SOME RESULTS ON HERMITE-HADAMARD INEQUALITIES

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ABSTRACT. In this paper, we establish Hermite-Hadamard type inequalities for uniformly p-convex functions and uniformly q-convex functions. Also, we obtain some new inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value belong to the class of uniformly p-convex functions.

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1. Introduction

Many of integral inequalities are based on a convexity assumption of a certain function and the theory of inequality is one of the most important field study of

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convex analysis and abstract analysis. The major inequalities in these fields are Hermite-Hadamard inequalities, which can be stated as follows:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then the following inequalities holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Recently, the generalizations, improvements, variations and applications for convexity and the Hermite-Hadamard inequality have attracted the attention of many researchers (see [2],[7],[8],[9]). For example:

1) A function f is called a p-function if for all $x, y \in [a, b]$ and $t \in [0, 1]$ one has the inequality ([4]):

$$f(tx + (1-t)y) \leq f(x) + f(y),$$

and Hermite-Hadamard inequality for a p-function f is

$$f(\frac{a+b}{2}) \leq \frac{2}{b-a}\int_a^b f(t)dt \leq 2(f(a) + f(b)).$$

2) A function $f : I = [a, b] \to (0, +\infty)$ is said to belong to the class $Q(I)$ if for all $x, y \in [a, b]$ and $t \in (0, 1)$, satisfies the inequality ([5]):

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}$$

and for a function $f \in Q(I)$, one has the inequality

$$f(\frac{a+b}{2}) \leq \frac{4}{b-a}\int_a^b f(t)dt.$$

2. Preliminaries on uniformly p-convex function and uniformly q-convex functions

In this section, we consider the basic concepts and results, which are needed to obtain our main results.

In [1, Definition 10.5], the class of uniformly convex functions is defined. In the following we generalize the definition of uniformly convex functions in two ways:

Definition 2.1. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Then, f is called uniformly p-convex function with modulus $\psi:[0,+\infty)\to[0,+\infty]$ if ψ is incresaing, ψ vanishes only at 0, and

$$f(tx+(1-t)y)+t(1-t)\psi(|x-y|)\leq f(x)+f(y), \tag{1}$$

for each $x,y\in\mathbb{R}$ and $t\in[0,1]$.

Definition 2.2. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Then, f is called a uniformly q-convex function with modulus $\psi:[0,+\infty)\to[0,+\infty)$ if ψ is incresaing, ψ vanishes only at 0, and

$$f(tx+(1-t)y)+t(1-t)\psi(|x-y|)\leq\frac{f(x)}{t}+\frac{f(y)}{1-t}, \tag{2}$$

for each $x,y\in\mathbb{R}$ and $t\in(0,1)$.

In order to prove our main theorems, we need the following lemma that has been proved in [3].

Lemma 2.3. *Let $f:I^o\to\mathbb{R}$ be a differentiable function on I^o, $a,b\in I^o$ with $a<b$. If $f'\in L^1[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt.$$

The next theorem gives a new result of the Hermite-Hadamard inequalities for uniformly p-convex functions:

Theorem 2.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be uniformly p-convex function with modulus ψ. Then,*

$$f(\frac{a+b}{2})+\frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt\leq\frac{2}{b-a}\int_a^b f(t)dt$$
$$\leq 2(f(a)+f(b))-\frac{1}{3}\psi(|a-b|).$$

Proof. In (1), set $t=\frac{1}{2}$, then we have

$$f(\frac{x+y}{2})+\frac{1}{4}\psi(|x-y|)\leq f(x)+f(y). \tag{3}$$

Now, set $x = ta + (1-t)b$ and $y = (1-t)a + tb$ in (8) and integrate on $[0,1]$ with respect to t. We get

$$f(\frac{a+b}{2}) + \frac{1}{4}\int_0^1 \psi(|(2t-1)(a-b)|)dt \le \int_0^1 f(ta+(1-t)b)dt + \int_0^1 f((1-t)a+tb)dt.$$

Now,

$$\begin{aligned}\frac{1}{4}\int_0^1 \psi(|(2t-1)(a-b)|)dt &= \frac{1}{4}\int_{b-a}^{a-b} \psi(|u|)\frac{du}{2(a-b)} \\ &= \frac{1}{8(b-a)}\int_{a-b}^{b-a} \psi(|t|)dt.\end{aligned}$$

