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ON PRODUCT STABLE QUOTIENT ORDER-HOMOMORPHISMS

N. Nazari and Gh. Mirhosseinkhani*

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Abstract. In this paper, we study the properties of some classes of quotient order-homomorphisms, as product stable in the category of topological fuzzes. We define the concept of a bi-quotient order-homomorphism and show that for Hausdorff topological fuzzes, a quotient order-homomorphism $f : L_1 \to L_2$ is product stable if and only if f is bi-quotient and L_2 is a core compact topological fuzz.

Keywords: Topological fuzz, quotient order-homomorphism, product stable.
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1. Introduction

In 1992, Wang introduced the theory of topological molecular lattices as a generalization of ordinary topological spaces, fuzzy topological spaces and L-fuzzy topological spaces in terms of closed elements, molecules, remote neighbourhoods and generalized order-homomorphisms [21]. Then, many authors characterized some topological notions in such spaces, such as convergence theories of molecular nets or ideals [3, 5], separation axioms [6, 10], generalized topological molecular lattices [9, 20] and other notions.

Topological fuzzes are an important class of topological molecular lattices for studying fuzzy topological spaces [12]. The category of all fuzzes with their homomorphisms is denoted by **Fuzz**, and the category of all topological fuzzes with their homomorphisms is denoted by **TopFuzz**. It is well known that these categories are both complete and cocomplete, and some categorical structures of them were introduced by many authors [4, 8, 12, 23]. The category **Top** of all topological spaces, as a full subcategory of **TopFuzz**, is a reflective and co-reflective subcategory, as stated in [15, 16]. So **TopFuzz** is a large category containing **Top** but it is not a cartesian closed category. Recall that an object A of a category **C** with finite products is called exponentiable if the product functor $A \times - : \mathbf{C} \to \mathbf{C}$ has a right adjoint. If all the objects of **C** are exponentiable, then **C** is called cartesian closed [1]. Some characterizations

*Corresponding author
E-mail: gh.mirhosseini@yahoo.com
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of the exponentiable objects in **TopFuzz** and the category of all topological molecular lattices were introduced in [2,14].

There are some characterizations of product stable quotient maps in **Top**. In particular, a topological space X is exponentiable if and only if the identity map $id_X : X \to X$ is product stable, i.e., the product map $id_X \times g$ is quotient for every quotient map g [7,11,17,18]. A quotient order-homomorphism f is called product stable if the product order-homomorphism $f \otimes g$ is quotient for every quotient order-homomorphism g. In this paper, we present some characterizations of product stable quotient order-homomorphisms. We first give some preliminaries. In section 3, we define the concept of a bi-quotient order-homomorphism and show that for Hausdorff topological fuzzes, a quotient order-homomorphism $f : L_1 \to L_2$ is product stable if and only if f is bi-quotient and L_2 is a core compact topological fuzz in the sense of Akbarpour and Mirhosseinkhani [2]. In section 4, we introduce some notions as core compact order-homomorphisms and locally compact topological fuzzes and show that for locally compact Hausdorff topological fuzzes, a quotient order-homomorphism f is product stable if and only if it is core compact. For general categorical background we refer to [1].

2. **Preliminaries**

In this section, we recall some definitions and properties of fuzzes and topological fuzzes.

An element a of a lattice L is called coprime, if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$, for every $b, c \in L$. We denote by $M(L)$ the set of all non-zero coprime elements of L. Non-zero coprime elements are also called molecules. If F is a completely distributive complete lattice, then F is $\vee$-generated by the set $M(F)$, i.e., every element of F is a join of some elements of $M(F)$. Thus, a completely distributive complete lattice is called a molecular lattice [21].

Definition 2.1. [12] A fuzz is a pair $(L, ')$ consisting of a molecular lattice L and order-reversing involution $' : L \to L$, that is, $x \leq y$ if and only if $y' \leq x'$ and $x'' = x$ for all $x, y \in L$.

