FUZZY POINTS IN BE-ALGEBRAS

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Abstract. The aim of this paper is to introduce the notion of fuzzy points and weak filters in BE-algebras and investigate their properties. We generalize some results of the set of fuzzy points in BCI-algebras to commutative and self-distributive weak BE-algebras. Then we establish some relations among filters, fuzzy filters and weak filters in BE-algebras.

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1. Introduction

K. Iséki introduced the concept of a BCK-algebra in 1966 [5]. H.S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a generalization of a BCK-algebra [9]. By using the concept of the upper sets, they provided conditions equivalent to a filter in BE-algebras.

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From them some mathematicians studied and developed many concepts in this algebraic structure, for instance, some properties of filters of BE-algebras were investigated by S. S. Ahn and K. S. So in [1].
Lately, in 2011, A. Rezaei and A. B. Saeid introduced the concept of fuzzy BE-algebras and perused their structure [11]. They got some of the theorems in fuzzy BE-algebras and level subalgebras and some characterizations of fuzzy subalgebras are also created.
Since the set of fuzzy points and weak filters are two important notions in the algebraic structures like BCI/BCK-algebras, we extend these notions to BE-algebras and discuss further properties of these concepts [6,10]. So, in this paper, we generalize the concepts of fuzzy points in BCI-algebras to BE-algebras. Finally, we present the concepts of fuzzification of weak BE-algebras and weak filters of a BE-algebra and some properties of these notions are investigated. We verify some useful properties of this notion in BE-algebras such as relation fuzzy filters and week filters. We use the notion of fuzzy points in BE-algebras to develop other new concepts such as m-polar fuzzy subalgebras, m-polar fuzzy filters in these structures [2,3,4]. We can also investigate the variety and some subvarieties of these specific type of BE-algebras.

2. Preliminaries

In this section, we review the definitions and specifications that might be used in this article. For more details, we refer the reader to [1,4,6,7].

Definition 2.1. [4] An algebra $(X, \to, 1)$ of type $(2, 0)$ is called a BE-algebra, if it satisfies the following axioms:
$(BE1)$ $x \to x = 1$,
$(BE2)$ $x \to 1 = 1$,
$(BE3)$ $1 \to x = x$,
$(BE4)$ $x \to (y \to z) = y \to (x \to z)$.
For all $x, y, z \in X$.

A binary relation $\leq$ defined on X by $x \leq y$ if and only if $x \to y = 1$. A non-empty subset A of a BE-algebra X is said to be a subalgebra of X if it is closed under the operation $\to$. Since that $x \to x = 1$ for all $x \in X$, it is clear that $1 \in A$. A BE-algebra X satisfies the following properties:

$(BE5)$ $x \to (y \to x) = 1$,

$(BE6)$ $x \to ((x \to y) \to y) = 1$.

A BE-algebra X is said to be commutative if $(x \to y) \to y = (y \to x) \to x$ for all $x, y \in X$. A BE-algebra X is called self-distributive if $x \to (y \to z) = (x \to y) \to (x \to z)$ for all $x, y, z \in X$. A nonempty subset F of a BE-algebra X is called a filter if [1]:

$i)$ $1 \in F$, $ii)$ $x \to y \in F$ and $x \in F$ imply $y \in F$.

We can show that a subset F of a BE-algebra X is a filter if:

(a) $x \in F$ and $y \in F$ imply $x \wedge y \in F$ and $y \wedge x \in F$,

(b) $x \in F$ and $x \leq y$ implies $y \in F$.

Where $x \wedge y = (y \to x) \to x$.

