TAXICAB SPHERICAL INVERSIONS IN TAXICAB SPACE

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ABSTRACT. In this paper, we define an inversion with respect to a taxicab sphere in the three dimensional taxicab space and prove several properties of this inversion. We also study cross ratio, harmonic conjugates and the inverse images of lines, planes and taxicab spheres in three dimensional taxicab space.

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1. INTRODUCTION

The inversion was introduced by Apollonius of Perga in his last book Plane Loci, and systematically studied and applied by Steiner about 1820s, [2]. During the following decades, many physicists and mathematicians independently rediscovered inversions, proving the properties that were most useful for their particular applications by defining a central cone, ellipse and circle inversion. Some of these features are inversion compared to the classical circle.

Inversion transformation and basic concepts have been presented in literature. The inversions with respect to the central conics in real Euclidean plane was introduced in [3]. Then the inversions with respect to ellipse was studied detailed in

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[13]. In three-dimensional space a generalization of the spherical inversion is given in [16]. Also, the inversions with respect to the taxicab distance, α-distance [18], [4], or in general a p-distance [11].

Taxicab geometry was founded by a gentleman named H. Minkowski. Minkowski was one of the developers in "non-Euclidean" geometry [8]. The taxicab plane geometry has been studied and improved by some mathematicians (for some references see [5], [6], [7], [7], [11], [14] and [15]). On circular inversion in taxicab plane was studied detailed in [17].

The circle inversion have been generalized in three-dimensional space by using a sphere as the circle of inversion [16]. An inversion with respect to a sphere is a transformation of the area that transforms the sphere from the inside out. That is, the points outside the sphere are mapped to the points inside the sphere and the points inside the sphere are mapped outside the sphere.

In this study, we define a notion of inversion valid in three dimensional taxicab space. In particular, we define an inversion with respect to a taxicab sphere and prove several properties of this new transformation. Also we introduce inverse points, cross ratio, harmonic conjugates and the inverse images of lines, planes and taxicab spheres in three dimensional taxicab space.

2. Taxicab Spherical Inversions

In this section, we introduce the inversion in a taxicab sphere and we show several known properties.

The taxicab space $\mathbb{R}^3_T$ is almost the same as the Euclidean space $\mathbb{R}^3$. The points and lines are the same, and the angles are measured the same way, but the distance function is different. In $\mathbb{R}^3$ the taxicab metric is defined using the distance function

$$d_T(A,B) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$$

where $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$ in $\mathbb{R}^3$. The unit ball in $\mathbb{R}^3$ is the set of points (x, y, z) in space which satisfy the equation $|x| + |y| + |z| = 1$.

We can define the notion of inversion in $\mathbb{R}^3_T$ as an analogue of inversion in $\mathbb{R}^3$.

Definition 2.1. *Let S be a taxicab sphere centered at a point O with radius r in $\mathbb{R}^3_T$. The inversion in the taxicab sphere S on the taxicab spherical inversion respect* *to S is the function such that*

$$I_{(O,r)} : \mathbb{R}_T^3 - \{O\} \to \mathbb{R}_T^3 - \{O\}$$

defined by $I_{(O,r)}(P) = P'$ for $P \neq O$ where P' is on the $\overrightarrow{OP}$ and

$$d_T(O,P) \cdot d_T(O,P') = r^2.$$

The point P' is said to be the taxicab spherical inverse of P in S, S is called the sphere of inversion and O is called the center of inversion.

The taxicab spherical inversions with respect to the sphere, like reflections, are involutions. The fixed points of $I_{(O,r)}$ are exactly those that constitute the taxicab sphere S centered at O with radius r.

Some basic properties about spherical inversion in the following items. Note that it is possible to extend every property of the taxicab circle inversion to taxicab spherical inversion.

