AN EFFICIENT NUMERICAL APPROACH FOR SOLVING THE VARIABLE-ORDER TIME FRACTIONAL DIFFUSION EQUATION USING CHEBYSHEV SPECTRAL COLLOCATION METHOD

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Abstract. In this paper we consider the one-dimensional variable-order time fractional diffusion equation where the order is $q(x,t) \in (0,1)$. One type of Caputo fractional derivative is introduced and to get a numerical technique, the time variable is discretized using a finite difference plan then we use a spectral collocation method to discretize the spatial derivative. In order to show the effectiveness and accuracy of this method, some test problems are considered, and it is shown that the obtained results are in very good agreement with exact solutions.

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1. Introduction

As far as we are concerned, the theory of fractional partial differential equations (FPDE), as a new and effective mathematical tool, is very popular and important in many scientific and engineering problems because it is more adequate than integer order models and provides an excellent tool for description of memory and heredity effects of various materials and processes like gas diffusion and heat diffusion in fractal porous media [2, 3], it can be refered to [1] for more information. Thanks to the increasing use of fractional derivative and fractional calculus in ordinary and partial differential equations and related problems, there is an interest for presenting efficient and reliable solutions for them.

Recently, researchers have understood that many dynamic processes have fractional-order behaviour that depend on time or space, therefore it is important to extend the concept of variable-order calculus. Currently, variable-order calculus has been applied in many different field such as riscoelastic mechanics [4], geographic data [5], signal and confirmation [6]. variable-order operator is a new development in science.
[24, 23] generalized the Riemann-Liouville and Marchaud fractional integration and differentiation of variable order and presented some properties. Different researchers have introduced different definition of variable-order differential operators, which each of them has specific meaning for specific golas.

Today, various numerical methods for variable order fractional differential equations are applied such as spectral method [16, 17, 18], finite difference method [7, 8, 9, 10, 11, 12, 13, 14, 15], reproducing kernel method [21, 22], matrix methods [19, 20] by many researchers.
The fractional models can be divided into three types: space fractional differential equation, time fractional differential equation and space-time fractional differential equation. [7] presented a numerical method for variable order time-space fractional-diffusion equation. Phanikumar et al. [8] considered an implicit Euler numerical method for the time variable fractional order mobile-immobile advection-dispersion model in [8]. [10] presented an implicit and explicit Euler approximation for the

variable order fractional advection-diffusion equation with a nonlinear source term.

In the existing literature, there is little work on higher-order numerical methods for the multi-term time-space variable-order fractional differential equations because complicated numerical analysis is involved. The aim of this paper is to consider a multi-term time-space variable-order fractional diffusion equations with initial-boundary value problem. The multi-term FPDEs have been applied to several models for describing the processes in practice; For instance, the oxygen delivery through a capillary to tissues [36], the underlying processes with loss [37], the anomalous diffusion in highly heterogeneous aquifers and complex viscoelastic materials [38], and so on.

In this paper, a meshless method is used to discretize the spatial derivative, then a numerical method is applied for the time derivative of variable-order. This method has been used for occasions where the order of derivative was frational and constant. We would like to show that this method is also suitable for the variable-order fractional derivative.

This paper contributes the following:

1) Definition of Gamma function, variable-order Caputo fractional derivative and Riemann-Liouville in Section 2.
2) Introducing the space-time spectral collocation method discussed in Section 3.
3) Discretizing the problem by Chebyshev-spectral-collocation method in Section 4.
4) Discretizing the problem in time by a proposed numerical method method in Section 5.
5) Several lemmas and theories used to obtain the error bound in Section 6, and then the error bound is computed.
6) Finally, multiple numerical examples to show the effectiveness of the method are provided in Section 7.

2. Preliminaries

In the literature of fractional calculus, several definition are found in [25] and [26]. Fractional calculus has many applications and looks suitable for physics phenomena. Now, we define some essential definitions.

The Gamma function is an extension of factorial function to real numbers.

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) dt, \qquad t > 0.$$

Some of properties are

$$\Gamma(1) = \Gamma(2) = 1, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi},$$
$$\Gamma(z+1) = z\Gamma(z),$$
$$\Gamma(n) = (n-1)!, \qquad n \in \mathbb{N}.$$

Suppose $\alpha \in (0,1)$ is given, the left and right Caputo fractional derivatives of order α of a function $f : [a,b] \to \mathbb{R}$ is defined as

$${}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f^{'}(\tau)}{(t-\tau)^\alpha} d\tau,$$

and

$${}_t^bD_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b \frac{f^{'}(\tau)}{(\tau-t)^\alpha} d\tau.$$

It is obvious that the Caputo fractional derivative of a constant is zero.