If X is a commutative BE algebra and $x \to y = y \to x = 1$, then $x = y$ for all $x, y \in X$. We note that $\leq$ is reflexive by (BE1). If X is self-distributive, then the relation $\leq$ is transitive. Because by assumption $x \leq y$ and $y \leq z$, we give $x \to z = 1 \to (x \to z) = (x \to y) \to (x \to z) = x \to (y \to z) = x \to 1 = 1$. Hence $x \leq z$. If X is a commutative, then $\leq$ is antisymmetric. So, if X is a commutative self-distributive BE algebra, then the relation $\leq$ is a partially ordered relation on X. If X is a self-distributive BE algebra, then for all $x, y, z \in X$ we obtain

(i) if $x \leq y$, then $z \to x \leq z \to y$ and $y \to z \leq x \to z$,

(ii) $y \to z \leq (z \to x) \to (y \to x)$.

In what follows, let $(X, \to, 1)$ or simply X would mean a BE-algebra, unless otherwise specified.

A fuzzy set μ in X is a map $\mu : X \to [0, 1]$. Let μ be a fuzzy set of X, for $t \in [0, 1]$ the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset of X. A fuzzy set μ of X is called a fuzzy BE-subalgebra of X if it satisfies:

$min\{\mu(x), \mu(y)\} \leq \mu(x \to y)$ for all $x, y \in X$.

Definition 2.2. A fuzzy set μ in X is called a fuzzy filter of X if for all $x, y \in X$ it satisfies [6]:

$(F1)$ $\mu(x) \leq \mu(1)$,

$(F2)$ $min\{\mu(x \to y), \mu(x)\} \leq \mu(y)$.

Let μ be a fuzzy filter of X. Then the filters μ_α, $\alpha \in [0, 1]$, are called level filters of X. Any filter of a BE-algebra X can be realized as a level filter of some fuzzy filters of X [13].

Let $(X, \rightarrow, 1)$ and $(Y, \longrightarrow, 1)$ be two BE-algebras. Then a mapping $f : X \longrightarrow Y$ is called a homomorphism if $f(x \rightarrow y) = f(x) \longrightarrow f(y)$ for all $x, y \in X$. It is clear that if $f : X \longrightarrow Y$ is a homomorphism, then $f(1) = 1$. If f is an onto homomorphism, then for any fuzzy set μ in Y we define a mapping $f : X \longrightarrow [0, 1]$ such that $\mu^f(x) = \mu(f(x))$ for all $x \in X$. Clearly the map μ^f is well-defined and fuzzy set in X [13].

Definition 2.3. [10] Let F be a non-empty subset of X and $\alpha \in (0, 1]$ we define a fuzzy set $\alpha_{\chi F}$ as

$$\alpha_{\chi F}(x) = \begin{cases} \alpha & x \in F \\ 0 & otherwise \end{cases}$$

Definition 2.4. [13] Let μ and ν be fuzzy sets in X. Then the cartesian product of μ and ν is defined by

$$(\mu \times \nu) = min\{\mu(x), \nu(y)\}$$

for all $x, y \in X$.

Theorem 2.5. [13] *Let μ be a fuzzy subset of X. Then the following conditions are equivalent:*
(1) μ is a fuzzy filter in X,
(2) for all $x, y, z \in X$, $x \rightarrow (y \rightarrow z) = 1$ implies $\mu(z) \geq min\{\mu(x), \mu(y)\}$,
(3) for any $\alpha \in (0, 1]$ the α-level subset $\mu_\alpha = \{x \in X : \mu(x) \geq \alpha\}$ is a filter, when $\mu_\alpha \neq \emptyset$.

3. Algebra structure of the set of fuzzy points in BE-algebras

In this section, we define the set of fuzzy points in BE-algebras and discuss their properties.

Definition 3.1. Suppose that λ is the family of all fuzzy sets in X. Then $x_\alpha \in \lambda$ is called a fuzzy point of X if:

$$x_\alpha(y) = \begin{cases} \alpha & x = y \\ 0 & x \neq y \end{cases}.$$

The set of all fuzzy points on X is denoted by $\tilde{X}$. So

$$\tilde{X} = \{x_\alpha : x \in X, \alpha \in (0,1]\}$$

We define a binary operation on $\tilde{X}$ as shown below:

