Journal of Mahani Mathematical Research Center
Print ISSN: 2251-7952 Online ISSN: 2645-4505

# SOME GENERALIZED RESULTS BASED ON DIFFERENTIAL SUBORDINATIONS OF ANALYTIC FUNCTIONS

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Article type: Research Article
(Received: 23 November 2020, Revised: 01 February 2021, Accepted: 04 February 2021)
(Communicated by N. Shajareh Poursalavati)

Abstract. For the function $f(z)$ analytic in the open unit disk and normalized by $f(0) = f'(0) - 1 = 0$, we consider the expression; $\alpha(\frac{zf'(z)}{f(z)} - 1) + 1 - (\frac{z}{f(z)})^\alpha$; $(\alpha > 0)$. Using differential subordination notion, we investigate properties of $(\frac{f(z)}{z})^\alpha$, as well as, sufficient conditions for univalence and starlikeness of $f(z)$. In the special case, for $\alpha = 1$, these results generalize and improve some previously results given in the literature.

*Keywords*: Differential subordination, Univalent, Starlike, Close-to-convex.
*2020 MSC*: Primary 30C45, Secondary 30C80.

## 1. Introduction

By $\mathcal{A}$ we denote the class of all analytic functions $f(z)$ in the open unit disk $\mathbb{D}$, which is normalized by $f(0) = f'(0) - 1 = 0$.

Let $f \in \mathcal{A}$. We say that $f(z)$ is strongly starlike of order $\gamma$, $0 < \gamma \leq 1$, if

$$\left| arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi\gamma}{2}; \quad (z \in \mathbb{D}).$$

We show by $SS^*(\gamma)$ the class of strongly starlike functions. In particular for $\gamma = 1, S^* = SS^*(1)$ is the class of starlike functions which is subclass of the class of univalent functions [1, 2]. Expressions such as $\left(\frac{f(z)}{z}\right)^\alpha$; $(\alpha > 0), (f'(z))^\alpha$ and $\left(\frac{z}{f(z)}\right)^\alpha f'(z)$ often appear in definitions of integral operators or other subclasses of analytic functions and use as a criteria for starlikeness (univalence), either in the condition or in the conclusion, for example see [2,3,5].

In the present paper, we study the expression,

$$\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^\alpha; \quad (\alpha > 0, z \in \mathbb{D}) \tag{1}$$

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DOI: 10.22103/jmmrc.2021.16816.1126

How to cite: H. Rahimpoor, P. Arjomandinia, *Some generalized results based on differential subordinations of analytic functions*, J. Mahani Math. Res. Cent. 2021; 10(1): 11-25.

its modulus and real part, and obtain conditions over them that lead to some properties of $\left(\frac{f(z)}{z}\right)^{\alpha}$, as well as to criteria of univalence and starlikeness. To get this purpose, we will use some definitions and techniques from the differential subordination theory [2].

**Definition 1.1.** For $f, g \in \mathcal{A}$ we say that $f(z)$ is subordinate to $g(z)$ and write $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in $\mathbb{D}$, $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In the special case if $g(z)$ is univalent in $\mathbb{D}$ then $f(z) \prec g(z)$, if and only if $f(0) = g(0)$ and $f(D) \subseteq g(D)$.

Let $\psi : \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$, ($\mathbb{C}$ is complex plane), and let $h(z)$ be univalent in $\mathbb{D}$. If the analytic function $p(z)$ satisfies the (first-order) differential subordination

$$\psi(p(z), zp'(z); z) \prec h(z) \tag{2}$$

then $p(z)$ is called a solution of the differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of (2) if $p \prec q$ for all $p$ satisfying (2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (2) is said to be the best dominant of (2).

We begin with the following lemmas that will be used in proving the next results.

**Lemma 1.2.** *( [4, 5]) Let* $b \in H(\mathbb{D}) \bigcap C^0(\bar{\mathbb{D}}), b(0) = 0, \sup_{z\in\mathbb{D}} |b(z)| = 1$ *and* $c = \sup_{z\in\mathbb{D}} \int_0^1 |b(tz)|dt$. *For* $0 < \gamma \leq 1$ *let*

$$\lambda(\gamma) = \frac{\sin\left(\frac{\gamma\pi}{2}\right)}{\sqrt{1 + 2c\cos\left(\frac{\gamma\pi}{2}\right) + c^2}}.$$

*If* $f \in \mathcal{A}$ *and*

$$|f'(z) - 1| \leq \lambda(\gamma)|b(z)|, \quad (z \in \mathbb{D})$$

*then* $f(z) \in SS^*(\gamma)$. *Additionally, if*

$$b(t) = \max_{0\leq\varphi\leq 2\pi} \left|b\left(te^{i\varphi}\right)\right|, \quad 0 \leq t \leq 1$$