In this paper, we consider the fractional derivative of variable order, which α depends on time and space. Some physical phenomena are better described when the order of the operator is variable. Now, we consider the order of derivative as $\alpha(x,t)$, taking values on the $[0,1] \times [0,1]$. Therefore we introduce the caputo derivative of variable-order.

Definition 2.1.

$${}_a^cD_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_a^t \frac{f^{'}(\tau)}{(t-\tau)^{\alpha(t)}} d\tau,$$

and

$${}_t^cD_b^{\alpha(t)} f(t) = -\frac{1}{\Gamma(1-\alpha(t))} \int_t^b \frac{f^{'}(\tau)}{(\tau-t)^{\alpha(t)}} d\tau.$$

In this paper, the following variable-order equation is considered .

$$
{}_0D_t^{q(x,t)}y(x,t) = \frac{\partial^2 y(x,t)}{\partial x^2} + f(x,t), \tag{1}
$$
$$
(x,t) \in \Omega = [0,L] \times [0,T],
$$

where the initial and boundary conditions are :

$$
y(x,0) = y_0(x), \qquad 0 \le x \le L,
$$
$$
y(0,t) = y(L,t) = 0, \qquad 0 \le t \le T,
$$

and also $0 < \underline{q} < q(x,t) < \bar{q} < 1$.

3. Chebyshev spectral collocation method

In this section, we present the Chebyshev spectral collocation method, briefly. The chebyshev-Gauss-Lobbato points in $\Lambda = [-1,1]$ is:

$$
\bar{x}_j = \cos(\frac{j\pi}{N}), \qquad j = 0,1,...,N.
$$

Now, we interpolate function F by these points,

$$
\bar{F}_N(\bar{x}) = \sum_{i=0}^{N} f_i \bar{L}_i(\bar{x}), \tag{2}
$$

where $\bar{L}_i(\bar{x})$ is the Lagrange interpolation polynomials assuring

$$
\bar{L}_i(\bar{x}_j) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}
$$

Let $\bar{F} = [f(\bar{x}_0), ..., f(\bar{x}_N)]$ and $\bar{F}^{(m)} = [f^{(m)}(\bar{x}_0), ..., f^{(m)}(\bar{x}_N)]^T$, By differentiating of Equation 2 and evaluating at $x = x_j$, we obtain

$$
\bar{F}_N^{(m)} = \sum_{i=0}^{N} f_i \bar{L}_i^{(m)}(\bar{x}), \qquad m \in \mathbb{N}, \tag{3}
$$

which can be rewritten as:

$$
\bar{F}_N^{(m)} = \bar{D}^{(m)}\bar{F}, \qquad m \in \mathbb{N}.
$$

$\bar{D}^{(m)}$ is the $(N+1) \times (N+1)$ matrix and

$$
\bar{D}_{ji}^{(m)} = \bar{L}_i^{(m)}(\bar{x}_j) \qquad j,i = 0,1,2,... \quad .
$$

$\bar{D}^{(1)} = \bar{D}$ is given as:

$$\bar{d}_{ij} = \begin{cases} \frac{2N^2+1}{6}, & i=j=0, \\ -\frac{c_i}{2c_j}\frac{(-1)^{j+i}}{sin((i+j)\frac{\pi}{2N})sin((i-j)\frac{\pi}{2N})}, & i \neq j, \\ -\frac{1}{2}\cos(\frac{i\pi}{N})(1+\cot^2(\frac{i\pi}{N})), & i=j, i\neq 0, N, \\ -\frac{2N^2+1}{6}, & i=j=N, \end{cases} \tag{4}$$

where

$$c_i = \begin{cases} 2, & i=0,N, \\ 1, & \text{o.w.} \end{cases}$$

Let $x_j = a + \frac{b-a}{2}(\bar{x}_j+1)$ are Chebyshev-Gauss-Lobatto points in $[a,b]$, such that

$$\begin{aligned} &\bar{x}_j = -1 + \frac{2}{b-a}(x_j - a), \\ &F^{(m)} = D^{(m)}F, \qquad m \in \mathbb{N}, \end{aligned}$$

where

$$F = [\bar{f}(-1+\frac{2}{b-a}(x_0-a)), ..., \bar{f}(-1+\frac{2}{b-a}(x_N-a))]^T,$$

and

$$D_{ji}^{(m)} = \bar{L}_i^{(m)}(-1+\frac{2}{b-a}(x_j-a)), \qquad j,i=0,1,...,N,$$

$$D_{ji}^{(m)} = (-\frac{2}{b-a})^m \bar{L}_i^{(m)}(\bar{x}_j), \qquad j,i=0,1,...N. \tag{5}$$

Hence, $D^{(m)} = (-\frac{2}{b-a})^m \bar{D}^{(m)}$.