$$x_\alpha \to y_\sigma = (x \to y)_{min(\alpha,\sigma)}$$

We can immediately see that $(\tilde{X}, \to)$ satisfies the following conditions:
for any $x_\alpha, y_\sigma, z_\gamma \in \tilde{X}$
$(BE1^{'})$ $x_\alpha \to x_\alpha = (x \to x)_{min(\alpha,\alpha)} = 1_{min(\alpha,\alpha)} = 1_\alpha$,
$(BE2^{'})$ $x_\alpha \to 1_\sigma = (x \to 1)_{min(\alpha,\sigma)} = 1_{min(\alpha,\sigma)}$,
$(BE3^{'})$ $1_\alpha \to x_\sigma = (1 \to x)_{min(\alpha,\sigma)} = x_{min(\alpha,\sigma)}$,
$(BE4^{'})$

$$\begin{aligned}
x_\alpha \to (y_\sigma \to z_\gamma) =& x_\alpha \to (y \to z)_{min(\sigma,\gamma)} \\
=&(x \to (y \to z))_{min(\alpha,min(\sigma,\gamma))} \\
=&(x \to (y \to z))_{min(\alpha,\sigma,\gamma)} \\
=&(y \to (x \to z))_{min(\alpha,\sigma,\gamma)} \\
=&y_\sigma \to (x \to z)_{min(\alpha,\gamma)} \\
=&y_\sigma \to (x_\alpha \to z_\gamma).
\end{aligned}$$

Example 3.2. Let $X = \{a, b, c, 1\}$. Define a binary operation $(\to)$ on X as follows:

$\to$	a	b	c	1
a	1	a	a	1
b	1	1	a	1
c	1	a	1	1
1	a	b	c	1

Then $(X, \to, 1)$ is a BE-algebra. In this example we give $\tilde{X} = \{a_\alpha, b_\sigma, c_\gamma, 1_\theta : \alpha, \sigma, \gamma, \theta \in (0,1]\}$. Obviously, $\tilde{X}$ is satisfied in (BE1$^{'}$) and (BE4$^{'}$). Assume that $\alpha = 0.5$, $\sigma = 0.7$ and $\gamma = 0.75$, we have $a_\alpha \to 1_\sigma = (a \to 1)_{min(0.5,0.7)} = 1_{min(0.5,0.7)} = 1_{0.5} \neq 1_\sigma$. Also $1_\gamma \to a_\sigma = (1 \to a)_{min(0.75,0.7)} = a_{min(0.75,0.7)} = a_{0.7} \neq a_\sigma$. Then (BE2$^{'}$) and (BE3$^{'}$) are not confirmed.

Remark 3.3. The conditions (BE2) and (BE3) does not hold in $(\tilde{X}, \to)$, generally. For this reason, we will call $(\tilde{X}, \to)$ a weak BE-algebra. Also, if X is a commutative BE-algebra, then the condition $x_\alpha \to y_\sigma = y_\sigma \to x_\alpha = 1_{min(\alpha,\sigma)}$ implies $x_\alpha = y_\sigma$ is not true in $(\tilde{X}, \to)$. Therefore the binary relation $\leq$ is not a partial order on $\tilde{X}$. Then we will call $(\tilde{X}, \to)$ a weak commutative BE-algebra.

We can also establish the following conditions: for any $x_\alpha, y_\sigma, z_\gamma \in \tilde{X}$.
(BE5$'$)

$$\begin{aligned} x_\alpha \to (y_\sigma \to x_\alpha) =& x_\alpha \to (y \to x)_{min(\alpha,\sigma)} \\ =& (x \to (y \to z))_{min(\alpha,min(\sigma,\gamma))} \\ =& 1_{min(\alpha,min(\sigma,\gamma)} \\ =& 1_{min(\alpha,\sigma)}. \end{aligned}$$