Also, we have $\int_0^1 f((1-t)a+tb)dt = \int_0^1 f((1-t)b+ta)dt = \frac{1}{b-a}\int_a^b f(t)dt$. Therefore

$$f(\frac{a+b}{2}) + \frac{1}{8(b-a)}\int_{a-b}^{b-a} \psi(|t|)dt \le \frac{2}{b-a}\int_a^b f(t)dt.$$

In the other hand, Put $x = a$ and $y = b$ in (1) and integrate on $[0,1]$ with respect to t. Then, we obtain

$$\int_0^1 f(ta+(1-t)b)dt + \int_0^1 t(1-t)\psi(|a-b|)dt \le \int_0^1 (f(a)+f(b))dt,$$

so

$$\frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{6}\psi(|a-b|) \le f(a)+f(b),$$

finally

$$\frac{2}{b-a}\int_a^b f(t)dt \le 2(f(a)+f(b)) - \frac{1}{3}\psi(|a-b|).$$

Which completes the proof.

□

If in Theorem 2.4, we set $\psi(t) = \frac{\beta}{2}t^2, \beta > 0$. Then, we obtain the following important inequality.

Corollary 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly p-convex function with modulus $\psi(t) = \frac{\beta}{2}t^2, \beta > 0$, then*

$$f(\frac{a+b}{2}) + \frac{\beta}{24}(b-a)^2 \le \frac{2}{b-a}\int_a^b f(t)dt \le 2(f(a)+f(b)) - \frac{\beta}{6}(b-a)^2. \tag{4}$$

Example 2.6. It is easy to see that, the following equality holds, for all $\alpha \in (0,1)$ and $x, y \in \mathbb{R}$.

$$(\alpha x + (1-\alpha)y)^2 + \alpha(1-\alpha)(x-y)^2 = \alpha x^2 + (1-\alpha)y^2. \tag{5}$$

Hence, we conclude that the function $f(t) = t^2$ for $t \in \mathbb{R}$ is uniformly p-convex with modulus $\psi(t) = t^2$ for all $t \geq 0$. Hence, in view of (4) for $a, b \in \mathbb{R}$ with $a > 0, b > 0$ one has

$$(\frac{a+b}{2})^2 + \frac{1}{12}(b-a)^2 \leq \frac{2}{3}(a^2 + ab + b^2) \leq \frac{5a^2 + 2ab + 5b^2}{3}.$$

Here we shall offer some applications of Lemma 2.3 connected with Hermite-Hadamard's integral inequality for convex functions which are very interesting.

Theorem 2.7. *Let $f : I^o \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ with $a < b$. If $|f'|$ is uniformly p-convex function with modulus ψ on I^o and $f' \in L^1[a,b]$. then the following inequality holds:*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{b-a}{4}(|f'(a)| + |f'(b)|) - \frac{b-a}{32}\psi(|a-b|).$$

Proof. In view of Lemma 2.3 and uniformly p-convexity of $|f'|$, one has

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||f'(ta+(1-t)b)|dt$$
$$\leq \frac{b-a}{2}\int_0^1 |1-2t|(|f'(a)| + |f'(b)| + t(t-1)\psi(|a-b|))dt$$
$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t||f'(a)|dt + \int_0^1 |1-2t||f'(b)|dt + \int_0^1 |1-2t|t(t-1)\psi(|a-b|))dt\right)$$
$$\leq \frac{b-a}{4}(|f'(a)| + (f'(b))) - \frac{b-a}{32}\psi(|a-b|).$$

Hence, The proof is complete. Also, note that

$$\int_0^1 |1-2t|dt = \frac{1}{2},$$
$$\int_0^1 |1-2t|t(t-1)\psi(|a-b|)dt = -\frac{1}{16}\psi(|a-b|).$$

□

Theorem 2.8. *Let $f : I^o \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ with $a < b$. Let $r, s > 1$ be such that $\frac{1}{r} + \frac{1}{s} = 1$. If $|f'|^s$ is uniformly p-convex function with modulus ψ on I^o and $f' \in L^1[a, b]$. then the following inequality holds:*

(6)

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{b-a}{2(r+1)^{\frac{1}{r}}}\left(|f'(a)|^s + |f'(b)|^s - \frac{1}{6}\psi(|a-b|)\right)^{\frac{1}{s}}.$$