Theorem 2.2. *Let S be an taxicab sphere with the center O in the taxicab inversion $I_{(O,r)}$. If the point P is in the exterior of S then the point P', the inverse of P, is interior to C, and conversely.*

Proof. Let the point P be in the exterior of S, then $d_T(O,P) > r$. If $P' = I_{(O,r)}(P)$; then $d_T(O,P) \cdot d_T(O,P') = r^2$. Hence $r^2 = d_T(O,P) \cdot d_T(O,P') > r \cdot d_T(O,P')$ and $d_T(O,P') < r$. □

The inversion $I_{(O,r)}$ is undefined at the point O. However, we can add to the taxicab space a single point at infinite O_∞, which is the inverse of the center O of taxicab inversion sphere S. So, the inversion $I_{(O,r)}$ is one-to-one map of extended taxicab sphere.

Theorem 2.3. *Let S be a taxicab sphere with the center $O = (0,0,0)$ and the radius r in $\mathbb{R}_T^3$. If $P = (x,y,z)$ and $P' = (x',y',z')$ are inverse points with respect to the taxicab spherical inversion $I_{(O,r)}$, then*

$$\begin{cases} x' = \frac{r^2 x}{(|x|+|y|+|z|)^2} \\ y' = \frac{r^2 y}{(|x|+|y|+|z|)^2} \\ z' = \frac{r^2 z}{(|x|+|y|+|z|)^2} \end{cases}.$$

Proof. The equation of S is $|x| + |y| + |z| = r$. Suppose that $P = (x, y, z)$ and $P' = (x', y', z')$ are inverse points with respect to the taxicab spherical inversion $I_{(O,r)}$. Since the points O, P and P' are collinear and the rays $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are same direction,

$$\overrightarrow{OP'} = k \cdot \overrightarrow{OP},\ k \in \mathbb{R}^+$$

$$(x', y', z') = (kx, ky, kz).$$

From $d_T(O, P) \cdot d_T(O, P') = r^2$, $k = \frac{r^2}{(|x|+|y|+|z|)^2}$. Replacing the value of k in $(x', y', z') = (kx, ky, kz)$, the equations of x', y' and z' are obtained. □

Corollary 2.4. *Let S be a taxicab sphere with the center $O = (a, b, c)$ and the radius r in $\mathbb{R}^3_T$. If $P = (x, y, z)$ and $P' = (x', y', z')$ are inverse points with respect to the taxicab spherical inversion $I_{(O,r)}$, then*

$$\begin{cases} x' = a + \frac{r^2(x-a)}{(|x-a|+|y-b|+|z-c|)^2} \\ y' = b + \frac{r^2(y-b)}{(|x-a|+|y-b|+|z-c|)^2} \\ z' = c + \frac{r^2(z-c)}{(|x-a|+|y-b|+|z-c|)^2} \end{cases}.$$

Proof. Since the translation preserve distances in the taxicab plane [6], [17] by translating in $\mathbb{R}^3_T$ $(0, 0, 0)$ to (a, b, c) one can easily get the value of x', y' and z'. □

Theorem 2.5. *Let P, Q and O be any three collinear different points in $\mathbb{R}^3_T$. If the taxicab spherical inversion $I_{(O,r)}$ transform P and Q into P' and Q' respectively, then*

$$d_T(P', Q') = \frac{r^2 \cdot d_T(P, Q)}{d_T(O, P) \cdot d_T(O, Q)}. \tag{1}$$

Proof. Assume first that O, P, Q are collinear. From the Definition 2.1 we conclude that $d_T(O, P) \cdot d_T(O, P') = r^2 = d_T(O, Q) \cdot d_T(O, Q')$. Since the ratios of the Euclidean and taxicab distances along a line are same,

$$\begin{aligned} d_T(P', Q') &= |d_T(O, P') - d_T(O, Q')| \\ &= \left| \frac{r^2}{d_T(O, P)} - \frac{r^2}{d_T(O, Q)} \right| \\ &= \frac{r^2 \cdot d_T(P, Q)}{d_T(O, P) \cdot d_T(O, Q)} \end{aligned}$$

is obtained. □

When O, P, Q are not collinear, the equality in (1) is not valid in taxicab space $\mathbb{R}^3_T$. For example, for $O = (0,0,0)$, $P = (1,0,0)$, $Q = (1,1,1)$ and $r = 2$, the inversion $I_{(O,r)}$ transform P and Q into $P' = (4,0,0)$ and $Q' = (\frac{4}{9},\frac{4}{9},\frac{4}{9})$. It follows that $d_T(P,Q) = 2$, $d_T(P',Q') = \frac{40}{9}$, $d_T(O,P) = 1$ and $d_T(O,Q) = 3$.

