac{1}{(l_y)^2}\left\{\sum_{l=1}^{2^J-1}a^n_{pl}\Gamma^2_{q-l}\right.\right.\\
&+ \left.\left. a^n_{p0}\sum_{l=2-N+q}^{0}\Gamma^2_{q-l} + a^n_{p2^J}\sum_{l=2^J}^{N-2+q}\Gamma^2_{q-l}\right\}\right). \qquad (19)
\end{aligned}
$$

Hence the truncated error will be found by subtracting Equations (18) and (19). i.e.,

$$E^n(a_{pq}) \quad = \quad O(\Delta t).$$

Here, we used spectral collocation scheme in space, clearly $E^n(a_{pq})$ vanishes as Δt tends to zero and l_X and l_Y tend to infinity [14]. Now, we show that the our method is stable.

In the pervious section, the Equation (13) can be represented as

$$a^{n+1} = Aa^n + B^n, \qquad n = 0, 1, 2, ... \ . \qquad (20)$$

where the vector B^n is generated by the boundary conditions. If each eigenvalue of A has a modulus ≤ 1, i.e. $\rho(A) \leq 1$, we say Equation (20) is stable. The eigenvalues of A can be evaluated numerically [12].

Therefore by the Lax-Richtmyer theorem, the scheme is convergent. □

5. Numerical results

Consider the following parameters:

$$
\begin{aligned}
&K_{lag} = 1, \ k_w = 20, \ D_w = 2.2 \times 10^{-9}, \ P_w = 10^{-8}, \ \epsilon_w = 0.61, \ k_b = 2 \times 10^{-14},\\
&\mu_b = 0.72 \times 10^{-2}, \ D_c = 1.0 \times 10^{-11}, \ P_c = 1.0 \times 10^{-8}, \ \Delta L = 5 \times 10^{-6}, \ k_c = 1,\\
&\epsilon_c = 0.1, \ c_0 = 1, \ p_1 = p_2 = 100, \ l_X = 1.8 \times 10^{-3}, \ l_Y = 0.4 \times 10^{-3}
\end{aligned}
$$

We solve numerically equation (5) by proposed method. The absolute value of

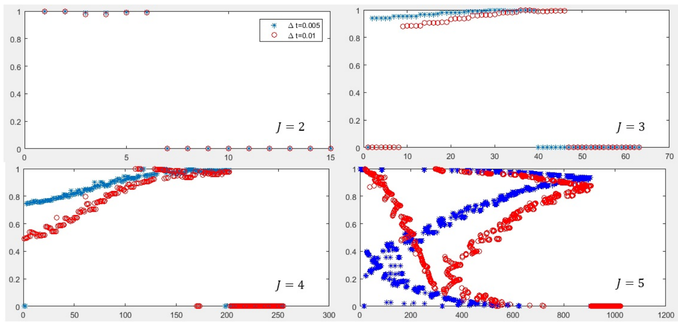

FIGURE 3. Eigenvalues of A at $\Delta t = 0.005$ and $\Delta t = 0.01$ for $J = 2, 3, 4, 5$.

eigenvalues of the matrix A is plotted in the Fig.3 . It shows that the upper bound for the absolute value of A's eigenvalues, for different J and Δt is 1. Thus the scheme is stable in time.

Let $J = 5$. The Figure 4 - Figure 7 show that for a fixed x, the value of a decreases as y increases. As we expected the value of a increases as t increases.