(BE6$'$)

$$\begin{aligned} x_\alpha \to ((x_\alpha \to y_\sigma) \to y_\sigma) = & x_\alpha \to (((x \to y)_{min(\alpha,\sigma)}) \to y_\sigma) \\ =& x_\alpha \to ((x \to y) \to y)_{min(\alpha,\sigma)} \\ =& (x \to ((x \to y) \to y))_{min(\alpha,\sigma)} \\ =& 1_{min(\alpha,\sigma)}. \end{aligned}$$

(BE7$'$) $x_\alpha \to (y_\sigma \wedge x_\alpha) = 1_{min(\alpha,\sigma)}$, where $(x_\alpha \wedge y_\sigma) = (y_\sigma \to x_\alpha) \to x_\alpha$.
We also recall that if μ is a fuzzy subset of a BE-algebra X, then we have:

$$\tilde{\mu} = \{x_\alpha : \mu(x) \geq \alpha, x_\alpha \in \tilde{X}, \alpha \in (0, 1]\}$$

for any $\alpha \in (0, 1]$

$$\tilde{X}_\alpha = \{x_\alpha : x_\alpha \in \tilde{X}\}, \qquad \tilde{\mu}_\alpha = \{x_\alpha : x_\alpha \in \tilde{\mu}\}$$

We have $\tilde{X}_\alpha \subseteq \tilde{X}, \tilde{\mu} \subseteq \tilde{X}, \tilde{\mu}_\alpha \subseteq \tilde{\mu}$ and $\tilde{\mu}_\alpha \subseteq \tilde{X}_\alpha$.

Theorem 3.4. *$(\tilde{X}_\alpha, \to, 1_\alpha)$ is a BE-algebra.*

proof. The proofs of (BE1), (BE2) and (BE3) are completely clear.
(BE4)

$$\begin{aligned}
x_\alpha \to (y_\alpha \to z_\alpha) =& x_\alpha \to (y \to z)_{min(\alpha,\alpha)}\\
=&(x \to (y \to z))_{min(\alpha,min(\alpha,\alpha))}\\
=&(x \to (y \to z))_{min(\alpha,\alpha,\alpha)}\\
=&(y \to (x \to z))_{min(\alpha,\alpha,\alpha)}\\
=&y_\alpha \to (x \to z)_{min(\alpha,\alpha)}\\
=&y_\alpha \to (x_\alpha \to z_\alpha).
\end{aligned}$$

Therefore $(\tilde{X}_\alpha, \to, 1_\alpha)$ is a BE-algebra, for any $\alpha \in (0, 1]$.

3.1. **Weak filters in BE-algebras.** Now, we introduce and discuss the notion of weak filters in a BE-algebra X.

Definition 3.5. Suppose that μ is a fuzzy set of X. Then $\tilde{\mu}$ is called a weak filter if:
i) $x_\alpha \wedge y_\sigma \in \tilde{\mu}$ and $y_\sigma \wedge x_\alpha \in \tilde{\mu}$, for any $x_\alpha, y_\sigma \in \tilde{\mu}$.
ii) $x_\alpha \in \tilde{\mu}$ and $x_\alpha \leq y_\sigma$ imply $y_{min(\alpha,\sigma)} \in \tilde{\mu}$.

Example 3.6. For BE-algebra thought out in Example 1 we define a fuzzy set $\mu : X \to [0, 1]$ as follows:

$$\mu(x) = \begin{cases} 1 & x = 1 \\ 0 & otherwise \end{cases}$$

Since $\mu(a) = \mu(b) = \mu(c) = 0$ and $\mu(1) = 1$, we have $\tilde{\mu} = \{1_\alpha\}$, for any $\alpha \in (0, 1]$. It can be easily verified that $\tilde{\mu}$ is a weak filter of $\tilde{X}$.