Definition 2.2. [12] A topological fuzz is a triple $(L, ', \tau)$ such that $(L, ')$ is a fuzz and $\tau \subseteq L$ is a topology, i.e., it is closed under finite meets, arbitrary joins and $0, 1 \in \tau$, where 0 and 1 are the smallest and the greatest elements of L, respectively. Every element of a topology τ is called open and every element of τ' is called closed, where $\tau' := \{a' \mid a \in \tau\}$.

Let $f : F \to G$ be a mapping between complete lattices such that preserves arbitrary joins. Then f has a right adjoint and denoted by $\hat{f}$. Moreover, $\hat{f}(y) = \bigvee\{x \in F \mid f(x) \leq y\}$ for every $y \in G$.

Definition 2.3. [22] A mapping $f : (L_1, ') \to (L_2, ')$ between fuzzes is called an order-homomorphism, if f preserves arbitrary joins and $\hat{f}$ preserves $'$.

Definition 2.4. [12] An order-homomorphism $f : (L_1, ', \tau) \to (L_2, ', \eta)$ between topological fuzzes is said to be continuous if $b \in \eta$ implies $\hat{f}(b) \in \tau$.

An extra order $\lhd$ on a complete L is defined by $a \lhd b$, if for every subset $S \subseteq L$, $b \leq \vee S$ implies that there exists $s \in S$ such that $a \leq s$ [15].

Theorem 2.5. *[21] A complete lattice L is a molecular lattice if and only if $b = \vee \lhd(b) = \vee(\lhd(b) \cap M(L))$ for every $b \in L$, where $\lhd(b) = \{a \in L \mid a \lhd b\}$.*

Remark 2.6. [23] The binary product of two topological fuzzes $(L_1, ', \tau_1)$ and $(L_2, ', \tau_2)$ is as follows: $L_1 \otimes L_2 = \{D \subseteq L_1 \times L_2 \mid D = \bigcup_{(x,y)\in D} \lhd(x) \times \lhd(y)\}$, $D' = \bigcap_{(x,y)\in D}\{(\lhd(x') \times \lhd(1)) \cup (\lhd(1) \times \lhd(y'))\}$ for every $D \in L_1 \otimes L_2$. The topology of $L_1 \otimes L_2$ is generated by the subbase $\{\hat{\pi}_1(x) \mid x \in \tau_1\} \cup \{\hat{\pi}_2(y) \mid y \in \tau_2\}$, where the projection order-homomorphisms π_1 and π_2 are defined by $\pi_1(D) = \bigvee\{x \in L_1 \mid \exists y \in L_2, (x, y) \in D\}$ and $\pi_2(D) = \bigvee\{y \in L_2 \mid \exists x \in L_1, (x, y) \in D\}$.

Let $(L, ', \tau)$ be a topological fuzz. A binary relation $\ll$ on L is defined by $a \ll b$, if for every subset $A \subseteq \tau$, $b \leq \vee A$ implies that there exists a finite subset D of A such that $a \leq \vee D$ [2].

Definition 2.7. [2] A topological fuzz $(L, ', \tau)$ is called core compact if $b = \bigvee\{a \in \tau \mid a \ll b\}$ for every $b \in \tau$.

Theorem 2.8. *[15]* **Top** *is a reflective and coreflective full subcategory of* **TopFuzz** *via the embedding power functor ρ :* **Top** $\to$ **TopFuzz** *defined by $\rho(X, \tau) = (\rho(X), \tau)$, where $\rho(X)$ is the power set of X and involution on $\rho(X)$ is the subset complement.*

Remark 2.9. [11] A topological space X is exponentiable in **Top** if and only if it is core compact, in the sense that any given neighbourhood V of a point x of X contains an open neighbourhood U of x with the property that every open cover of V has a finite subcover of U. Thus, a topological space X is core compact if and only if $\rho(X)$ is a core compact topological fuzz.

Definition 2.10. [12] An element a of a topological fuzz L is said to be compact if $a \ll a$, and L is called compact if every its closed element is compact.