Definition 3.1. [31, 32] *Suppose $C=(c_{ij})_{m\times n}$ and D are two arbitary matrices. The matrix*

$$C \otimes D = \begin{bmatrix} c_{11}D & c_{12}D & \cdots & c_{1n}D \\ c_{21}D & c_{22}D & \cdots & c_{2n}D \\ \vdots & \vdots \ddots & \vdots & \\ c_{m1}D & c_{m2}D & \cdots & c_{mn}D \end{bmatrix}$$

is named Kronocker product of C and D.

Definition 3.2. *Suppose $C = (c_{ij})_{m\times n}$ is a given matrix, then* $\mathrm{vec}(C)$ *is a column vector made of the row of C stacked a top one another from left to right that its size is $m \times n$ and*

$$\mathrm{vec}(C) = (c_{11}, c_{12}, ..., c_{1n}, c_{21}, c_{22}, ..., c_{m1}, ..., c_{mn})^T.$$

4. Discretizing in space

To discretize Equation 1 in space, we define:

$$y(x,t) = \sum_{i=0}^{N} y(x_i,t)L_i(x),$$

so we have

$$y(x_j,t) = \sum_{i=0}^{N} y(x_i,t)L_i(x_j).$$

By second derivative of $y(x,t)$ with respect to x, we obtain:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \sum_{i=0}^{N} y(x_i,t)\frac{\partial^2 L_i(x)}{\partial x^2},$$

$$\frac{\partial^2 y(x_j,t)}{\partial x^2} = \sum_{i=0}^{N} y(x_i,t)d_{ji}^{(2)}.$$

$y(x_0,t) = y(x_N,t) = 0$, hence we have

$$ {}_0D_t^{q(x_j,t)}y(x_j,t) = \sum_{i=1}^{N-1} y(x_i,t)d_{ji}^{(2)} + f(x_j,t), \tag{6}$$

and

$$\begin{bmatrix} {}_0D_t^{q(x_2,t)}y(x_2,t) \\ \vdots \\ {}_0D_t^{q(x_{N-1},t)}y(x_{N-1},t) \end{bmatrix} = D_1^{(2)} \begin{bmatrix} y(x_2,t) \\ \vdots \\ y(x_{N-1},t) \end{bmatrix} + \begin{bmatrix} f(x_2,t) \\ \vdots \\ f(x_{N-1},t) \end{bmatrix}.$$

$D_1^{(2)}$ is the matrix of second derivative which the first and last rows and columns are eliminated.

5. Discretizing in time

For positive integer number N_t, let $\triangle t = \frac{T}{N_t}$ denotes the step size time variable, $t_k = k\triangle t, k = 0, 1, ..., N_t$. In this section, we present the following lemma for discretization of time fractional derivative.

Lemma 5.1. *Suppose* $0 \leq \alpha \leq 1$ *and* $g(t) \in C^2[0, t_n]$, *it holds that*

$$\left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{g^{'}(t)}{(t_n - t)^{\alpha}} \mathrm{dt} - c \left[b_0 g(t_n) - \sum_{m=1}^{n-1} (b_{n-m-1} - b_{n-m}) g(t_m) - b_{n-1} g(t_0) \right] \right| \leq \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha}) \right] \max_{0 \leq t \leq t_k} |g^{''}(t_n)| \tau^{2-\alpha},$$

where

$$b_m = (m+1)^{1-\alpha} - m^{1-\alpha}, \qquad c = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}.$$

Therefore, for $n = 1$,

$$\partial_t^{\alpha} g(t_1) \simeq c \left[g(t_1) - g(t_0) \right]$$

and for $2 \leq n \leq N_t$,

$$\partial_t^{\alpha} g(t_n) \simeq c \left[b_0 g(t_n) - \sum_{m=1}^{n-1} (b_{n-m-1} - b_{n-m}) g(t_m) - b_{n-1} g(t_0) \right].$$

Equation 1 is in the following form at point (x_j, t_n)

$$D_t^{q(x_j,t_n)} y(x_j, t_n) = \sum_{i=1}^{N-1} y(x_i, t_n) d_{ji}^{(2)} + f(x_j, t_n), \qquad 1 \leq n \leq N_t.$$

By Lemma 5.1, we achieve

$$\frac{1}{\Gamma(1 - q(x_j, t_n))} \int_0^{t_n} \frac{y^{'}(x_j, t)}{(t_n - t)^{q(x_j,t_n)}} \mathrm{dt} \simeq$$