Theorem 3.7. *Let F be a non-empty subset of X and $\alpha \in (0, 1]$. Then the following conditions are equivalent:*
i) F is a filter of X,
ii) α_{χ_F} is a fuzzy filter,
iii) $\tilde{\alpha_{\chi_F}}$ is a weak filter.

proof. $i \implies ii$) Suppose that F is a filter of X. Let $x, y \in X$ be such that $x, y \in F$. We have $\alpha_{\chi_F}(x) = \alpha$ and $\alpha_{\chi_F}(y) = \alpha$. Since F is a filter, $x \wedge y \in F$ and

$y \wedge x \in F$ and hence $\alpha_{\chi F}(x \wedge y) = \alpha = \alpha_{\chi F}(y \wedge x)$ such that

$$\alpha_{\chi F}(x \wedge y) \geq min(\alpha_{\chi F}(x), \alpha_{\chi F}(y))$$

and

$$\alpha_{\chi F}(y \wedge x) \geq min(\alpha_{\chi F}(x), \alpha_{\chi F}(y)).$$

If $x \notin F$ or $y \notin F$, $min(\alpha_{\chi F}(x), \alpha_{\chi F}(y)) = 0$ such that

$$\alpha_{\chi F}(x \wedge y) \geq min(\alpha_{\chi F}(x), \alpha_{\chi F}(y))$$

and

$$\alpha_{\chi F}(y \wedge x) \geq min(\alpha_{\chi F}(x), \alpha_{\chi F}(y)).$$

Now, let $x, y \in X$ and $x \leq y$, we must show that $\alpha_{\chi F}(x) \leq \alpha_{\chi F}(y)$. If $x \in F$, we have $\alpha_{\chi F}(x) = \alpha$, since F is a filter $y \in F$ and $\alpha_{\chi F}(x) = \alpha$ such that $\alpha_{\chi F}(x) \leq \alpha_{\chi F}(y)$. If $x \notin F$, $\alpha_{\chi F}(x) = 0$ and we obtain $\alpha_{\chi F}(x) \leq \alpha_{\chi F}(y)$.
$ii \Longrightarrow iii)$ Suppose that $\alpha_{\chi F}$ is a fuzzy filter of X, and let $x_\beta, y_\sigma \in \tilde{\alpha}_{\chi_F}$, for $\beta, \sigma \in (0,1]$. Hence $\alpha_{\chi F}(x) \geq \beta$ and $\alpha_{\chi F}(y) \geq \sigma$. Since $\alpha_{\chi F}$ is a fuzzy filter, $\alpha_{\chi F}(x \wedge y) \geq min\{\alpha_{\chi F}(x), \alpha_{\chi F}(y)\} \geq min(\beta, \sigma)$ and $\alpha_{\chi F}(y \wedge x) \geq min\{\alpha_{\chi F}(x), \alpha_{\chi F}(y)\} \geq min(\beta, \sigma)$ such that $x_\beta \wedge y_\sigma \in \tilde{\alpha}_{\chi_F}$ and $y_\sigma \wedge x_\beta \in \tilde{\alpha}_{\chi_F}$. Now, let $x_\beta \in \tilde{\alpha}_{\chi_F}$ and $x_\beta \leq y_\sigma$. Therefore $\alpha_{\chi F}(x) \geq \beta$ and $x \leq y$. Since $\alpha_{\chi F}$ is a fuzzy filter $\alpha_{\chi F}(y) \geq \alpha_{\chi F}(x) \geq \beta \geq min(\beta, \sigma)$ such that $y_{min(\beta,\sigma)} \in \tilde{\alpha}_{\chi_F}$.
$iii \Longrightarrow i)$ If $1 \notin F$, then $\alpha_{\chi F}(1) = 0$. But $\tilde{\alpha}_{\chi_F}$ is a weak filter, then $\alpha_{\chi F}(x) \leq \alpha_{\chi F}(1) = 0$, for all $x \in X$. Hence $\alpha_{\chi F}(x) = 0$, for every $x \in F$. That means F is empty. This is a contradiction. Therefore $1 \in F$.
Now, let $x, y \in X$ be such that $x, x \longrightarrow y \in F$, but $y \notin F$. Then $\alpha_{\chi F}(x) = \alpha$ and $\alpha_{\chi F}(x \to y) = \alpha$ and $\alpha_{\chi F}(y) = 0$. Since $\alpha_{\chi F}$ is a fuzzy set, $min\{\alpha_{\chi F}(x), \alpha_{\chi F}(x \to y)\} = \alpha \leq \alpha_{\chi F}(y) = 0$. Hence $\alpha \leq 0$. As $\alpha \in (0,1]$, we have a contradiction. So $y \in F$ and hence F is a filter of X.