Definition 2.11. [13] A topological fuzz L is called T_1 if the following conditions hold:

(1) L is T_0, i.e., every $a \in L$ can be written in the form $a = \bigwedge_{i\in I} \bigvee_{j\in J_i} u_{ij}$, where u_{ij} is open or closed for every $i \in I$ and $j \in J_i$,
(2) L is R_0, i.e., every open element of L is a supremum of closed elements.

Definition 2.12. [13] A topological fuzz L is called Hausdorff (T_2) if the following conditions hold:

(1) L is T_0,
(2) L is R_1, i.e., every $a \in L$ can be written in the form $a = \bigwedge_{i\in I} \bigvee_{j\in J_i} u_{ij} = \bigwedge_{i\in I} \bigvee_{j\in J_i} \bar{u}_{ij}$, where u_{ij} is open and $\bar{u}_{ij}$ is the closure of u_{ij} for every $i \in I$ and $j \in J_i$.

3. Product stability

In this section, we define the concept of a bi-quotient order-homomorphism and give a characterization of product stable quotient order-homomorphisms by this concept. Similar to topological maps, a surjective order-homomorphism $f : (L_1, ', \tau_1) \to (L_2, ', \tau_2)$ is called quotient, if $\tau_2 = \{a \in L_2 \mid \hat{f}(a) \in \tau_1\}$. A quotient order-homomorphism $f : L_1 \to L_2$ is called product stable, if $f \otimes g$ is quotient for every quotient order-homomorphism g.

Let $(L, ', \tau)$ be a topological fuzz and $a \in L$. The interior of a is denoted by $int(a)$, where $int(a) = \bigvee\{t \in \tau \mid t \leq a\}$. We say that a is a neighbourhood of an element $m \in M(L)$, if $m \in \lhd(int(a))$. Also, the element $\bigwedge\{t \in \tau \mid a \leq t\}$ is denoted by $a^\bigstar$.

Theorem 3.1. *Let $f : (L_1, ', \tau_1) \to (L_2, ', \tau_2)$ be a continuous surjective order-homomorphism. Then the order-homomorphism $f \bigotimes id_Z : L_1 \bigotimes Z \to L_2 \bigotimes Z$ is quotient for every identity order-homomorphism id_Z if and only if f is quotient and satisfies the following condition:*
Given $m \in M(L_2)$, a neighbourhood v of m, and an open covering $\{b_\alpha\}_{\alpha\in\Lambda}$ of $\hat{f}(v)$, there exists a finite set $\{\alpha_1, ..., \alpha_n\} \subset \Lambda$ such that $m \in int(\{f(b_{\alpha_1}) \vee ... \vee f(b_{\alpha_n})\}^\bigstar)$.

Proof. Let $m \in M(L_2)$, $m \in \lhd(int(v))$ and $\hat{f}(v) \leq \bigvee_{\alpha\in\Lambda} b_\alpha$. We may without loss of generality suppose v to be open and $b_\alpha = b_\alpha \wedge \hat{f}(v)$, i.e., $b_\alpha \leq \hat{f}(v)$. Let $Z = \rho(\tau_2)$ be the power set of τ_2, and for each $\alpha \in \Lambda$ set $h_\alpha = \{W \in Z \mid f(b_\alpha) \leq w, \forall w \in W\}$. Note that $v \in h_\alpha$ for every α. Let C be the filter on Z generated by all the h_α for $\alpha \in \Lambda$, and give Z a topology by taking $q \in \tau_Z$ if and only if either $v \notin q$ or $q \in C$. In $L_1 \bigotimes Z$ consider the following subset:

$$S = \bigcup\{\lhd(p) \times \{a\} \mid p \in M(L_1), a \in \tau_2, f(p) \in \lhd(a)\}.$$