(7)

$$\frac{\Delta t^{-q(x_j,t_n)}}{\Gamma(2 - q(x_j, t_n))} \left[b_0^{jn} y(x_j, t_n) - \sum_{m=1}^{n-1} (b_{n-m-1}^{jn} - b_{n-m}^{jn}) y(x_j, t_m) - b_{n-1}^{jn} y(x_j, t_0) \right],$$

where

$$b_m^{jn} = (m+1)^{1-q(x_j,t_n)} - m^{1-q(x_j,t_n)}, \qquad j = 1, ..., N-1, \qquad n = 1, ..., N_t.$$

For more details you can see [7]. We define a $N_t \times N_t$ matrix B^j and a vector A^j as:

$$B^j = \begin{bmatrix} c^{j1}b_0^{j1} & 0 & 0 & \cdots & 0 \\ c^{j2}(b_1^{j2} - b_0^{j2}) & c^j b_0^{j2} & 0 & \cdots & 0 \\ c^{j3}(b_2^{j3} - b_1^{j3}) & c^{j3}(b_1^{j3} - b_0^{j3}) & c^{j3}b_0^{j3} & \cdots & 0 \\ \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ c^{jN_t}(b_{K-1}^{jN_t} - b_{K-2}^{jN_t}) & c^{jN_t}(b_{K-2}^{jN_t} - b_{K-3}^{jN_t}) & \cdots & c^{jN_t}(b_1^{jN_t} - b_0^{jN_t}) & c^{jN_t}b_0^{jN_t} \end{bmatrix},$$

$$A^j = \begin{bmatrix} c^{j1}b_0^{j1}y_0(x_j) \\ c^{j2}b_1^{j2}y_0(x_j) \\ c^{j3}b_2^{j3}y_0(x_j) \\ \vdots \\ c^{jN_t}b_{Nt-1}^{jN_t}y_0(x_j) \end{bmatrix}, \qquad c^{ji} = \frac{\Delta t^{-q(x_j,t_i)}}{\Gamma(2 - q(x_j,t_i))}$$

and

$$B = \left[B^1, \cdots, B^{N-1}\right], \qquad A = \left[A^1, \cdots A^{N-1}\right].$$

Equation 7 can be rewritten as:

$$\left[B \otimes I_{(N-1),(N-1)} - D \otimes I_{N_t,N_t}\right] \mathrm{vec}(y) = \mathrm{vec}(f) - \mathrm{vec}(A),$$

where

$$\mathrm{vec}(y) = \left[y(x_1,t_1), \cdots, y(x_1,t_{N_t}), y(x_1,t_1), \cdots, y(x_1,t_{N_t}), \cdots, y(x_{N-1},t_1), \cdots, y(x_{N-1},t_{N_t})\right],$$

$$\mathrm{vec}(f) = \left[f(x_1,t_1), \cdots, f(x_1,t_{N_t}), f(x_1,t_1), \cdots, f(x_1,t_{N_t}), \cdots, f(x_{N-1},t_1), \cdots, f(x_{N-1},t_{N_t})\right]$$

6. Error Estimate

Many lemmas and thorems have been developed to obtain the error bound. In order to use these theorems, we need to define discrete inner product and discrete norm.

Definition 6.1. *The discrete inner product and discrete norm are introduced as*

$$(f,g)_N = \sum_{j=0}^{N} f(\bar{x}_j)g(\bar{x}_j)w_j,$$

where $\bar{x}_j$ are the Chebyshev-Gauss-Lobatto points, weights w_j are positive and given as following

$$w_j = \frac{\pi}{d_j N}, \quad \textit{where} \quad w_j = \begin{cases} 2, & j = 0, N, \\ 1, & \text{o.w} \end{cases}$$

By the Gauss type integration in (Canuto & Hussaini & Quarteroni & Zang, 1987; Ben-yu, 1998) we have

$$(f,g)_N = (f,g), \quad \forall f,g \in P_{2N-1},$$

where $P_N = \operatorname{span}\{L_0(\bar{x}), L_1(\bar{x}), \cdots, L_N(\bar{x})\}$ and $(f,g) = \int_\Omega f(\bar{x}_j)g(\bar{x}_j)\mathrm{d\bar{x}}$. Denote

$$\|f\| = \|f\|_{L^2} = \sqrt{(f,f)}, \quad \|f\|_r = \|f\|_{H^r} = \sqrt{\sum_{|\alpha|\leqslant r} \|D^\alpha f\|^2}.$$

In the below, the bochner space $L^p(J;B)$ endowed with the norm is presented

$$\|f\|_{L^p;B} = \begin{cases} \left(\int_J \|f\|_B^p dt\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t\in J} \|f\|_B, & p = \infty. \end{cases}$$

Lemma 6.2. *(21, 22, 23, 24).For any $p \in P_N$, we have*

$$\|f\| \leqslant \|f\|_N \leqslant \sqrt{3}\|f\|,$$

and also the following inequality is obtained,

$$|(f,g) - (f,g)_N| \leqslant CN^{-s}\|f\|_s\|g\|.$$

If $f \in H^s(\Omega)$ for some $s > \frac{1}{2}$ and $v \in P_N$.