Theorem 3.8. *A fuzzy subset μ of X is a fuzzy filter iff $\tilde{\mu}$ is a weak filter.*

proof. Suppose that $\tilde{\mu}$ is a week filter, we desire to prove that μ is a fuzzy filter. Let $x, y \in X$ and $\alpha = min\{\mu(x), \mu(y)\}$. Then $x_\alpha, y_\alpha \in \tilde{\mu}$ and because $\tilde{\mu}$ is a weak filter, we have $x_\alpha \wedge y_\alpha \in \tilde{\mu}$ and $y_\alpha \wedge x_\alpha \in \tilde{\mu}$ such that $\mu(x \wedge y) \geq \alpha = min\{\mu(x), \mu(y)\}$ and $\mu(y \wedge x) \geq \alpha = min\{\mu(x), \mu(y)\}$. By assumption $x, y \in X$ and $x \leq y$, we must show that $\mu(x) \leq \mu(y)$. If $\alpha = \mu(x)$, then $x_\alpha \in \tilde{\mu}$ and since $x \leq y$, we have

$x \to y = 1$. Hence $(x \to y)_\alpha = 1_\alpha$. Then we obtain $x_\alpha \to y_\alpha = 1_\alpha$. By use these fact that $\tilde{\mu}$ is a weak filter, we give $y_\alpha \in \tilde{\mu}$ such that $\mu(y) \geq \alpha = \mu(x)$.
Conversely, suppose that μ is a fuzzy filter of X, we show that $\tilde{\mu}$ is a weak filter. Let $x_\alpha, y_\sigma \in \tilde{\mu}$. Hence $\mu(x) \geq \alpha$ and $\mu(y) \geq \sigma$. Since μ is a fuzzy filter, $\mu(x \wedge y) \geq min\{\mu(x), \mu(y)\} \geq min(\alpha, \sigma)$ and $\mu(y \wedge x) \geq min\{\mu(x), \mu(y)\} \geq min(\alpha, \sigma)$ such that $x_\alpha \wedge y_\sigma \in \tilde{\mu}$ and $y_\sigma \wedge x_\alpha \in \tilde{\mu}$. Now, let $x_\alpha \in \tilde{\mu}$ and $x_\alpha \to y_\alpha = 1_\alpha$. Therefore $\mu(x) \geq \alpha$ and $x \leq y$. Since μ is a fuzzy filter $\mu(y) \geq \mu(x) \geq \alpha \geq min(\alpha, \sigma)$ such that $y_{min(\alpha,\sigma)} \in \tilde{\mu}$.

Remark 3.9. A weak filter $\tilde{\mu}$ has the following property:
$x_\alpha \to y_\beta = 1_{min(\alpha,\beta)}$ and $x_\alpha \in \tilde{\mu} \Longrightarrow x_{min(\alpha,\beta)} \in \tilde{\mu}$.

Clearly, let $x_\alpha, y_\beta \in \tilde{X}$ such that $x_\alpha \to y_\beta = 1_{min(\alpha,\beta)}$ and $x_\alpha \in \tilde{\mu}$.
Now, $x_\alpha \in \tilde{\mu}$ implies that $\alpha \leq \mu(x)$. Suppose that $\mu(x) = t$. By use definition we obtain $1_t \in \tilde{\mu}$. Therefore $t \leq \mu(1)$. But $t = \mu(x) \geq \beta \geq min(\alpha, \beta)$. So $1_{min(\alpha,\beta)} \in \tilde{\mu}$. Using definition we obtain $x_{min(\alpha,\beta)} \in \tilde{\mu}$.