We show that S is open in $L_1 \bigotimes Z$. Let $\lhd(p) \times \{a\} \leq S$ for $a \in \tau_2$ and $p \in M(L_1)$. We consider two cases: $a = v$ or not. First let $a \neq v$ and set $D = \{b \mid a \leq b\} - \{v\}$. It is obvious that $D \in \tau_Z$ and $\lhd(p) \times \{a\} \leq \hat{\pi}_1(\hat{f}(a)) \wedge \hat{\pi}_2(D)$. On the other hand, if $\lhd(p') \times \{a'\} \leq \hat{\pi}_1(\hat{f}(a)) \wedge \hat{\pi}_2(D)$ for $a' \in \tau_2$ and $p' \in M(L_1)$ then $f(p') \in \lhd(a')$ and so $\lhd(p') \times \{a'\} \leq S$. Therefore $\hat{\pi}_1(\hat{f}(a)) \wedge \hat{\pi}_2(D) \leq S$, which shows that S is open.

Now, let $a = v$. Since $f(p) \in \lhd(a)$, there is $\alpha_0 \in \Lambda$ such that $p \leq b_{\alpha_0}$. Hence $\{v\} \in h_{\alpha_0}$ and $\lhd(p) \times \{v\} \leq \hat{\pi}_1(b_{\alpha_0}) \wedge \hat{\pi}_2(h_{\alpha_0}) \leq S$. Thus S is open in $L_1 \bigotimes Z$.

It is easy to check that S is the inverse image under $f \otimes id_Z$ of the subset

$$T = \bigcup\{\lhd(y) \times \{a\} \mid y \in \lhd(a) \cap M(L_2), a \in \tau_2\}.$$

Thus T is open because $f \otimes id_Z$ is quotient. Since $\lhd(m) \times \{v\} \leq T$, there are open sets $O \in \tau_2$ and $\bigcap_{i=1}^{i=n} h_{\alpha_i} \in \tau_z$ such that

$$\lhd(m) \times \{v\} \leq \hat{\pi}_1(O) \times \hat{\pi}_2(h_{\alpha_1} \cap ... \cap h_{\alpha_n}) \leq T.$$

On the other hand, $\{O\} \subseteq \bigcap_{i=1}^{i=n} h_{\alpha_i}$. In other word, if $w \in \tau_2$ and $f(b_{\alpha_1}) \vee ... \vee f(b_{\alpha_n}) \leq w$, then $\{w\} \in h_{\alpha_1} \cap ... \cap h_{\alpha_n}$ and so $m \leq O \leq w$. Therefore $m \in int((f(b_{\alpha_1}) \vee ... \vee f(b_{\alpha_n}))^{\bigstar})$.

Conversely, Let $B \subseteq L_2 \otimes Z$ such that $C = (\widehat{f \otimes id_Z})(B)$ be open in $L_1 \otimes Z$. It is suffices to show that B is open. Suppose $y \in M(L_2)$, $z \in M(Z)$, we prove that there exists $O \in \tau_{L_2 \otimes Z}$ such that $\lhd(y) \times \lhd(z) \leq O \leq B$. For $d \in M(Z)$ we define

$$C_d := \bigvee \{m \in M(L_1) \mid \lhd(m) \times \lhd(d) \leq C\}$$

and

$$B_d := \bigvee \{\eta \in M(L_2) \mid \lhd(\eta) \times \lhd(d) \leq B\}.$$

It is easy to check that $C_d = \hat{f}(B_d)$. Then C_d is open because C is open, and hence B_d is open because f is quotient. For each $x \in C_z$, $\lhd(x) \times \lhd(z) \leq C$, so there are open sets $t_x \in \tau_1$ and $u_x \in \tau_2$ such that $x \leq t_x$, $z \leq u_x$, and