Theorem 2.6. *Let P, Q and O be any three non-collinear different points in $\mathbb{R}^3_T$ and $I_{(O,r)}$ be the inversion such that transform P and Q into P' and Q' respectively. If P and Q lie on the lines with any direction*

$$D_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$$

or

$$D_2 = \{(1,1,1),(-1,1,1),(1,-1,1),(1,1,-1)\}$$

or

$$D_3 = \{(1,1,0),(1,0,1),(0,1,1),(1,-1,0),(1,0,-1),(0,1,-1)\}$$

through O

$$d_T(P',Q') = \frac{r^2 \cdot d_T(P,Q)}{d_T(O,P)\cdot d_T(O,Q)}.$$

Proof. Since translations preserve the taxicab distances, it is enough to consider the center O of the inversion sphere as the origin. Let P, $Q \in D_1$ in $\mathbb{R}^3_T$. If $P = (p,0,0)$ and $Q = (0,q,0)$, easily calculate that $d_T(O,P) = p$, $d_T(O,Q) = q$. Then the images of P and Q respect to $I_{(O,r)}$ are $P' = (\frac{r^2}{p},0,0)$ and $Q' = (0,\frac{r^2}{q},0)$. It follows that

$$d_T(P',Q') = \frac{r^2}{|p|} + \frac{r^2}{|q|} = \frac{r^2(|p|+|q|)}{|p|\,|q|} = \frac{r^2 \cdot d_T(P,Q)}{d_T(O,P)\cdot d_T(O,Q)}.$$

Let P, $Q \in D_2$ in $\mathbb{R}^3_T$. If $P = (p,p,p)$ and $Q = (-q,q,q)$ it is easily calculate that $d_T(O,P) = 3p$, $d_T(O,Q) = 3q$. Then the images of P and Q respect to $I_{(O,r)}$ are $P' = \left(\frac{r^2}{9p},\frac{r^2}{9p},\frac{r^2}{9p}\right)$ and $Q' = \left(-\frac{r^2}{9q},\frac{r^2}{9q},\frac{r^2}{9q}\right)$. It follows that

$$d_T(P',Q') = 2\frac{r^2\,|q-p|}{9\,|p|\,|q|} + \frac{r^2\,|q+p|}{9\,|p|\,|q|} = \frac{r^2 \cdot d_T(P,Q)}{d_T(O,P)\cdot d_T(O,Q)}.$$

Let P, $Q \in D_3$ in $\mathbb{R}^3_T$. If $P = (p,p,0)$ and $Q = (q,0,q)$, easily calculate that $d_T(O,P) = 2p$, $d_T(O,Q) = 2q$. Then the images of P and Q respect to $I_{(O,r)}$ are $P' = \left(\frac{r^2}{4p},\frac{r^2}{4p},0\right)$ and $Q' = \left(\frac{r^2}{4q},0,\frac{r^2}{4q}\right)$. It follows that

$$d_T(P',Q') = r^2\frac{|q-p|+|q|+|p|}{4\,|p|\,|q|} = \frac{r^2 \cdot d_T(P,Q)}{d_T(O,P)\cdot d_T(O,Q)}.$$

□

3. Cross Ratio and Harmonic Conjugates

The taxicab directed distance from the point A to the point B along a line l in $\mathbb{R}_T^2$ is denoted by $d_T[AB]$. If the ray with initial point A containing B has the positive direction of orientation, $d_T[AB] = d_T(A,B)$ and if the ray has the opposite direction, $d_T[AB] = -d_T(A,B)$ [11]. The taxicab cross ratio is preserved by the inversion in the taxicab circle in [17].