*then the constant* $\lambda(\gamma)$ *cannot be replaced by any larger number without violating the conclusion.*

**Lemma 1.3.** *( [2]) Let* $q(z)$ *be univalent in* $\mathbb{D}$ *and let* $\theta(w)$ *and* $\varphi(w)$ *be analytic in a domain* $D$ *containing* $q(\mathbb{D})$, *with* $\varphi(w) \neq 0$, *for* $w \in q(\mathbb{D})$. *Set* $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, *and suppose that either:*
*(i)* $h$ *is convex, or*
*(ii)* $Q$ *is starlike.*
*(iii)* $Re\frac{zh'(z)}{Q(z)} = Re\left(\frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0; \quad (z \in \mathbb{D}).$
*If* $p(z)$ *is analytic in* $\mathbb{D}$, *with* $p(0) = q(0), p(\mathbb{D}) \subseteq D$, *and*

$$\theta\left(p(z)\right) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) = h(z)$$

*then* $p \prec q$ *and* $q$ *is the best dominant.*

Our motivation in defining this class of functions is to use Lemma 1.3 and generalize [5, Lemma (2)]. In special cases for $\alpha$, we can obtain some geometric conditions, such as starlikeness and close-to-convexity, for the functions which satisfy in special inequalities, [example, 2.11]. Using Lemma 1.3 we state and prove the following theorem that will be used to prove other results of the paper.

## 2. Main results

**Theorem 2.1.** *Let* $q(z)$ *be univalent in* $\mathbb{D}, q(0) = 0, q(z) \neq -1$ *for all* $z \in \mathbb{D}$. *Also, suppose that :*

*(i)* $Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{1+q(z)}\right) > 0; \quad z \in \mathbb{D},$

*(ii)* $Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)-1}{1+q(z)}\right) > 0; \quad z \in \mathbb{D}.$

*If* $\alpha > 0$, $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ *for all* $z \in \mathbb{D}$, *and*

$$\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{zq'(z) + q(z)}{1+q(z)} \tag{3}$$

*then* $\left(\frac{f(z)}{z}\right)^{\alpha} - 1 \prec q(z)$, *and* $q(z)$ *is the best dominant of (3). All powers in (3) are principle ones.*

*Proof.* Consider the functions $\theta(w) = \frac{w}{1+w}$ and $\varphi(w) = \frac{1}{1+w}$, then the functions $\theta$ and $\varphi$ are analytic in the domain $D = \mathbb{C} - \{-1\}$, which contains $q(\mathbb{D})$ and $\varphi(w) \neq 0$ for $w \in q(\mathbb{D})$. We have

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{zq'(z)}{1+q(z)}$$

which is starlike, since

$$Re\frac{zQ'(z)}{Q(z)} = Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{1+q(z)}\right) > 0; \quad (z \in \mathbb{D}).$$

Also, for the function $h(z) = \theta(q(z)) + Q(z) = \frac{q(z) + zq'(z)}{1+q(z)}$, we have;

$$Re\frac{zh'(z)}{Q(z)} = Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)-1}{1+q(z)}\right) > 0.$$

Now, we choose the function $p(z) = \left(\frac{f(z)}{z}\right)^{\alpha} - 1$. $p(z)$ is analytic in $\mathbb{D}$, $p(0) = 0$ and $p(z) \neq -1$ for all $z \in \mathbb{D}$ (which is equivalent to $p(\mathbb{D}) \subseteq D$). A simple calculation shows that subordinations (3) and $\theta(p(z)) + zp'(z)\varphi(p(z)) \prec h(z)$ are equivalent. So, by Lemma 1.3, we conclude that

$$\left(\frac{f(z)}{z}\right)^{\alpha} - 1 = p(z) \prec q(z),$$

and $q(z)$ is the best dominant. □

Differential subordination (3) in Theorem 2.1 is a generalization of the basic Briot-Bouquet differential subordination, $p(z)+\dfrac{zp'(z)}{\beta q(z)+\gamma} \prec h(z)$, in [2], where $h(z)$ is univalent in $\mathbb{D}$. In this theorem, first we select the dominant $q(z)$ and then find the appropriate $h(z)$ correspondingto this $q$, i.e. we find the largest class of univalent function $h(z)$ such that the relation (3) in Theorem 2.1 holds.

Let us examine the conditions of Theorem 2.1 by an example.