Theorem 3.10. *Let $f : X \longrightarrow Y$ be onto homomorphism. For a fuzzy set μ in Y, $\tilde{\mu}$ is a weak filter in Y if and only if $\tilde{\mu}^f$ is a weak filter in X.*

proof. At first we show that the fuzzy set μ in Y is a fuzzy filter if and only if μ^f is a fuzzy filter in X. Assume that μ is a fuzzy filter of Y. For any $x, y \in X$, we have
$\mu^f(1) = \mu(f(1)) = \mu(1) \leq \mu(f(x)) = \mu^f(x)$. Also

$$\begin{aligned}
\mu^f(y) =&\mu(f(y)) \\
\geq& min\{\mu(f(x)), \mu(f(x) \to f(y))\} \\
=& min\{\mu(f(x)), \mu(f(x \to y))\} \\
=& min\{\mu^f(x), \mu^f(x \to y)\}.
\end{aligned}$$

Hence μ^f is a fuzzy filter of X.
Conversely, assume that μ^f is a fuzzy filter of X. Let $y \in Y$. As f is onto, then there exists $x \in X$ such that $f(x) = y$. Then $\mu(1) = \mu(f(1)) = \mu^f(1) \geq \mu^f(x) = \mu(f(x)) = \mu(y)$.
Now, let $x, y \in Y$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$.

Hence we get

$$\begin{aligned}\mu(y) =&\mu(f(b))\\ =&\mu^f(b)\\ \geq& min\{\mu^f(a), \mu^f(f(a \to b)\}\\ =&min\{\mu(f(a)), \mu(f(a \to b))\}\\ =&min\{\mu(f(a)), \mu(f(a) \to f(b))\}\\ =&min\{\mu(x), \mu(x \to y)\}.\end{aligned}$$

So μ is a fuzzy filter in Y. Hence for a fuzzy set μ in Y by use Theorem 3.8, $\tilde{\mu}$ is a weak filter in Y if and only if μ is a fuzzy filter in Y if and only if μ^f is a fuzzy filter in X if and only if $\tilde{\mu}^f$ is a weak filter in X.
The appendix theorem describe weak fuzzy filters in self-distributive BE-algebras.

Theorem 3.11. *Let μ be a fuzzy set of a self-distributive BE-algebra X. Then $\tilde{\mu}$ is a weak filter iff $\tilde{\mu}$ satisfies in the following conditions:*
i) $x_\alpha \in \tilde{\mu}$ and $y_{min(\alpha,\sigma)} \notin \tilde{\mu}$ implies $x_\alpha \to y_\sigma \in \tilde{\mu}$.
ii) $x_\alpha \in \tilde{\mu}$ and $x_\alpha \to y_\sigma \in \tilde{\mu}$ implies $y_{min(\alpha,\sigma)} \in \tilde{\mu}$.