Now, we show the properties related to the taxicab cross ratio and harmonic conjugates in $\mathbb{R}_T^3$.

Definition 3.1. *Let A, B, C and D be four distinct points an oriented line in $\mathbb{R}_T^3$. We define the their taxicab cross ratio $(AB,CD)_T$ by*

$$(AB,CD)_T = \frac{d_T[AC]}{d_T[AD]} \cdot \frac{d_T[BD]}{d_T[BC]}.$$

It is known that the cross ratio is positive if both C and D are between A and B or if neither C nor D is between A and B, whereas the cross ratio is negative if the pairs $\{A,B\}$ and $\{C,D\}$ separate each other. Also, the cross ratio is an invariant under inversion in a sphere whose center is not any of the four points A, B, C and D, [11]. Similarly, this property is valid in taxicab sphere.

Theorem 3.2. *The inversion in a taxicab sphere in $\mathbb{R}_T^3$ preserves the taxicab cross ratio.*

Proof. Let A, B, C and D be four collinear points with the center of the inversion $I_{(O,r)}$ in $\mathbb{R}_T^3$. Let $I_{(O,r)}$ transform A, B, C and D into A', B', C' and D', respectively. The taxicab spherical inversion reverses the taxicab directed distance from the point A to the point B along a line l in $\mathbb{R}_T^3$ to the taxicab directed distance from the point B' to the point A' and preserves the separation or non separation of the pair A, B and CD. Hence it is suffices to show that. $\left|(A'B',C'D')_T\right| = |(AB,CD)_T|$. This follows Theorem 2.2

$$\begin{aligned}\frac{d_T\left(A',C'\right)}{d_T\left(A',D'\right)} \cdot \frac{d_T\left(B',D'\right)}{d_T\left(B',C'\right)} &= \frac{\frac{r^2 \cdot d_T(A,C)}{d_T(O,A)\cdot d_T(O,C)}}{\frac{r^2 \cdot d_T(A,D)}{d_T(O,A)\cdot d_T(O,D)}} \cdot \frac{\frac{r^2 \cdot d_T(B,D)}{d_T(O,B)\cdot d_T(O,D)}}{\frac{r^2 \cdot d_T(B,C)}{d_T(O,B)\cdot d_T(O,C)}} \\ &= \frac{d_T(A,C)}{d_T(A,D)} \cdot \frac{d_T(B,D)}{d_T(B,C)}.\end{aligned}$$

□

Definition 3.3. *Let A and B be two points on a line l in $\mathbb{R}^3_T$, any pair C and D on l for which*

$$\frac{d_T\,[AC]}{d_T\,[CB]} = \frac{d_T\,[AD]}{d_T\,[DB]}$$

is said to divide A and B harmonically. The points C and D are called taxicab harmonic conjugates with respect to A and B, and the taxicab harmonic set of points is denoted by $H(AB, CD)_T$.

It is clear that two distinct points C and D are taxicab harmonic conjugates with respect to A and B if and only if $(AB, CD)_T = -1$.

Theorem 3.4. *Let S be a taxicab sphere with the center O, and the line segment $[AB]$ a diameter of S in $\mathbb{R}^3_T$. Let P and P' be distinct points of the ray OA, which divide the segment $[AB]$ internally and externally. Then P and P' are taxicab harmonic conjugates with respect to A and B if and only if P and P' are inverse points with respect the taxicab spherical inversion $I_{(O,r)}$.*