**Example 2.2.** *Suppose that $q(z) = e^z - 1$, $f(z) = ze^z$ and $\alpha = \frac{1}{2}$. Then $q(z)$ is univalent in $\mathbb{D}$, $q(0) = 0$, $q(z) \neq -1$. Also we have,*
*(i)* $Re\left(1 + \dfrac{zq''(z)}{q'(z)} - \dfrac{zq'(z)}{1+q(z)}\right) = Re\left(1 + \dfrac{ze^z}{e^z} - \dfrac{ze^z}{e^z}\right) = 1 > 0,$
*(ii)* $Re\left(1 + \dfrac{zq''(z)}{q'(z)} - \dfrac{zq'(z)}{1+q(z)}\right) + Re\left(\dfrac{1}{1+q(z)}\right) = 1 + Re(e^{-z}) = 1 + e^{-x}cosy > 0$ *where, $z = x+iy \in \mathbb{D}$. For the function $f(z) = ze^z$, we have $f \in \mathcal{A}$, $\dfrac{f(z)}{z} = e^z \neq 0$, and*

$$\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha} = \frac{1}{2}z + 1 - e^{-\frac{1}{2}z},$$

$$\frac{zq'(z)+q(z)}{1+q(z)} = z + 1 - e^{-z}.$$

*So, we obtain*

$$\frac{1}{2}z + 1 - e^{-\frac{1}{2}z} \prec z + 1 - e^{-z}$$

*and the condition (3) of Tehorem 2.1 is also satisfied. Therefore, we conclude that $\left(\dfrac{f(z)}{z}\right)^{\frac{1}{2}} - 1 \prec q(z)$, and $q(z)$ is the best dominant of (3).* □

Taking $\alpha = 1$ in Theorem 2.1, we obtain the result given in [5].
Now, we study the modulus of the expression (1) and obtain conclusions over $\left(\dfrac{f(z)}{z}\right)^{\alpha}$ and $\alpha\left(\dfrac{zf'}{f} - 1\right) + 1 - \left(\dfrac{z}{f(z)}\right)^{\alpha}$ that will lead to sufficient conditions for starlikeness and univalence. Applying Theorem 2.1 we obtain the following result.

**Corollary 2.3.** *Suppose that* $\alpha > 0$, $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ *for all* $z \in \mathbb{D}$ *and* $0 < \lambda \leq 1$. *If*

$$\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{2\lambda z}{1+\lambda z} = h_1(z) \tag{4}$$

*then*

$$\left(\frac{f(z)}{z}\right)^{\alpha} - 1 \prec \lambda z \tag{5}$$

*and* $\lambda z$ *is the best dominant of (4). In addition*

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha} - 1\right| < \lambda \quad (z \in \mathbb{D}). \tag{6}$$

*and this inequality is sharp, i.e., in (6)* $\lambda$ *cannot be replaced by a smaller number so that the implication holds.*

*Proof.* The function $q(z) = \lambda z$ satisfies all conditions of Theorem 2.1, and the subordinations (3) and (4) are equivalent. So, (5) follows directly from Theorem 2.1. For the sharpness part assume that (4) and $\left|\left(\frac{f(z)}{z}\right)^{\alpha} - 1\right| < \lambda_1$; $(z \in \mathbb{D})$, i.e., $\left(\frac{f(z)}{z}\right)^{\alpha} - 1 \prec \lambda_1 z$ hold. Because $\lambda z$ is the best dominant of (4), we have that $\lambda z \prec \lambda_1 z$, or equivalently $\lambda \leq \lambda_1$. □

It is easy to see that for $h_1(z) = \frac{2\lambda z}{1+\lambda z}$ with $0 < \lambda < 1$, $h_1(\mathbb{D})$ is an open disk with center $c = \frac{h_1(1)+h_1(-1)}{2} = \frac{-2\lambda^2}{1-\lambda^2}$ and radius $r = h_1(1) - c = \frac{2\lambda}{1-\lambda^2}$, and that for $\lambda = 1$ is a half plane containing all points $(x, y)$ , with $x < 1$. Therefore, we can rewrite Corollary 2.3 in the following equivalent form.

**Corollary 2.4.** *Suppose that* $f \in \mathcal{A}$ *and* $\frac{f(z)}{z} \neq 0$ *in* $\mathbb{D}$.

*(i) If* $0 < \lambda < 1$ *and* $\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha} + \frac{2\lambda^2}{1-\lambda^2}\right| < \frac{2\lambda}{1-\lambda^2}$, $(z \in \mathbb{D})$ *then*

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha} - 1\right| < \lambda; \quad (z \in \mathbb{D}). \tag{7}$$

*(ii) If* $\alpha Re\left(\frac{zf'(z)}{f(z)} - 1\right) < Re\left(\frac{z}{f(z)}\right)^{\alpha}$ *then*

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha} - 1\right| < 1; \quad (z \in \mathbb{D}). \tag{8}$$