Proof. See [30, 29, 27, 28]. □

Lemma 6.3. *For all* $g \in H_0^1(\Omega)$

$$\|g\| \leqslant C\|g_{\bar{x}}\|.$$

Proof. See [28]. □

Assume the interpolation operator $\bar{I}_N$ from $H^1(\Omega)$ onto P_N assuring

$$\bar{I}_N f(\bar{x}) = \sum_{j=0}^{N} f(\bar{x}_j)\bar{L}_j(\bar{x}).$$

Then we have the following lemma.

Lemma 6.4. *Any* $f \in H^s(\Omega), s > \frac{1}{2}$, *we have*

$$\|f - I_N f\| \leqslant CN^{-s}\|f\|_s.$$

Proof. See [30, 29, 27, 28]. □

In the next step, a bilinear operator on the space $H_0^1(\Omega) \times H_0^1(\Omega)$ is introduced, which is coercive continuous,

$$a(f, g) = (f_{\bar{x}}, g_{\bar{x}}) = \int_{-1}^{1} f_{\bar{x}} g_{\bar{x}} \mathrm{d}\bar{\mathrm{x}}.$$

Now, the projection operator P_N^0 from $H_0^1(\Omega)$ onto V_N is used which is described by

$$(f - P_N^0 f, g) = 0, \quad \forall g \in V_N,$$

This shows P_N^0 is the orthogonal projection of y upon the space V_N, where $V_N = \mathrm{span}\{L_0(\bar{x}), \cdots, L_N(\bar{x})\}$ [30, 29, 27, 28].

Lemma 6.5. *For all* $f \in H_0^1(\Omega) \bigcap H^s(\Omega)$ *and* $s \geqslant 1$, *the following inequality holds*

$$\|f - P_N^0 f\|_1 + N\|f - P_N^0 f\| \leqslant CN^{1-s}\|f\|_s.$$

Proof. See [30, 29, 27, 28]. □

Now, we write (6), (1) as

$$\begin{cases} {}_0D_t^{q(x_j,t)} y_N(x_j, t) = \frac{\partial^2 y_N}{\partial x^2}(x_j, t) + f(x_j, t), \\ y_N(x_0, t) = y_N(x_N, t) = 0, \qquad t \in (t_0, T) \\ y_N(x_j, t_0) = y_0(x). \end{cases} \tag{8}$$

$$(9)\qquad \begin{cases} {}_0D_t^{q(x,t)}y(x,t) = \frac{\partial^2 y}{\partial x^2}(x,t) + f(x,t), \\ y(x_0,t) = y(x_N,t) = 0, \qquad t \in (t_0,T) \\ y(x,t_0) = y_0(x). \end{cases}$$

where $y_N(x) = \bar{y}_N(\bar{x}) = I_N\bar{y}(\bar{x})$ and $x = a + \frac{b-a}{2}(\bar{x}+1)$. Put $\bar{y}(\bar{x}) = y(a + \frac{b-a}{2}(\bar{x}+1))$, $\bar{q}(\bar{x}) = q(a + \frac{b-a}{2}(\bar{x}+1))$ and $\bar{f}(\bar{x}) = f(a + \frac{b-a}{2}(\bar{x}+1))$.
Now, the equations (8)and (9) can be written as

$$(10)\qquad \begin{cases} {}_0D_t^{\bar{q}(\bar{x}_j,t)}\bar{y}_N(\bar{x}_j,t) = (\frac{2}{b-a})^2\frac{\partial^2 \bar{y}_N}{\partial x^2}(\bar{x}_j,t) + \bar{f}(\bar{x}_j,t), \\ \bar{y}_N(\bar{x}_0,t) = \bar{y}_N(\bar{x}_N,t) = 0, \qquad t \in (t_0,T) \\ \bar{y}_N(\bar{x}_j,t_0) = \bar{y}_0(\bar{x}_j). \end{cases}$$

$$(11)\qquad \begin{cases} {}_0D_t^{\bar{q}(\bar{x},t)}\bar{y}(\bar{x},t) = (\frac{2}{b-a})^2\frac{\partial^2 \bar{y}}{\partial x^2}(\bar{x},t) + \bar{f}(\bar{x},t), \\ \bar{y}(\bar{x}_0,t) = \bar{y}(\bar{x}_N,t) = 0, \qquad t \in (t_0,T) \\ \bar{y}(\bar{x},t_0) = \bar{y}_0(\bar{x}). \end{cases}$$

