FRACTIONAL q-DIFFERINTEGRAL OPERATOR RELATED TO UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

SH. NAJAFZADEH
DEPARTMENT OF MATHEMATICS, PAYAME NOOR UNIVERSITY
POST OFFICE BOX: 19395–3697, TEHRAN, IRAN
E-MAIL:NAJAFZADEH1234@YAHOO.IE

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Abstract. In this paper, we introduce a new subfamily of univalent functions defined in the open unit disk involving a fractional q–differintegral operator. Some results on coefficient estimates, weighted mean, convolution structure and convexity are discussed.

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1. Introduction

Let $\mathcal{A}_n$ denote the family of analytic and univalent functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, expressed in the type:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \qquad \big(a_k \geqslant 0, \quad n \in \mathbb{N} = \{1, 2, \ldots\}\big). \tag{1}$$

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The q–shifted factorial is defined for $w, q \in \mathbb{C}$, by:

$$(w,q) = \begin{cases} 1 & , \quad n=0, \\ (1-w)(1-wq)\cdots(1-wq^{n-1}) & , \quad n \in \mathbb{N}, \end{cases} \tag{2}$$

and according to the basic analogue of the gamma function:

$$(q^w,q)_n = \frac{\Gamma_q(w+n)(1-q)^n}{\Gamma_q(w)}, \qquad (n>0), \tag{3}$$

where the q–gamma function is given by:

$$\Gamma_q(y) = \frac{(q,q)_\infty(1-q)^{1-y}}{(q^y,q)_\infty}, \qquad (0<q<1). \tag{4}$$

If $|q|<1$, the relation (2) is meaningful for $n=\infty$ as a convergent product defined by:

$$(w,q)_\infty = \prod_{j=0}^{\infty}(1-wq^j). \tag{5}$$

The corresponding relation for q–gamma function is given by:

$$\Gamma_q(1+y) = \frac{(1-q^y)\Gamma_q(y)}{1-q}. \tag{6}$$

Jackson's q–derivative and q–integral of a function $f(z)$ defined on a subset of $\mathbb{C}$, respectively introduced by:

$$D_{q,z}f(z) = \frac{f(z)-f(qz)}{z(1-q)}, \qquad (z \neq 0, \quad q \neq 0), \tag{7}$$

and

$$\int_0^z f(t)d(t,q) = z(1-q)\sum_{k=0}^{\infty} q^k f\big(zq^k\big). \tag{8}$$

See [3] and [4]. Also [1, 6] and [8] are useful.

According to the relation:

$$\lim_{q\to 1^-} \frac{(q^w,q)_n}{(1-q)^n} = (w)_n, \tag{9}$$

we conclude that the q–shifted factorial (1) reduces to the familiar Pochhammer symbol:

$$(w)_n = w(w+1)\cdots(w+n-1). \tag{10}$$

The fractional q–integral operator $I^{w}_{q,z}f(z)$ of a function $f(z)$ of order w is given by:

$$
\begin{aligned}
I^{w}_{q,z}f(z) &= D^{-w}_{q,z}f(z) \\
&= \frac{1}{\Gamma_q(w)}\int_0^z (z-tq)_{w-1}f(t)d(t,q), \qquad (w>0),
\end{aligned} \tag{11}
$$

where $f(z)$ is holomorphic in a simply connected region around the origin. On the other hand, the q–binomial function $(z-tq)_{w-1}$ is single–valued when:

$$
\left|\arg\left(\frac{-tq^w}{z}\right)\right| < \pi, \quad \left|\frac{tq^w}{z}\right| < 1, \quad \text{and} \quad |\arg z| < \pi. \tag{12}
$$

The fractional q–derivative operator $D^{w}_{q,z}f(z)$ of a function $f(z)$ of order w is introduced as:

$$
\begin{aligned}
D^{w}_{q,z}f(z) &= D_{q,z}I^{1-w}_{q,z}f(z) \\
&= \frac{1}{\Gamma_q(1-w)}D_{q,z}\int_0^z (z-tq)_{-w}f(t)d(t,q), \qquad (0 \leqslant w < 1).
\end{aligned} \tag{13}
$$

The extended fractional q–derivative operator for a function $f(z)$ of order w is given as follows, see [7]:

$$
D^{-w}_{q,z}f(z) = D^{m}_{q,z}I^{m-w}_{q,z}f(z), \qquad \big(m-1 \leqslant w < n, \quad m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}\big). \tag{14}
$$

Now, we consider a fractional q–differintegral operator $\Omega^{w}_{q,z}$ for a function $f(z)$ of the form (10) by:

$$
\begin{aligned}
\Omega^{w}_{q,z}f(z) &= \frac{\Gamma_q(2-w)}{\Gamma_q(2)}z^{w-1}D^{w}_{q,z}f(z) \\
&= 1 - \sum_{k=n+1}^{\infty}\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}a_k z^{k-1}.
\end{aligned} \tag{15}
$$

For more details see [5], see also [2].