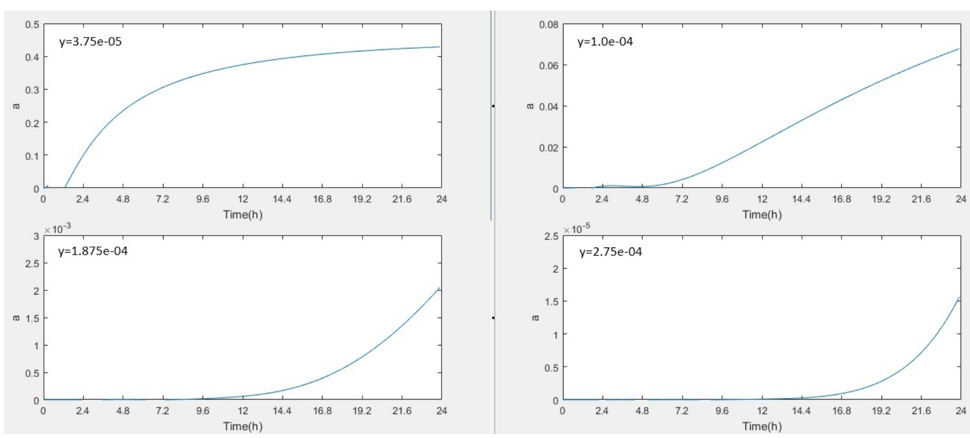

FIGURE 4. Comparison of $a(x, y, t)$ at $x = 1.125e - 04$, for different y.

6. Conclusion

In this paper we constructed a wavelet-finite difference approximation for solving two-dimensional model of drug release in the cardiovascular tissue from the stent. A double tensor product is used to rich a two dimensional wavelet, converted the

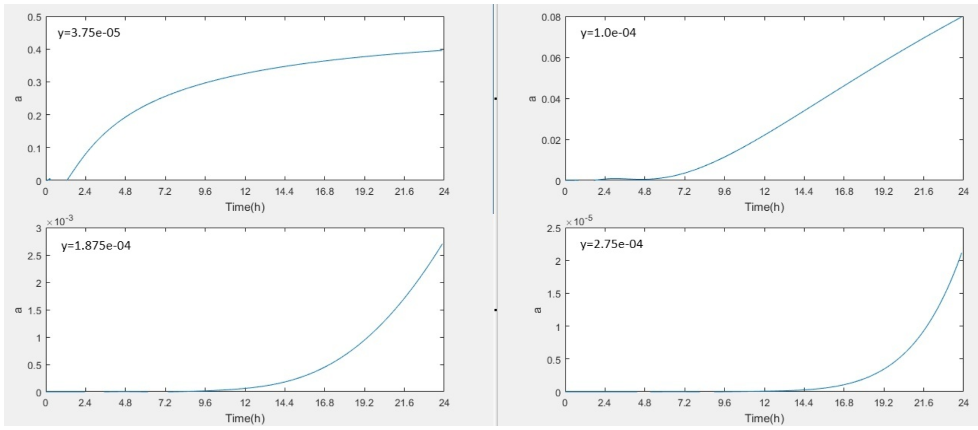

FIGURE 5. Comparison of $a(x, y, t)$ at $x = 7.875e - 04$, for different y.

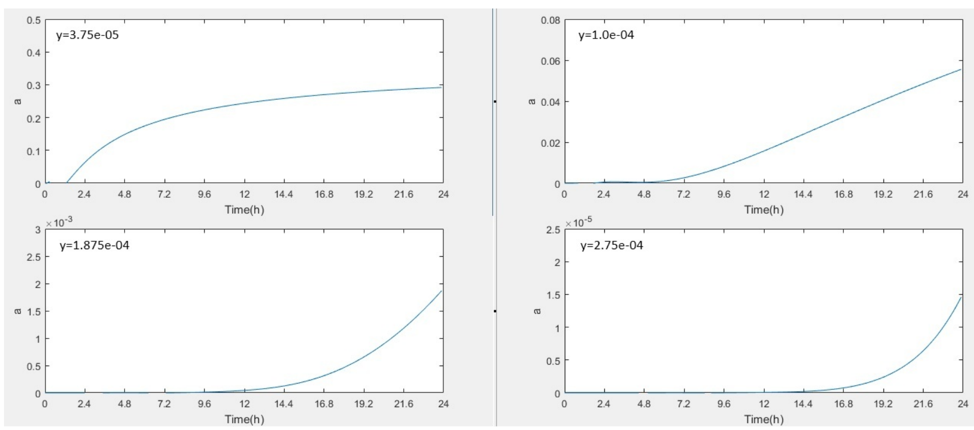

FIGURE 6. Comparison of $a(x, y, t)$ at $x = 1.35e - 03$, for different y.

drug release model to a system of matrix equations. We applied the Lax-Richtmyer theorem to prove the convergent of our system. Finally we constructed several examples to show the sharpness of our method. In fact these examples show that for a fixed x the value of a, the free drug inside the arterial wall, decreases as the value of y increases. Also the graphs show that the value of a increases as the value of t, time, increases.