*Also, both of the inequalities (7) and (8) are sharp i.e., in each case the radius of the open disk from the conclusion is the smallest possible so that the corresponding implication holds.*

**Corollary 2.5.** *Let $\alpha > 0$ and $0 < \lambda \leq 1$. If for the function $f \in \mathcal{A}$, we have,*

$$\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha}\right| < \lambda; \quad (z \in \mathbb{D})$$

*then*

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha} - 1\right| < \frac{\lambda}{2-\lambda} \equiv \gamma; \quad (z \in \mathbb{D}).$$

*Proof.* In the case $0 < \lambda < 1$ (i.e., $0 < \gamma < 1$) we have

$$\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha} + \frac{2\gamma^2}{1-\gamma^2} - \frac{2\gamma^2}{1-\gamma^2}\right| < \frac{2\gamma}{1+\gamma}$$

that leads to

$$\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha} + \frac{2\gamma^2}{1-\gamma^2}\right| < \frac{2\gamma}{1+\gamma} + \frac{2\gamma^2}{1-\gamma^2} = \frac{2\gamma}{1-\gamma^2}$$

Now, the result follows from Corollary 2.4 (i). On the other hand, for $\lambda = \gamma = 1$, we have

$$\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha}\right| < 1$$

which gives $\alpha\left(Re\left(\frac{zf'(z)}{f(z)}\right) - 1\right) < Re\left(\frac{z}{f(z)}\right)^{\alpha}$ and the result follows from Corollary 2.4 (ii). □

**Example 2.6.** *For the function $f(z) = \frac{z}{(1+cz)^{\frac{1}{\alpha}}}$, with $\alpha > 0$ and $0 < c \leq \frac{3-\sqrt{5}}{2}$, we obtain the maximum modulus over expressions*

$$\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha}, \quad \left(\frac{f(z)}{z}\right)^{\alpha} - 1$$

*as following:*

$$\begin{aligned}\max_{|z|=1}\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha}\right| &= \max_{|z|=1}\left|\frac{-cz}{1+cz} + 1 - (1+cz)\right| \\ &= \max_{|z|=1}\left|\frac{cz(2+cz)}{1+cz}\right| \\ &= \frac{c(2-c)}{1-c} \equiv \lambda \in (0,1]\,.\end{aligned}$$

*and*

$$\max_{|z|=1}\left|\left(\frac{f(z)}{z}\right)^{\alpha}-1\right| = \max_{|z|=1}\left|\frac{-cz}{1+cz}\right|$$
$$= \frac{c}{1-c} < \gamma = \frac{\lambda}{2-\lambda}$$
$$= \frac{c(2-c)}{c^2-4c+2}.$$

□

From the Example 2.6 raises a question that whether the result obtained in Corollary 2.5 is sharp or not, i.e., does there exist $\gamma < \dfrac{\lambda}{2-\lambda}$ such that the implication from the corollary holds? This is still an open problem.

Making use of Corollary 2.5 we obtain the following implications.

**Corollary 2.7.** *Suppose that* $f \in \mathcal{A}$, $\alpha > 0$ *and* $0 < \lambda \leq 1$. *If*

$$\left|\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1-\left(\frac{z}{f(z)}\right)^{\alpha}\right| < \lambda; \quad (z \in \mathbb{D})$$

*then*

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha}\left(\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1\right)-1\right| < \frac{2\lambda}{2-\lambda}; \quad (z \in \mathbb{D})$$

*and*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 1-\frac{5\lambda-\lambda^2}{2\alpha}; \quad (z \in \mathbb{D}).$$

*Proof.* All conditions of Corollary 2.5 are satisfied, so we have

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha}-1\right| < \frac{\lambda}{2-\lambda} \equiv \gamma; \quad (z \in \mathbb{D})$$

which gives,

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha}\right| < 1+\gamma; \quad (z \in \mathbb{D})$$

and

$$0 \leq 1-\gamma < Re\left(\frac{f(z)}{z}\right)^{\alpha} < 1+\gamma; \quad (z \in \mathbb{D}).$$

Also,

$$Re\left(\frac{z}{f(z)}\right)^{\alpha} = \left|\left(\frac{z}{f(z)}\right)^{\alpha}\right|^2 Re\left(\frac{f(z)}{z}\right)^{\alpha}$$
$$> \frac{1}{(1+\gamma)^2}(1-\gamma) = \frac{1-\gamma}{(1+\gamma)^2}.$$