The solutions $\bar{y}_N$ of (10) verifies

$$({}_0D_t^{\bar{q}_N}\bar{y}_N, v)_N + (\frac{2}{b-a})^2((y_N)_{\bar{x}}, v_{\bar{x}})_N = (\bar{f}_N, v)_N,$$

and by Gauss type integration, we get

$$({}_0D_t^{\bar{q}_N}\bar{y}_N, v) + (\frac{2}{b-a})^2((y_N)_{\bar{x}}, v_{\bar{x}}) = (\bar{f}_N, v),$$

and also, we have

$$({}_0D_t^{\bar{q}}\bar{y}, v) + (\frac{2}{b-a})^2((y)_{\bar{x}}, v_{\bar{x}}) = (\bar{f}, v),$$

The error can be written as a sum of two terms

$$\bar{y}_N - \bar{y} = (\bar{y}_N - P_N^0(\bar{y})) + (P_N^0(\bar{y}) - \bar{y}) = \theta + \rho$$

where

$$\theta = \bar{y}_N - P_N^0(\bar{y}), \qquad \rho = P_N^0(\bar{y}) - \bar{y}$$

Lemma 6.6. *For $\rho = P_N^0(\bar{y}) - \bar{y}$, $\bar{y} \in L^\infty(J, H^s(\Omega))$ we have*

$$a(\rho, \nu) = 0 \quad \forall t \in J,$$
$$\|\rho\| \leqslant CN^{-s}\|\bar{y}\|_s, \quad \forall t \in J,$$
$$\|\rho_t\| \leqslant CN^{-s}\|\bar{y}\|_s, \quad \forall t \in J,$$

Proof. See (Liu & Boying & Jiebao, 2015). □

We should mention that

$$0 < \underline{q} < \bar{q}(x,t) < \bar{q} < 1 \Rightarrow \frac{1}{\Gamma(1-\bar{q})} < \frac{1}{\Gamma(1-\bar{q}(x,t))} < \frac{1}{\Gamma(1-\underline{q})}, \tag{12}$$

$$D_t^{\bar{q}(\bar{x},t)} y(x,t) = \frac{1}{\Gamma(1-\bar{q}(x,t))}\int_{t_0}^{t} \frac{y_t(x,\tau)d\tau}{(t-\tau)^{\bar{q}(x,t)}}, \tag{13}$$

$$0 < t-\tau < 1 \Rightarrow 0 < (t-\tau)^{\bar{q}(x,t)} < 1 \Rightarrow \frac{1}{(t-\tau)^{\bar{q}(x,t)}} > 1, \tag{14}$$

therefore,

$${}_0D^{\bar{q}(\bar{x},t)} y(x,t) > \frac{1}{\Gamma(1-\bar{q})} y(t), \qquad D^{\bar{q}_N(\bar{x},t)} y_N(x,t) > \frac{1}{\Gamma(1-\bar{q})} y_N(t). \tag{15}$$

Theorem 6.7. *Suppose $\bar{y}$ and $\bar{y}_N$ be the solutions of* (10), (11) *and $\bar{y} \in L^\infty(J, H^s(\Omega))$. Then, it is obtained that*

$$\|\bar{y}_N(t) - \bar{y}(t)\| \leqslant CN^{-s}\|\bar{y}(t)\|_s, \quad for\ t \in J,$$

Proof. By $\bar{y_N}$ and $\bar{y}$ which are true in Equations (10), (11) and by (15) and we obtain:

$$\frac{1}{\Gamma(1-\bar{q})}(y_N, v) + (\frac{2}{b-a})^2 a(y_N, v) \leq (f_N, v),$$
$$\frac{1}{\Gamma(1-\bar{q})}(y, v) + (\frac{2}{b-a})^2 a(y, v) \leq (f, v).$$

Therefore

$$\frac{1}{\Gamma(1-\bar{q})}(\theta + \rho, v) + (\frac{2}{b-a})^2 a(\theta, v) \leq (f_N - f, v),$$