We say that $f \in \mathcal{A}_n$ is in the class $\Omega^{w}_{q,z}(\alpha,\beta,\gamma,t)$, if it satisfies the inequality:

$$
\left|\frac{z^2\big(\Omega^{w}_{q,z}f(z)\big)'}{\big(\gamma+(\alpha-\gamma)(1-\beta)\big)f_t(z)+\gamma z\Omega^{w}_{q,z}f(z)}\right| < 1, \tag{16}
$$

where $0 \leq t \leq 1$, $-1 \leqslant \gamma < \alpha \leqslant 1$, $0 < \beta < 1$ and $f_t(z) = (1-t)z + tf(z)$.

2. Main results

In this section, the sharp coefficient bounds and weighted mean for functions in the family $\Omega_{q,z}^{w}(\alpha,\beta,\gamma,t)$ are found.

Theorem 2.1. *Let* $f(z)=z-\sum\limits_{k=n+1}^{\infty}a_k z^k$ *be analytic in* $\mathbb{U}$*. Then* $f\in\Omega_{q,z}^{w}(\alpha,\beta,\gamma,t)$ *if and only if:*

$$\sum_{k=n+1}^{\infty}\left[\frac{\Gamma_q(2-w)\Gamma_q(k+1)(k-1-\gamma)}{\Gamma_q(2)\Gamma_q(k+1-w)(\alpha-\gamma)(1-\beta)}+t\right]a_k\leqslant 1. \tag{17}$$

Proof. Let $|z|=1$ and (17) holds true. So we have:

$$\begin{aligned}&\left|z^2\big(\Omega_{q,z}^{w}f(z)\big)'\right|-\left|\big(\gamma+(\alpha-\gamma)(1-\beta)\big)f_t(z)-\gamma z\Omega_{q,z}^{w}f(z)\right|\\&=\left|-\sum_{k=n+1}^{\infty}\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}(k-1)a_k z^k\right|\\&-\left|(\alpha-\gamma)(1-\beta)z-\sum_{k=n+1}^{\infty}\Big(t(\alpha-\gamma)(1-\beta)-\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}\gamma\Big)a_k z^k\right|\\&\leqslant\left|\sum_{k=n+1}^{\infty}\Big(\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}(k-1-\gamma)+t(\alpha-\gamma)(1-\beta)\Big)a_k-(\alpha-\gamma)(1-\beta)\right|.\end{aligned}$$

By (17), the above inequality is less than or equal to zero, so $f(z)\in\Omega_{q,z}^{w}(\alpha,\beta,\gamma,t)$.

To prove the converse, let $f(z)\in\Omega_{q,z}^{w}(\alpha,\beta,\gamma,t)$, thus:

$$\begin{aligned}&\left|\frac{z^2\big(\Omega_{q,z}^{w}f(z)\big)'}{\big(\gamma+(\alpha-\gamma)(1-\beta)\big)f_t(z)+\gamma z\Omega_{q,z}^{w}f(z)}\right|\\&=\left|\frac{-\sum\limits_{k=n+1}^{\infty}\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}(k-1)a_k z^k}{(\alpha-\gamma)(1-\beta)z-\sum\limits_{k=n+1}^{\infty}\Big(t(\alpha-\gamma)(1-\beta)-\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}\gamma\Big)a_k z^k}\right|<1.\end{aligned}$$

Since for all $z\in\mathbb{U}$, $\mathrm{Re}\{z\}\leqslant|z|$, so we get:

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=n+1}^{\infty}\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}(k-1)a_k z^k}{(\alpha-\gamma)(1-\beta)z-\sum\limits_{k=n+1}^{\infty}\Big(t(\alpha-\gamma)(1-\beta)-\gamma\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}\Big)a_k z^k}\right\}<1.$$