From here, we conclude that

$$\begin{aligned}
&\left|\left(\frac{f(z)}{z}\right)^{\alpha}\left(\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1\right)-1\right| \\
&=\left|\left(\frac{f(z)}{z}\right)^{\alpha}\right|\left|\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1-\left(\frac{z}{f(z)}\right)^{\alpha}\right| \\
&< \lambda(1+\gamma)=\lambda\left(1+\frac{\lambda}{2-\lambda}\right)=\frac{2\lambda}{2-\lambda}.
\end{aligned}$$

Also, we have:

$$\begin{aligned}
Re\left(\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1\right) &= Re\left(\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1-\left(\frac{z}{f(z)}\right)^{\alpha}\right)+Re\left(\frac{z}{f(z)}\right)^{\alpha} \\
&> -\lambda+\frac{1-\gamma}{(1+\gamma)^2} \\
&= -\lambda+\frac{(1-\lambda)(2-\lambda)}{2}=\frac{\lambda^2-5\lambda+2}{2},
\end{aligned}$$

and finally, from the last relations, we obtain

$$Re\left(\frac{zf'(z)}{f(z)}\right)>1-\frac{(5\lambda-\lambda^2)}{2\alpha}.$$

□

Combining Lemma 1.2 and Corollary 2.7, we obtain the following result.

**Corollary 2.8.** *Let $f\in\mathcal{A}$, $0<\mu\le 1$ and*

$$\lambda(\mu)=\frac{2\sin(\frac{\pi}{2}\mu)}{\sqrt{5+4\cos(\frac{\pi}{2}\mu)}}.$$

*If*

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha}\left(\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1\right)-1\right|<\frac{2\lambda(\mu)}{2+\lambda(\mu)}\left|\left(\frac{f(z)}{z}\right)^{\alpha}\right|$$

*then*

$$\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha}\left(\alpha\left(\frac{tf'(t)}{f(t)}-1\right)+1\right)dt\in SS^*(\mu).$$

*Proof.* Using $\lambda=\dfrac{2\lambda(\mu)}{2+\lambda(\mu)}$ in Corollary 2.7, we obtain

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha}\left(\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1\right)-1\right|<\frac{2\lambda}{2-\lambda}=\lambda(\mu);\quad (z\in\mathbb{D}).$$

An application of Schwartz Lemma shows that

$$\left|\left(\frac{f(z)}{z}\right)^{\alpha}\left(\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1\right)-1\right|\le\lambda(\mu)|z|;\quad (z\in\mathbb{D}).$$

Finally, taking $b(z) = z$ in Lemma 1.2 we conclude that

$$c = \sup_{z\in\mathbb{D}} \int_0^1 |b(tz)|\, dt = \sup_{z\in\mathbb{D}} \frac{|z|}{2} = \frac{1}{2}$$

and

$$\int_0^z \left(\frac{f(t)}{t}\right)^\alpha \left(\alpha\left(\frac{tf'(t)}{f(t)} - 1\right) + 1\right) dt \in SS^*(\mu).$$

□

**Example 2.9.** *Let $f(z) = ze^z$, $z \in \mathbb{D}$ and $\mu = 1$, then we obtain:*

$$\lambda(\mu) = \frac{2sin\left(\frac{\pi}{2}\right)}{\sqrt{5 + 4cos\left(\frac{\pi}{2}\right)}} = \frac{2}{\sqrt{5}}, \quad \frac{2\lambda(\mu)}{2+\lambda(\mu)} = \frac{\sqrt{5}-1}{2}$$

*Let $n \in \mathbb{N}$. An easy computation shows that*

$$lim_{n\to+\infty}\left(\frac{1}{n}z + 1 - e^{-\frac{z}{n}}\right) = 0.$$

*So, we can find $N \in \mathbb{N}$, such that*

$$\left|\frac{1}{N}z + 1 - e^{-\frac{z}{N}}\right| < \frac{\sqrt{5}-1}{2}.$$

*Now, let $\alpha = \frac{1}{N}$, then we obtain that:*

$$\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^\alpha\right| = \left|\frac{1}{N}z + 1 - e^{-\frac{z}{N}}\right| < \frac{\sqrt{5}-1}{2}.$$

*So, we conclude that*

$$\int_0^z \left(\frac{f(t)}{t}\right)^\alpha \left(\alpha\left(\frac{tf'(t)}{f(t)} - 1\right) + 1\right) dt = ze^{\frac{z}{N}} \in SS^*(1) = S^*.$$

□

*Remark* 2.10. In Example 2.6 we conclude that Corollary 2.5 is not sharp. This implies that Corollaries 2.7 and 2.8 are not sharp too.

In the following example, we obtain conclusions that can be obtained from the previous results by taking the special values for $\alpha$ and $\lambda$.