Since P_N^0 is an orthogonal projection then $(\rho, v) = 0$. Put $v = \theta$, we have

$$\frac{1}{\Gamma(1-\bar{q})}\|\theta\|^2 + (\frac{2}{b-a})^2\|\theta\|^2 \leq \|f_N - f\|\|\theta\|,$$

$$\|\theta(t)\| \leq \frac{1}{c}\|f_N - f\|,$$

where

$$c = (\frac{2}{b-a})^2 + \frac{1}{\Gamma(1-\bar{q})}.$$

Therefore

$$\|\theta(t)\| \leq CN^{-s}\|f\|_s.$$

The proof is completed by Lemma 5.1 and equation 7. □

Theorem 6.8. *Suppose y, y_N are the solution of (8), (9) and $y, p \in L^\infty(J, H^s(\Omega))$. Then the following inequalities are obtained*

$$||y_N(t) - y(t)|| \leqslant CN^{-s},$$

7. Numerical example

In this section, the following variable-order fractional diffusion equation is considered:

Example 7.1. [7]

$$\begin{aligned}
&{}_0D_t^{q(x,t)}y(x,t) = \frac{\partial^2 y(x,t)}{\partial x^2} + f(x,t), \qquad (x,t) \in \Omega = [0,1] \times [0,1],\\
&y(x,0) = 10x^2(1-x), \qquad 0 \leq x \leq 1,\\
&y(0,t) = y(1,t) = 0, \qquad 0 \leq t \leq 1,
\end{aligned}$$

where $q(x,t) = \frac{2+\sin(tx)}{4}$ *(satisfies* $0 < q(x,t) < 1$*)*

$$f(x,t) = 20x^2(1-x)[\frac{t^{2-q(x,t)}}{\Gamma(3-q(x,t))} + \frac{t^{1-q(x,t)}}{\Gamma(2-q(x,t))}] - 20(t+1)^2(1-3x).$$

The exact solution is

$$y(x,t) = 10x^2(1-x)(t+1)^2.$$

Table 1 shows the numerical solution and exact solution of the equation 1 for $N = 11$ and $N_t = 30$.

Figure 1 shows the exact and numerical solution for $N_t = 30$ and $N = 20$ and Figure 2 shows the error of the solution.

TABLE 1. Comparison between the numerical solution and exact solution of Example 7.1, $N = 11$ and $N_t = 30$

x_i	*Numerical solution*	*Exact solution*	*Error*
0.0203	*0.0161*	*0.0161*	*0.0318*$\times 10^{-3}$
0.0794	*0.2321*	*0.2320*	*0.1246*$\times 10^{-3}$
0.1726	*0.9859*	*0.9856*	*0.2700*$\times 10^{-3}$
0.2923	*2.4190*	*2.4185*	*0.4473*$\times 10^{-3}$
0.4288	*4.2022*	*4.2016*	*0.6110*$\times 10^{-3}$
0.5712	*5.5966*	*5.5959*	*0.6914*$\times 10^{-3}$
0.7077	*5.8564*	*5.8558*	*0.6328*$\times 10^{-3}$
0.8274	*4.7264*	*4.7259*	*0.4483*$\times 10^{-3}$
0.9206	*2.6911*	*2.6909*	*0.2248*$\times 10^{-3}$
0.9797	*0.7777*	*0.7777*	*0.0588*$\times 10^{-3}$

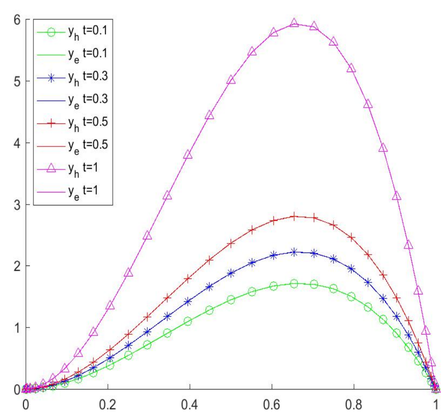

FIGURE 1. Comparisons between the numerical solution and exact solution of $y(x,t)$ at $t = 0.1$s, $t = 0.3$s, $t = 0.5$s, $t = 1$ in Example 7.1.

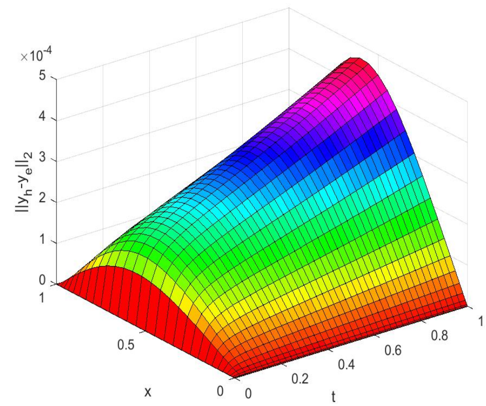

FIGURE 2. Plots of the maximum error of the exact solution and numerical solution $\|y_h - y_e\|$ in Example 7.1.