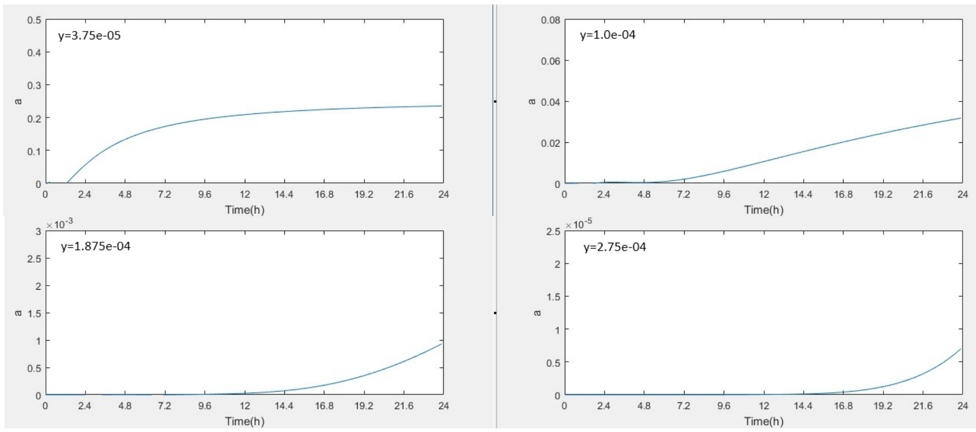

FIGURE 7. Comparison of $a(x, y, t)$ at $x = 1.6875e-03$, for different y.

Appendix

General Matrices Form

In general, we assume every element of the matrix is corresponding to a collocation point. For the collocation points (ξ_p, η_q), and (ξ_k, η_l), that are corresponding elements of the square matrix B_1, we can write

$$(B_1)_{i,j} = \begin{cases} 1+\frac{K_{lag}}{k_w}\frac{k_b}{\mu_b}2^J\Delta t\left(\frac{p_1}{l_x}+\frac{p_2}{l_y}\right)\Gamma_0^1 - D_w 2^{2J}\left(\frac{1}{(l_x)^2}+\frac{1}{(l_y)^2}\right)\Gamma_0^2, & p=k,\ q=l, \\ \frac{K_{lag}}{k_w}\frac{k_b}{\mu_b}2^J\Delta t\frac{p_2}{l_y}\Gamma^1_{q-l} - D_w 2^{2J}\frac{1}{(l_y)^2}\Gamma^2_{q-l}, \quad p=k,\ q\neq l, & |q-l|\leq N-2, \\ \frac{K_{lag}}{k_w}\frac{k_b}{\mu_b}2^J\Delta t\frac{p_1}{l_x}\Gamma^1_{p-k} - D_w 2^{2J}\frac{1}{(l_x)^2}\Gamma^2_{p-k}, \quad p\neq k,\ q=l, & |p-k|\leq N-2, \\ \frac{K_{lag}}{k_w}\frac{k_b}{\mu_b}2^J\Delta t\frac{p_2}{l_y}\sum_{l=2^J}^{N-2+2^J}\Gamma^1_{2^J-l} - D_w 2^{2J}\frac{1}{(l_y)^2}\sum_{l=2^J}^{N-2+2^J}\Gamma^2_{2^J-l}, & p=k,\ q=2^J, \\ \frac{K_{lag}}{k_w}\frac{k_b}{\mu_b}2^J\Delta t\frac{p_2}{l_y}\sum_{l=2-N}^{0}\Gamma^1_{-l} - D_w 2^{2J}\frac{1}{(l_y)^2}\sum_{l=2-N}^{0}\Gamma^2_{-l}, & p=k,\ q=0. \end{cases}$$