**Example 2.11.** *Let $f \in \mathcal{A}$.*
*(i) If $0 < \alpha \leq 2$ and $\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^\alpha\right| < \frac{5-\sqrt{25-8\alpha}}{2}$, then $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ in $\mathbb{D}$, which implies that $f(z)$ is starlike, hence univalent,*

*($\lambda = \dfrac{5-\sqrt{25-8\alpha}}{2}$ in Corollary 2.7).*

*(ii) If $\left|(f'(z))^{\alpha-1}(\alpha z f''(z)+f'(z))-1\right| < |f'(z)|^{\alpha}$, then $Re(1+\dfrac{zf''(z)}{f'(z)}) > 1-\dfrac{2}{\alpha}$, $(z\in\mathbb{D})$, which for $\alpha \geq \dfrac{4}{3}$ implies that $Re\left(1+\dfrac{zf''(z)}{f'(z)}\right) > -\dfrac{1}{2}$, $(z\in\mathbb{D})$, and $f$ is close-to-convex, hence univalent. (Taking $zf'(z)$ instead of $f(z)$ and $\lambda=1$ in Corollary 2.7).* □

Next, we obtain some results over the real part of

$$\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1-\left(\frac{z}{f(z)}\right)^{\alpha};\quad (z\in\mathbb{D}).$$

Taking $q(z)=\dfrac{2\mu z}{1-z}$, in Theorem 2.1 we obtain the following result.

**Corollary 2.12.** *For $\alpha>0$, $f\in\mathcal{A}$, $\dfrac{f(z)}{z}\neq 0$ in $\mathbb{D}$ and $0<\mu\leq 1$, assume that*

$$\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1-\left(\frac{z}{f(z)}\right)^{\alpha}\prec h_2(z) \tag{9}$$

*with $h_2(z)=\dfrac{2\mu z}{1-(1-2\mu)z}\left(1+\dfrac{1}{1-z}\right)$ then*

$$\left(\frac{f(z)}{z}\right)^{\alpha}-1\prec\frac{2\mu z}{1-z} \tag{10}$$

*and $\dfrac{2\mu z}{1-z}$ is the best dominant of (9). In addition*

$$Re\left(\frac{f(z)}{z}\right)^{\alpha}>1-\mu;\quad (z\in\mathbb{D}) \tag{11}$$

*and the number $1-\mu$ obtained in (11) is sharp, i.e., in (11) $1-\mu$ cannot be replaced by a larger number such that the implication holds.*

*Proof.* It is claer that $q(z)=\dfrac{2\mu z}{1-z}$ is univalent in $\mathbb{D}$, $q(0)=0$ and $q(z)\neq -1$ for all $z\in\mathbb{D}$. Now, for $z\in\mathbb{D}$ and $-1\leq\lambda=1-2\mu<1$ we obtain:

$$\begin{aligned} Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{1+q(z)}\right) &= Re\left(\frac{z}{1-z}+\frac{1}{1-\lambda z}\right)\\ &> -\frac{1}{2}+\frac{1}{1+|\lambda|}=\frac{1-|\lambda|}{2(1+|\lambda|)}\geq 0, \end{aligned}$$

and so, condition (i) in Theorem 2.1 is satisfied. Also, it is easy to see that condition (ii) from Theorem 2.1 is also satisfied:

$$Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)-1}{1+q(z)}\right)>Re\left(\frac{1}{1+q(z)}\right)>0\ \ (z\in\mathbb{D}).$$

Finally, easy computations show that

$$\frac{zq'(z)+q(z)}{1+q(z)} = h_2(z).$$

Therefore, all conditions of Theorem 2.1 are satisfied and (10) follows directly. Further, inequality (11) follows easily from the subordination condition (10) by writing:

$$Re\left(\frac{f(z)}{z}\right)^{\alpha} > Re\left(\frac{2\mu z}{1-z}\right) + 1 > 1-\mu.$$

Now, we investigate the sharpness of (11). Assume that (9) and $Re(\frac{f(z)}{z})^{\alpha} > 1-\mu_1$ for all $z \in \mathbb{D}$, hold. Equivalently, we have

$$\left(\frac{f(z)}{z}\right)^{\alpha} - 1 \prec \frac{2\mu_1 z}{1-z}.$$

But $\frac{2\mu z}{1-z}$ is the best dominant of (9), which implies that $\frac{2\mu z}{1-z} \prec \frac{2\mu_1 z}{1-z}$, i.e., $-\mu_1 \leq -\mu$ and $1-\mu_1 \leq 1-\mu$. □

Now using the definition of subordination and the properties of $h_2(\mathbb{D})$ and $q(\mathbb{D})$ in Corollary 2.12, we obtain some results over the real part of

$$\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha}.$$

Before to state the next corollary, note that for the function $h_2(z)$ defined by expression (9) we have:

$$h_2(z) = \begin{cases} 1 + \frac{1}{1-z} - \frac{1}{\lambda} + \left(\frac{1}{\lambda} - 2\right)\frac{1}{1-\lambda z}; & \lambda \neq 0, \big(or\ \mu \neq \frac{1}{2}\big) \\ \frac{1}{1-z} - 1 + z; & \lambda = 0, \big(or\ \mu = \frac{1}{2}\big) \end{cases}$$

where $0 < \mu \leq 1$ and $\lambda = 1 - 2\mu$.