$$\lhd(x) \times \lhd(z) \leq \hat{\pi_1}(t_x) \wedge \hat{\pi_2}(u_x) \leq C.$$

Then $C_z = \hat{f}(B_z) \leq \bigvee_{x \in C_z} t_x$ and $y \leq B_z$. Thus by hypothesis there are $x_1, ..., x_n \in C_z$ such that $y \in int((f(t_{x_1}) \vee ... \vee f(t_{x_n}))^{\bigstar})$.
Set $O = \lhd(int((f(t_{x_1}) \vee ... \vee f(t_{x_n}))^{\bigstar})) \times \lhd(u_{x_1} \wedge ... \wedge u_{x_n})$. If $d \in \lhd(u_{x_1} \wedge ... \wedge u_{x_n})$, then $x \leq t_{x_i} \leq C_d$ for $1 \leq i \leq n$, and so $f(t_{x_i}) \leq f(C_d) = B_d$. Thus $f(t_{x_1}) \vee ... \vee f(t_{x_n}) \leq B_d$ and since B_d is open, it follows that $int(((f(t_{x_1}) \vee ... \vee f(t_{x_n}))^{\bigstar}) \leq B_d$. Hence $\lhd(y) \times \lhd(z) \leq O \leq B$, which completes the proof. □

Theorem 3.2. *[2] Let L be a topological fuzz. Then the following statements are equivalent:*

(1) L is exponentiable.
(2) L is core compact.
(3) $f \bigotimes id_L : L_1 \bigotimes L \to L_2 \bigotimes L$ is quotient for every quotient order-homomorphism $f : L_1 \to L_2$.

Theorem 3.3. *A quotient order-homomorphism $f : L_1 \to L_2$ is product stable if and only if L_2 is core compact and the product map $f \bigotimes id_Z$ is quotient for every identity order-homomorphism id_Z.*

Proof. Let $f : L_1 \to L_2$ be a product stable quotient order-homomorphism. Then $id_{L_2} \bigotimes f$ is quotient and so L_2 is core compact by Theorem 3.2. Conversely, for every quotient order-homomorphism $g : Z \to Y$, $f \bigotimes g = (id_{L_2} \otimes g) \circ (f \otimes id_Z)$. Since L_2 is core compact, by Theorem 3.2, $id_{L_2} \otimes g$ is quotient. On the other hand, by hypothesis $f \bigotimes id_Z$ is quotient. Thus $f \bigotimes g$ is quotient for every quotient order-homomorphism g. □

By Theorems 3.1, 3.2 and 3.3, we have the following result.

Corollary 3.4. *A quotient order-homomorphism $f : L_1 \to L_2$ is product stable if and only if the following statements hold:*

(1) L_2 *is core compact.*
(2) *For every* $y \in M(L_2)$*, every neighbourhood* v *of* y*, and every open covering* $\{u_i\}_{i\in I}$ *of* $\hat{f}(v)$*, there exists a finite subset* F *of* I *such that* $y \in int((\bigvee_{i\in F} f(u_i))^\star)$*.*

Definition 3.5. Let $f : L_1 \to L_2$ be a continuous surjective order-homomorphism. Then f is called bi-quotient, if for every $m \in M(L_2)$ and each open covering $\{u_i\}_{i\in I}$ of $\hat{f}(m)$ there exists a finite subset F of I such that $m \in int(\bigvee_{i\in F} f(u_i))$.

If L is a T_1 topological fuzz, then $a^\star = a$ for every $a \in L$. Thus, by Corollary 3.4 and Definition 3.5, we have the following result.

Corollary 3.6. *Let* $f : L_1 \to L_2$ *be an order-homomorphism such that* L_2 *be a Hausdorff topological fuzz. Then* f *is product stable if and only if* f *is bi-quotient and* L_2 *is core compact.*

4. Compactness and product stability

In this section, we present some notions as f-relatively compactness, core compact order-homomorphism which are a generalization of the notions introduced in [18, 19]. We give some characterizations of product stability by these concepts.

Definition 4.1. Let $f : L_1 \to L_2$ be an order-homomorphism, $a \in L_2$ and $b \in L_1$. We say that a is f-relatively compact in b, written $a \ll_f b$, if for every open covering $\{u_i\}_{i\in I}$ of b, there exists a finite subset F of I such that $a \leq \bigvee_{i\in F} f(u_i)$.