By letting $z \to 1$ through positive values and choosing the values of z such that $z^2\big(\Omega^w_{q,z}f(z)\big)'$, $z\Omega^w_{q,z}f(z)$ and $f_t(z)$ are real, we conclude:

$$\sum_{k=n+1}^{\infty}\Big((k-1)\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}\Big)a_k$$
$$\leqslant(\alpha-\gamma)(1-\beta)-\sum_{k=n+1}^{\infty}\Big(t(\alpha-\gamma)(1-\beta)-\gamma\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}\Big)a_k.$$

So, we get:

$$\sum_{k=n+1}^{\infty}\Big(\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}(k-1-\gamma)+t(\alpha-\gamma)(1-\beta)\Big)a_k\leqslant(\alpha-\gamma)(1-\beta),$$

and this complete the proof. ☐

Remark 2.2. *We note that the function:*

$$F(z)=z-\sum_{k=n+1}^{\infty}\frac{(\alpha-\gamma)(1-\beta)}{\frac{\Gamma_q(2-w)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-w)}(k-1-\gamma)+t(\alpha-\gamma)(1-\beta)}z^k, \tag{18}$$

shows that the inequality (17) *is sharp.*

Also by applying Theorem 2.1, if $f(z)\in\Omega^w_{q,z}(\alpha,\beta,\gamma,t)$, then for $k\geqslant n+1$:

$$a_k\leqslant\frac{\Gamma_q(2)\Gamma_q(k+1-w)(\alpha-\gamma)(1-\beta)}{\Gamma_q(2-w)\Gamma_q(k+1)(k-1-\gamma)+t\Gamma_q(2)\Gamma_q(k+1-w)(\alpha-\gamma)(1-\beta)}. \tag{19}$$

Now, we introduce weighted mean property.

Theorem 2.3. *If f and g belong to $\Omega^w_{q,z}(\alpha,\beta,\gamma,t)$, then the weighted mean of f and g is also in the same class.*

Proof. We have to prove that:

$$h_m(z)=\frac{(1-m)f(z)+(1+m)g(z)}{2},$$

is in the class $\Omega^w_{q,z}(\alpha,\beta,\gamma,t)$.

Since $f(z)=z-\sum_{k=n+1}^{\infty}a_kz^k$ and $g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k$, so:

$$h_m(z)=z-\sum_{k=n+1}^{\infty}\Big(\frac{(1-m)a_k+(1+m)b_k}{2}\Big)z^k.$$

To prove $h_m(z)\in\Omega^w_{q,z}(\alpha,\beta,\gamma,t)$, by Theorem 2.1, we need to show that:

$$L=\sum_{k=n+1}^{\infty}\Big(\frac{\Gamma_q(2-w)\Gamma_q(k+1)(k-1-\gamma)}{\Gamma_q(2)\Gamma_q(k+1-w)(\alpha-\gamma)(1-\beta)}+t\Big)\Big(\frac{(1-m)a_k+(1+m)b_k}{2}\Big)<1.$$

But, for this we have:

$$L = \Big(\frac{1-m}{2}\Big) \sum_{k=n+1}^{\infty} \Big(\frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)}{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} + t\Big) a_k$$
$$+ \Big(\frac{1+m}{2}\Big) \sum_{k=n+1}^{\infty} \Big(\frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)}{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} + t\Big) b_k.$$

By (17), we get:

$$L < \frac{1-m}{2} + \frac{1+m}{2} = 1.$$

Hence the result follows. □

3. Convolution preserving and convexity

In this section, we show that the family $\Omega^w_{q,z}(\alpha,\beta,\gamma,t)$ is closed under convolution. Also we conclude that this class is a convex set.

Theorem 3.1. *Let* $f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k$ *and* $g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k$ *be in the class* $\Omega^w_{q,z}(\alpha,\beta,\gamma,t)$, *then* $(f*g)(z)$ *defined by:*

$$(f*g)(z) = z - \sum_{k=n+1}^{\infty} a_k b_k z^k, \tag{20}$$

belongs to $\Omega^w_{q,z}(\alpha,\hat{\beta},\gamma,t)$, *where:*

$$\hat{\beta} \leqslant 1 - \frac{(k-1-\gamma)(\alpha-\gamma)(1-\beta)^2\Gamma_q(2-w)\Gamma_q(k+1)}{W^2 - t(\alpha-\gamma)^2(1-\beta)^2\Gamma_q(2)\Gamma_q(k+1-q)}, \tag{21}$$

and

$$W = (k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w). \tag{22}$$

Proof. By Theorem 2.1, it is sufficient to show that:

$$\sum_{k=n+1}^{\infty} \Big(\frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)}{(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k+1-w)} + 1\Big) a_k b_k \leqslant 1.$$