**Example 2.13.** *Let $\alpha = \mu = \frac{1}{2}$, and $f(z) = \frac{z}{(1-z)^2}$ be the "Koebe" function [1, 2]. It is easy to see that :*

$$\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha} = \frac{z(2-z)}{1-z} = h_2(z). \tag{12}$$

*So, we conclude that*

$$\left(\frac{f(z)}{z}\right)^{\alpha} - 1 \prec \frac{z}{1-z},$$

*and $\frac{z}{1-z}$ is the best dominant of (12).* □

**Corollary 2.14.** *Let $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$ and $0 < \mu \leq \frac{1}{2}$. If*
(13)

$$Re\left(\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1-\left(\frac{z}{f(z)}\right)^{\alpha}\right) > C = \frac{3}{2}\begin{cases} 1-\frac{1}{1-\mu}; & 0<\mu\leq\frac{1}{4} \\ 1-\frac{1}{3\mu}; & \frac{1}{4}\leq\mu\leq\frac{1}{2} \end{cases}$$

*then* $Re\left(\frac{f(z)}{z}\right)^{\alpha} > 1-\mu$.

*Proof.* We use Corollary 2.12 by showing that the inequality (13) implies subordination (9). The definition of $h_2(z)$ shows that $h_2(0) = 0$, and that $h_2(z)$ is close-to-convex (hence univalent) in $\mathbb{D}$. This is easy to check, because the proofs of Corollary 2.12 and Theorem 2.1 show that $Q(z) = \frac{zq'(z)}{1+q(z)}$ is starlike and $Re\frac{zh_2'(z)}{Q(z)} > 0$ in $\mathbb{D}$. Therefore the subordination (9) is equivalent to the following condition:

$$\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+1-\left(\frac{z}{f(z)}\right)^{\alpha} \in h_2(\mathbb{D}); \quad (z \in \mathbb{D}).$$

Now, we analyze $h_2(\mathbb{D})$. For $\lambda \neq 0$, we have:

$$\begin{aligned} Reh_2\left(e^{i\theta}\right) &= 1-\frac{1}{\lambda}+Re\left(\frac{1}{1-e^{i\theta}}\right)+\left(\frac{1}{\lambda}-2\right)Re\left(\frac{1}{1-\lambda e^{i\theta}}\right) \\ &= \frac{3}{2}-\frac{1}{\lambda}+\left(\frac{1}{\lambda}-2\right)Re\left(\frac{1}{1-\lambda e^{i\theta}}\right). \end{aligned}$$

Note that

$$0 < \frac{1}{1+\lambda} \leq Re\left(\frac{1}{1-\lambda e^{i\theta}}\right) \leq \frac{1}{1-\lambda}; \quad 0 \leq \lambda < 1.$$

So, for $0 \leq \lambda \leq \frac{1}{2}$; (i.e., $\frac{1}{4} \leq \mu \leq \frac{1}{2}$), we obtain:

$$\begin{aligned} Reh_2\left(e^{i\theta}\right) &\geq \frac{3}{2}-\frac{1}{\lambda}+\left(\frac{1}{\lambda}-2\right)\frac{1}{1+\lambda} \\ &= \frac{3}{2}\left(1-\frac{2}{1+\lambda}\right) = \frac{3}{2}\left(1-\frac{1}{1-\mu}\right). \end{aligned}$$

Also, if $\frac{1}{2} < \lambda < 1$; (i.e., $0 < \mu < \frac{1}{4}$), we have:

$$\begin{aligned} Reh_2\left(e^{i\theta}\right) &\geq \frac{3}{2}-\frac{1}{\lambda}+\left(\frac{1}{\lambda}-2\right)\frac{1}{1-\lambda} \\ &= \frac{3}{2}-\frac{1}{1-\lambda} = \frac{3}{2}\left(1-\frac{1}{3\mu}\right). \end{aligned}$$

Therefore we have:

$$Reh_2\left(e^{i\theta}\right) \geq \begin{cases} k_1 = \dfrac{3}{2}\left(1 - \dfrac{1}{3\mu}\right); & 0 < \mu \leq \dfrac{1}{4} \\ k_2 = \dfrac{3}{2}\left(1 - \dfrac{1}{1-\mu}\right); & \dfrac{1}{4} \leq \mu \leq \dfrac{1}{2}. \end{cases}$$

For the imaginary part of $h_2\left(e^{i\theta}\right)$, we know that the image of the function $\dfrac{1}{1-z}$ for all $z \in \mathbb{D}$ is a half plane with $Re\left(\dfrac{1}{1-z}\right) > \dfrac{1}{2}$. So, we conclude that $Imh_2(z)$ takes values within the whole set of real numbers.