Example 7.2. [35]

$$
\begin{aligned}
&{}_0D_t^{q(x,t)}y(x,t) = K\frac{\partial^2 y(x,t)}{\partial x^2} + f(x,t), \qquad (x,t)\in\Omega = [0,L]\times[0,T],\\
&y(x,0) = 0, \qquad 0 \le x \le L,\\
&y(0,t) = y(1,t) = 0, \qquad 0 \le t \le T,
\end{aligned}
$$

where $q(x,t) = 0.8 + \frac{0.2xt}{LT}$ *(satisfies* $0 < q(x,t) < 1$*),* $T = 0.5, L = 10, K = 0.01$

$$f(x,t) = \frac{2}{\Gamma(3-q(x,t))}t^{2-q(x,t)}\sin(\frac{x\pi}{L}) + K\frac{\pi^2 t^2}{L^2}\sin(\frac{x\pi}{L}).$$

The exact solution is

$$y(x,t) = t^2\sin(\frac{x\pi}{L}).$$

Table 2 shows the numerical solution and exact solution of the equation 1 for $N = 11$ *and* $N_t = 30$ *at* $t = 0.1$.

Figure 3 shows the exact and numerical solution for $N_t = 30$ *and* $N = 40$ *and Figure 4 shows the error of the solution.*

8. Conclusion

In this paper we consider the one-dimensional variable-order time fractional diffusion equation. We use spectral collocation method and finite difference method

TABLE 2. Comparison between the numerical solution and exact solution of Example 7.1, $N = 11$ and $N_t = 30$

x_i	*Numerical solution*	*Exact solution*	*Error*
0.2025	*0.0007*	*0.0006*	*0.0571*$\times 10^{-3}$
0.7937	*0.0027*	*0.0025*	*0.2228*$\times 10^{-3}$
1.7257	*0.0056*	*0.0052*	*0.4697*$\times 10^{-3}$
2.9229	*0.0087*	*0.0079*	*0.7308*$\times 10^{-3}$
4.2884	*0.0107*	*0.0098*	*0.9075*$\times 10^{-3}$
5.7116	*0.0107*	*0.0098*	*0.9187*$\times 10^{-3}$
7.0771	*0.0087*	*0.0079*	*0.7574*$\times 10^{-3}$
8.2743	*0.0057*	*0.0052*	*0.4969*$\times 10^{-3}$
9.2063	*0.0027*	*0.0025*	*0.2395*$\times 10^{-3}$
9.7975	*0.0007*	*0.0006*	*0.0620*$\times 10^{-3}$

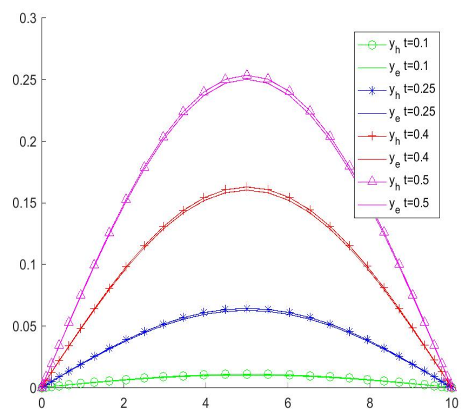

FIGURE 3. Comparisons between the numerical solution and exact solution of $y(x,t)$ (1) and at $t = 0.1$s, $t = 0.25$s, $t = 0.4$s, $t = 0.5$ in Example7.2.

to discretize the spatial variable and time variable, respectively. The used technique is applied to solve a test problem and the resulting solutions are in good

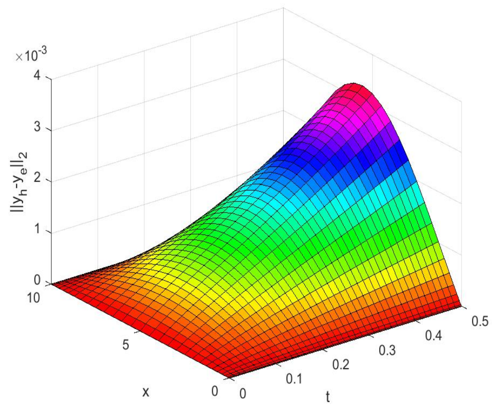

FIGURE 4. Plots of the maximum error of the exact solution and numerical solution $\|y_h - y_e\|$ in Example 7.2.

agreement with the known exact solutions. For the sake of simplicity, we only considered the one-dimensional case with standard initial and boundary conditions, but the method can be extended to multi-dimensional cases with even non-classic boundary conditions which is the subject of the authors.

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