By applying Cauchy-Schwarz inequality, from (17), we get:

$$\sum_{k=n+1}^{\infty} \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1) + t(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)}{t(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} \sqrt{a_k b_k} \leqslant 1.$$

Hence, we find the largest $\hat{\beta}$ such that:

$$\sum_{k=n+1}^{\infty} \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)+t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k+1-w)}{(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k+1-w)} a_k b_k$$
$$\leqslant \sum_{k=n+1}^{\infty} \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)+t(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)}{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} \sqrt{a_k b_k} \leqslant 1,$$

or equivalently:

$$\sqrt{a_k b_k} \leqslant \frac{\left[\dfrac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)+t(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)}{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)}\right]}{\left[\dfrac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)+t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k+1-w)}{(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k+1-w)}\right]}$$
$$= \frac{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)+t(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)}{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)+t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k+1-w)} \times \frac{1-\hat{\beta}}{1-\beta}.$$

This inequality holds if,

$$\frac{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)}{W}$$
$$\leqslant \frac{W}{1-\beta} \times \frac{1-\hat{\beta}}{(k+1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)+t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k+1-w)},$$

where W is given in (22). So

$$\frac{1-\hat{\beta}}{(k-1-\gamma)\Gamma_q(2-w)\Gamma_q(k+1)+t(\alpha-\gamma)(1-\hat{\beta})\Gamma_q(2)\Gamma_q(k+1-w)}$$
$$\geqslant \frac{(\alpha-\gamma)(1-\beta)^2}{W^2}.$$

After a simple calculation, we obtain the required result. □

Remark 3.2. *With the same assumptions of Theorem 3.1,* $(f*g)(z)$ *belongs to* $\Omega^w_{q,z}(\alpha,\beta,\hat{\gamma},t)$*, where:*

$$\hat{\gamma} \leqslant \frac{\alpha X-k+1}{X-1},$$
$$X = \frac{W^2 - t(1-\beta^2)(\alpha-\gamma)^2\Gamma_q(2)\Gamma_q(k+1-w)}{(\alpha-\gamma)^2(1-\beta)\Gamma_q(2-w)\Gamma_q(k+1)},$$

and W *is given in* (22).

Theorem 3.3. *The class* $\Omega^w_{q,z}(\alpha,\beta,\gamma,t)$ *is a convex set.*

Proof. It is enough to show that if $f_j(z)$ $(j = 1, 2, \dots, m)$ be in the class $\Omega^w_{q,z}(\alpha, \beta, \gamma, t)$, then the function:

$$H(z) = \sum_{j=1}^{m} \delta_j f_j(z), \tag{23}$$

is also in $\Omega^w_{q,z}(\alpha, \beta, \gamma, t)$, with $\delta_j \geqslant 0$ and $\sum_{j=1}^{m} \delta_j = 1$. By (23), we obtain:

$$\begin{aligned} H(z) &= \sum_{j=1}^{m} \delta_j \Big(z - \sum_{k=n+1}^{\infty} a_{k,j} z^k \Big) \\ &= z - \sum_{k=n+1}^{\infty} \Big(\sum_{j=1}^{m} \delta_j a_{k,j} \Big) z^k. \end{aligned}$$

But by Theorem 2.1, we have:

$$\begin{aligned} &\sum_{k=n+1}^{\infty} \Big(\frac{(k-1-\gamma)\Gamma_q(z-w)\Gamma_q(k+1)}{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} + t \Big) \Big(\sum_{j=1}^{m} \delta_j a_{k,j} \Big) \\ &= \sum_{j=1}^{m} \Big(\sum_{k=n+1}^{\infty} \Big(\frac{(k-1-\gamma)\Gamma_q(z-w)\Gamma_q(k+1)}{(\alpha-\gamma)(1-\beta)\Gamma_q(2)\Gamma_q(k+1-w)} + t \Big) a_{k,j} \Big) \delta_j \end{aligned}$$

by (17), we have:

$$\begin{aligned} &< \sum_{j=1}^{m} \delta_j \\ &= 1. \end{aligned}$$

So by Theorem 2.1, we get the required result. □

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