proof. Obviously, (i) and (ii) are equivalent. Suppose that $\tilde{\mu}$ is a weak filter, we must prove that $\tilde{\mu}$ satisfies (i). Assume $x_\alpha \in \tilde{\mu}$ and $y_{min(\alpha,\sigma)} \notin \tilde{\mu}$. Using (BE5$'$) and self-distributivity of X, we obtain $1_\alpha \to (x_\alpha \to y_\sigma) = (1_\alpha \to x_\alpha) \to (1_\alpha \to y_\sigma) = [1_\alpha \to (1_\alpha \to y_\sigma)] \to x_\alpha$.
If $x_\alpha \to y_\sigma \in \tilde{\mu}$, then $([1_\alpha \to (1_\alpha \to y_\sigma)] \to x_\alpha \in \tilde{\mu}$. So $[1_\alpha \to (1_\alpha \to y_\sigma)] \to x_\alpha \in \tilde{\mu}$. Since $x_\alpha \in \tilde{\mu}$ and $\tilde{\mu}$ is a weak filter, we give $1_\alpha \to (1_\alpha \to y_\sigma) \in \tilde{\mu}$. But from $[1_\alpha \to (1_\alpha \to y_\sigma)] \to y_\sigma = 1_{min(\alpha,\sigma)}$ and since $\tilde{\mu}$ is a weak filter, we obtain $y_{min(\alpha,\sigma)} \in \tilde{\mu}$ which is a contradiction. Thus $\tilde{\mu}$ satisfies (i).
Conversely, suppose that $\tilde{\mu}$ satisfies (ii), we must show that $\tilde{\mu}$ is a weak filter. Let $x_\alpha \in \tilde{\mu}$ and $y_\sigma \in \tilde{\mu}$. From (BE7$'$) $y_\sigma \to (x_\alpha \wedge y_\sigma) = 1_{min(\alpha,\sigma)}$ and $x_\alpha \to (y_\sigma \wedge x_\alpha) = 1_{min(\alpha,\sigma)}$. Because $\tilde{\mu}$ is a weak filter, $(x_\alpha \wedge y_\sigma) \in \tilde{\mu}$ and $(y_\sigma \wedge x_\alpha) \in \tilde{\mu}$. Now, let $x_\alpha \in \tilde{\mu}$ and $x_\alpha \to y_\sigma = 1_{min(\alpha,\sigma)}$. Since $\tilde{\mu}$ is a weak filter, $x_\alpha \to y_\sigma \in \tilde{\mu}$. Using (ii), we obtain $y_{min(\alpha,\sigma)} \in \tilde{\mu}$.

Example 3.12. Let $X = \{0, a, b, c, d, 1\}$ be a BE-algebra with the following Cayley table

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

It is not difficult to verify that $(X, \to, 1)$ is a commutative self-distributive BE-algebra. Define a mapping $\mu : X \to [0, 1]$ by $\mu(1) = \mu(a) = \mu(b) = 1$ and $\mu(0) = \mu(c) = \mu(d) = 0.75$ Then μ is a fuzzy filter of X. By routine calculations $\tilde{\mu} = \{1_\alpha, a_\sigma, b_\gamma\}$ for all $\alpha, \beta, \gamma \in (0, 1]$. $\tilde{\mu}$ is satisfies in condition (ii) of Theorem 3.11. Therefore $\tilde{\mu}$ is a weak filter.

Lemma 3.13. *The cartesian product of two weak filters is again a weak filter.*

proof. Let μ and ν be fuzzy sets in X such that $\tilde{\mu}$ and $\tilde{\nu}$ are two weak filters. Then μ and ν are fuzzy filters of X. Hence $\mu \times \nu$ is a fuzzy filter. By use Theorem 3.8 we obtain $\mu \times \nu$ is a weak filter.

Corollary 3.14. *Let μ be a fuzzy subset of X. Then $\tilde{\mu}_\alpha$ is a weak filter in $\tilde{X}$ if and only if μ_α is a filter of X, when $\mu_\alpha \neq \emptyset$ and $\alpha \in (0, 1]$.*

proof. Let μ be a fuzzy subset of X. By use Theorems 2.5, 3.8 $\tilde{\mu}_\alpha$ is a weak filter in $\tilde{X}$ if and only if μ is a fuzzy filter of X if and only if μ_α is a filter of X.

4. Conclusion

The aim of this paper is to develop the fuzzy filter of BE-algebras. In this paper, we introduced the concept of the set fuzzy points of a BE-algebra X and investigated some related properties. Applying this concept, we considered weak BE-algebra. Also, we introduced a characterization of the weak fuzzy filters in BE-algebras. Some important issues for future work are:

We can discussed relations between m-polar fuzzy subalgebras, m-polar fuzzy filters with weak fuzzy filters in BE-algebras. The concepts proposed in this article may be extended further to various kind of filters in CI-algebras, for example, (positive) implicative filters, n-fold (positive) implicative filters. Furthermore, the work presented in this paper may be extended to several algebraic structures, for example CI-algebras, Q-algebras, semigroups, semirings and lattice implication algebras.

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