The previous calculations over real and imaginary parts of $h_2\left(e^{i\theta}\right)$ yield that:

$$\{x + iy : x > C_1, y \in \mathbb{R}\} \subseteq h_2(\mathbb{D})$$

where $C_1 = \max\{k_1, k_2\}$. Finally note that, by easy computations we can show that, $C_1 = C < 0$ for $0 < \mu \leq \frac{1}{2}$, $\max\{k_1, k_2\} = k_1$, for $\frac{1}{4} \leq \mu \leq \frac{1}{2}$ and $\max\{k_1, k_2\} = k_2$ when $0 < \mu \leq \frac{1}{4}$, which proves that (13) implies (9). Now we see that all conditions of (Corollary 2.12) are satisfied and we conclude that $Re\left(\dfrac{f(z)}{z}\right)^{\alpha} > 1 - \mu$. □

Finally, taking $\mu = 1$ in Corollary 2.12, we obtain the following result.

**Corollary 2.15.** *Let $f \in \mathcal{A}$, $\dfrac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. If $\alpha > 0$ and*

$$\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + 1 - \left(\frac{z}{f(z)}\right)^{\alpha} \in \mathbb{C}\backslash\left\{1 + iy : y \in \mathbb{R}, |y| \geq \sqrt{3}\right\} \equiv \Omega, \quad (z \in \mathbb{D})$$

*then $Re\left(\dfrac{f(z)}{z}\right)^{\alpha} > 0$, and the result is sharp, i.e., the number zero in the conclusion cannot be replaced by a larger number such that the implication holds.*

*Proof.* To prove the result, it is sufficient to show that $h_2(\mathbb{D}) = \Omega$. For $\mu \neq \frac{1}{2}$, we have

$$h_2(z) = 1 + \frac{1}{1-z} - \frac{1}{\lambda} + (\frac{1}{\lambda} - 2)\frac{1}{1-\lambda z}; \quad (\lambda = 1 - 2\mu)$$

So, taking $\mu = 1$; (or $\lambda = -1$) in $h_2(z)$ we obtain:

$$h_2(z) = 2 + \frac{1}{1-z} - \frac{3}{1+z}$$

Now, by simple calculations we determine $h_2\left(e^{i\theta}\right)$:

$$\begin{aligned} h_2\left(e^{i\theta}\right) &= 2+\frac{1-e^{-i\theta}}{\left|1-e^{i\theta}\right|^2}-\frac{3\left(1+e^{-i\theta}\right)}{\left|1+e^{i\theta}\right|^2} \\ &= 2+\frac{1-\cos\theta+i\sin\theta}{2\left(1-\cos\theta\right)}-\frac{3\left(1+\cos\theta-i\sin\theta\right)}{2\left(1+\cos\theta\right)} \\ &= 1+\frac{1}{2}\sin\theta\left(\frac{3}{1+\cos\theta}+\frac{1}{1-\cos\theta}\right)i. \end{aligned}$$

Analyzing $Reh_2\left(e^{i\theta}\right)$ and $Imh_2\left(e^{i\theta}\right)$, we obtain: $Reh_2\left(e^{i\theta}\right)=1$ and

$$\begin{aligned} f(\theta)=Imh_2\left(e^{i\theta}\right) &= \frac{1}{2}\sin\theta\left(\frac{3}{1+\cos\theta}+\frac{1}{1-\cos\theta}\right) \\ &= \frac{2-\cos\theta}{\sin\theta}. \end{aligned}$$

Determining the maximum and minimum of $f(\theta)$ we have $\left|Imh_2\left(e^{i\theta}\right)\right|\geq\sqrt{3}$. So, for $\mu=1$, $h_2(\mathbb{D})=\Omega$, and all conditions of Corollary 2.12 are satisfied. Therefore the implication $Re\left(\frac{f(z)}{z}\right)^{\alpha}>0;\quad(z\in\mathbb{D})$ and its sharpness part follow directly from Corollary 2.12. □

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## 3. Acknowledgment

The authors would like to thank the referees for their useful suggestions and comments during the work on this paper.

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