Proof. By Lemma 2.3 and Hölder's inequality, we conclude

$$\begin{aligned}
&|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||f'(ta+(1-t)b)|dt \\
&\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^r dt\right)^{\frac{1}{r}}\left(\int_0^1 |f'(ta+(1-t)b)|^s dt\right)^{\frac{1}{s}} \\
&\leq \frac{b-a}{2}\frac{1}{(r+1)^{\frac{1}{r}}}\left(|f'(a)|^s\int_0^1 dt + |f'(b)|^s\int_0^1 dt + \psi(|a-b|)\int_0^1 t(t-1)dt\right)^{\frac{1}{s}} \\
&\leq \frac{b-a}{2(r+1)^{\frac{1}{r}}}\left(|f'(a)|^s + |f'(b)|^s - \frac{1}{6}\psi(|a-b|)\right)^{\frac{1}{s}}.
\end{aligned}$$

Hence, the proof is complete. □

Example 2.9. Let $f(t) = t^{\frac{2+s}{s}}$, for all $t \in \mathbb{R}$. Hence, $|f'(t)|^s = \left(\frac{2+s}{s}\right)|t|^2$. In view of example 2.6, $|f'(t)|^s$ is uniformly p-convex with modulus $\psi(t) = \left(\frac{2+s}{s}\right)t^2$. Let $r, s > 1$ be such that $\frac{1}{r} + \frac{1}{s} = 1$. Hence, from theorem 2.8 and inequality (6), one has

$$\begin{aligned}
&|\frac{a^{\frac{2+s}{s}} + b^{\frac{2+s}{s}}}{2} - \frac{s}{(b-a)(2+2s)}\left(b^{\frac{2+2s}{s}} - a^{\frac{2+2s}{s}}\right)| \\
&\leq \frac{(b-a)(2+s)}{2(r+1)^{\frac{1}{r}}s}\left(a^2 + b^2 - \frac{1}{6}(a-b)^2\right)^{\frac{1}{s}}
\end{aligned}$$

3. Hermite-Hadamard inequality for uniformly q-convex functions

The next theorem gives a new result of the Hermite-Hadamard inequalities for uniformly q-convex functions:

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly q-convex function with modulus ψ. Then,*

$$f(\frac{a+b}{2}) + \frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt \leq \frac{4}{b-a}\int_a^b f(t)dt. \tag{7}$$

Proof. In (2), set $t = \frac{1}{2}$, then we have

$$f(\frac{x+y}{2}) + \frac{1}{4}\psi(|x-y|) \leq 2f(x) + 2f(y). \tag{8}$$

Now, set $x = ta + (1-t)b$ and $y = (1-t)a + tb$ in (8) and integrate on $[0,1]$ with respect to t. We get

$$f(\frac{a+b}{2}) + \frac{1}{4}\int_0^1 \psi(|(2t-1)(a-b)|)dt \leq 2\int_0^1 f(ta+(1-t)b)dt + 2\int_0^1 f((1-t)a+tb)dt.$$

Hence

$$\begin{aligned}\frac{1}{4}\int_0^1 \psi(|(2t-1)(a-b)|)dt &= \frac{1}{4}\int_{b-a}^{a-b} \psi(|u|)\frac{du}{2(a-b)} \\ &= \frac{1}{8(b-a)}\int_{a-b}^{b-a} \psi(|t|)dt.\end{aligned}$$

Also, we have $\int_0^1 f((1-t)a+tb)dt = \int_0^1 f((1-t)b+ta)dt = \frac{1}{b-a}\int_a^b f(t)dt$. Therefore

$$f(\frac{a+b}{2}) + \frac{1}{8(b-a)}\int_{a-b}^{b-a} \psi(|t|)dt \leq \frac{4}{b-a}\int_a^b f(t)dt.$$

that completes the proof. □

Example 3.2. In view of example 2.6, we know that the function $f(t) = t^2$ for $t \in \mathbb{R}$ is uniformly q-convex with modulus $\psi(t) = t^2$ for all $t \geq 0$. Hence, in view of (7) for $a, b \in \mathbb{R}$ with $a > 0, b > 0$, one has

$$(\frac{a+b}{2})^2 + \frac{1}{12}(b-a)^2 \leq \frac{4}{3}(a^2+ab+b^2).$$

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