Consider the matrix $A_1 = (a_{pq})$ with $a_{pq} = 0$ for $p \neq k$ and $q \neq l$ and $a_{pq} = 1$.
In a similar way, we denote the boundary conditions by matrices A_2 and A_3 defined by

$$(A_2)_{i,j} = \begin{cases} D_w 2^J\frac{1}{l_x}\Gamma^1_{p-k}, & q=0,\ 1\leq k\leq 2^J-1,\ |p-k|\leq N-2, \\ D_w 2^J\frac{1}{l_x}\Gamma^1_0+\alpha(t_{n+1}), & q=0,\ p=k. \end{cases}$$

$$(A_3)_{i,j} = \begin{cases} D_w 2^J\frac{1}{l_x}\Gamma^1_{p-k}, & q=2^J,\ 1\leq k\leq 2^J-1,\ |p-k|\leq N-2, \\ D_w 2^J\frac{1}{l_x}\Gamma^1_0+\frac{P_w}{\varepsilon_w k_w}, & q=2^J,\ p=k. \end{cases}$$

Now we put matrices A_2 and A_3 in the last rows of matrix A_1 and call the resulting matrix $\bar{A}$. So corresponding to the changed rows, we set the corresponding elements in the matrix B_1 equal zero and call the resulting matrix $\bar{B}$.
Finally, we define vector $\bar{C}$ with dimension equal to the dimension of $\bar{B}$, where all elements except the corresponding elements to the matrix A_2 are zero and the non-zero elements are $\beta(t_{n+1})C_0$.

References

[1] A. Borghi, E. Foa, R. Balossino, F. Migliavacca, G. Dobini, Modeling drug elution from stents: effect of reversible binding in the vascular wall and degradable polymeric matrix, Computer Methods in Biomechanics and Biomedical Engineering, Vol. 11, Number 4, 367-377 (2008).

[2] A. Raval, J. Parikh, C. Engineer, Mechanism of controlled release kinetics from medical devices, Brazilian Journal of Chemical Engineering, Vol. 27, Number 2, 211-225 (2010).

[3] H. Q. Zhao, D. Jayasinghe, S. Hossainy, L. B. Schwartz, A theoretical model to characterize the drug release behavior of drug-eluting stents with durable polymer matrix coating, Journal of Biomedical Materials Research, Part A, Vol. 100A, 120-124 (2012).

[4] S. Hossainy, S. Prabhu, A mathematical model for predicting drug release from a biodurable drugeluting stent coating, Journal of Biomedical Materials Research, Part A, Vol. 87A, 487-493 (2008).

[5] M. Prosi, P. Zunino, K. Perktold, A. Quarteroni, Mathematical and numerical models for transfer of low-density lipoproteins through the arterial walls: a new methodology for the model set up with applications to the study of disturbed lumenal flow, Journal of Biomechanics, Vol. 38, 903-917 (2005).

[6] S. Minicini, Mathematical and Numerical Modeling of Controlled Drug Release, Ph.D Thesis, Politecnico di Milano, Italy (2009).

[7] A. Borghi, E. Foa, R. Balossino, F. Migliavacca, G. Dobini, Modeling drug elution from stents: effect of reversible binding in the vascular wall and degradable polymeric matrix, Computer Methods in Biomechanics and Biomedical Engineering, Vol. 11, Number 4, 367-377 (2008).

[8] I. Daubechies, Ten Lectures on Wavelets, CBMSNSF Series in Applied Mathematics (Society for Industrial Applied Mathematics, Philadelphia), 1992.

[9] G. Beylkin and N. Saito, Wavelets, their autocorrelation functions, and multiresolution representation of signals, Expanded abstract in Proceedings ICASSP-92, Vol. 4, 381-384 (1992).

[10] Z. Kalateh Bojdi, A. Askari Hemmat, Wavelet collocation methods for solving the Pennes bioheat transfer equation, Optik, International Journal for Light and Electron Optics, Vol. 132, 80-88 (2017).

[11] E. Hernandez, G. Weiss, A First Course on Wavelets, CRC press, 1996.

[12] GD. Smith, Numerical Solution of Partial Differential Equations Finite Difference Methods, Third ed., Oxford, Oxford University Press, 1985.

[13] C. Canuto, MY. Hussaini, A. Quarteroni, TA. Zang, Spectral Methods in Fluid Dynamics, Berlin, Springer Series in Computational Physics, 1988.

[14] C. Canuto, MY. Hussaini, A. Quarteroni, TA. Zang, Spectral Methods: Fundamentals in Single Domains, Berlin, Springer, 2006.