Proof. Suppose that P and P' are taxicab harmonic conjugates with respect to A and B. Then

$$(AB, PP')_T = -1$$

$$\frac{d_T\,[AP]}{d_T\,[AP']} \cdot \frac{d_T\,[BP']}{d_T\,[BP]} = -1.$$

Since P divides the line segment $[AB]$ internally and P is on the ray OB, $d_T(P, B) = r - d_T(O, P)$ and $d_T(A, P) = r + d_T(O, P)$. Since P' divides the line segment $[AB]$ externally and P' is on the ray OB, $d_T(A, P') = d_T(O, P') + r$ and $d_T(B, P') = d_T(O, P') - r$. Hence

$$\frac{r + d_T(O, P)}{d_T(O, P') + r} \cdot \frac{d_T(O, P') - r}{d_T(O, P) - r} = -1$$

$$(r + d_T(O, P)) \cdot (d_T(O, P') - r) = (d_T(O, P') + r) \cdot (r - d_T(O, P))\,.$$

Simplifying the last equality, $d_T(O, P) \cdot d_T(O, P') = r^2$ is obtained. Therefore P and P' are taxicab inverse points with respect to the taxicab spherical inversion $I_{(O,r)}$. Conversely, if P and P' are taxicab inverse points with respect to the taxicab spherical inversion $I_{(O,r)}$, the proof is similar. □

4. Taxicab Spherical Inversions of Lines, Planes and Taxicab Spheres

It is well known that inversions with respect to circle transform lines and circles into lines and/ or circles in Euclidean plane and Hyperbolic plane.

The following features are well known for inversion in Euclidean plane:

i) Lines passing through the inversion center are invariant.

ii) Lines that do not pass through the center of inversion transform circles passing through the center of inversion.

iii) Circles passing through the center of inversion transform lines does not pass through the center of the inversion.

iv) Circles not passing through the center of inversion transform circles does not pass through the center of the inversion.

v) Circles with center of inversion transform circles with center of inversion.

In this section, we study the taxicab spherical inversion of lines, planes and taxicab spheres. The taxicab spherical inversion $I_{(O,r)}$ maps the lines, planes passing through O onto themselves.

The taxicab spherical inversion $I_{(O,r)}$ maps taxicab spheres with centered O onto taxicab spheres. But the taxicab spherical inversion of a sphere not passing through the centre of inversion is another taxicab sphere that does not contain the centre of inversion.

Theorem 4.1. *Consider the inversion $I_{(O,r)}$ in a taxicab sphere S with centre O. Every plane containing O is invariant under the inversion.*

Proof. It is clear that the straight lines containing O onto themselves. Let S be a taxicab sphere of inversion with equation $|x| + |y| + |z| = r$ and the plane $Mx + Ny + Tz = 0$. Applying $I_{(O,r)}$ to this plane gives

$$M\frac{r^2x'}{(|x'| + |y'| + |z'|)^2} + N\frac{r^2y'}{(|x'| + |y'| + |z'|)^2} + T\frac{r^2z'}{(|x'| + |y'| + |z'|)^2} = 0.$$

So, $Mx' + Ny' + Tz' = 0$ is obtained □

The inverse of a plane not containing O is not a taxicab sphere containing O.

Theorem 4.2. *The inverse of a taxicab sphere with the centre O with respect to the taxicab spherical inversion $I_{(O,r)}$ is a taxicab sphere containing O.*

Proof. Since the translation preserve distance in $\mathbb{R}^3_T$, we can take a taxicab sphere S of inversion with the equation $|x| + |y| + |z| = r$ and S the taxicab sphere $|x| + |y| + |z| = k$, $k \in \mathbb{R}^+$. Applying $I_{(O,r)}$ to S gives

$$|x'| + |y'| + |z'| = \frac{r^2}{k}.$$

Note that this is a taxicab sphere with the centre O. □

Theorem 4.3. *The inversion $I_{(O,r)}$ in a taxicab sphere S with centre O. Every edges, vertices and faces of taxicab sphere is invariant under the inversion.*

Proof. The points of taxicab sphere are mapped by $I_{(O,r)}$ back onto taxicab sphere from the Definition 2.1. Hence every edges, vertices and faces of taxicab sphere is invariant under $I_{(O,r)}$. □

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