Definition 4.2. Let $f : (L_1, \tau_1) \to (L_2, \tau_2)$ be a surjective order-homomorphism. We say that f is a core compact order-homomorphism, if $b = \bigvee\{a \in \tau_2 \mid a \ll_f \hat{f}(b)\}$ for every $b \in \tau_2$.

It is easy to check that if $f : L_1 \to L_2$ is a core compact order-homomorphism, then L_2 is a core compact topological fuzz. For every open order-homomorphism f, we have $a \ll b$ in L_2 if and only if $a \ll_f \hat{f}(b)$. Thus, an open order-homomorphism f is core compact if and only if L_2 is a core compact topological fuzz. In particular, an identity order-homomorphism $id_L : L \to L$ is core compact if and only if L is a core compact topological fuzz.

Theorem 4.3. *Every core compact quotient order-homomorphism is product stable.*

Proof. Let $f : (L_1, \tau_1) \to (L_2, \tau_2)$ be a core compact quotient order-homomorphism, $m \in \lhd(b) \cap M(L_2)$ and $b \in \tau_2$. Then $b = \bigvee\{a \in \tau_2 \mid a \ll_f \hat{f}(b)\}$. If $\hat{f}(b) \leq \bigvee_{i\in I} u_i$, then there exists a finite subset F of I such that $m \leq a \leq \bigvee_{i\in F} f(u_i) \leq (\bigvee_{i\in F} f(u_i))^\star$. Thus by Theorem 3.1, the result follows. □

Now, we define the concept of locally compact topological fuzzes and show that the converse of Theorem 4.3, is also true for locally compact Hausdorff topological fuzzes.

Definition 4.4. A topological fuzz L is said to be locally compact, if for each $b \in \tau$ and each compact element $m \in M(L) \cap \lhd(b)$, there exists a compact neighbourhood c of m such that $c \leq b$.

Clearly, if X is a locally compact (compact) topological space, then $\rho(X)$ is a locally compact (compact) topological fuzz. If $a \leq c \leq b$ such that c is compact, then $a \ll b$. This implies that every locally compact topological fuzz is core compact.

Lemma 4.5. *Every compact Hausdorff topological fuzz is locally compact.*

Proof. Let $(L, {}', \tau)$ be a compact Hausdorff topological fuzz, $b \in \tau$ and $m \in \lhd(b) \cap M(L)$. Since $\lhd$ satisfies the interpolation property [14], it follows that there exists $d \in L$ such that $m \lhd d \lhd b$. By assumption, $d = \bigwedge_{i\in I} \bigvee_{j\in J_i} u_{ij} = \bigwedge_{i\in I} \bigvee_{j\in J_i} \bar{u}_{ij}$, where u_{ij} is open. For each $i \in I$, $d \leq \bigvee_{j\in J_i} u_{ij} \leq \bigvee_{j\in J_i} \bar{u}_{ij}$. Thus $m \leq v_i \leq \bar{v}_i \leq \bigvee_{j\in J_i} \bar{u}_{ij}$, where $v_i = u_{ij}$ for some $j \in J_i$. Then $m \leq \bigwedge_{i\in I} v_i \leq \bigwedge_{i\in I} \bar{v}_i \leq d \leq b$. Hence $b' \leq \bigvee_{i\in I} \bar{v}_i'$. By compactness, there exists a finite subset F of I such that $\bigwedge_{i\in F} \bar{v}_i \leq b$. Let $a = \bigwedge_{i\in F} v_i$. Then $a \in \tau$ and $m \leq a \leq \bar{a} \leq b$. □

Theorem 4.6. *Let $f : (L_1, \tau_1) \to (L_2, \tau_2)$ be a quotient order-homomorphism and L_2 be a locally compact Hausdorff topological fuzz. Then f is core compact if and only if it is product stable.*

Proof. By Theorem 4.3, every core compact quotient order-homomorphism is product stable. Conversely, let $b \in \tau_2$ and $y \in \lhd(b) \cap M(L_2)$. Since L_2 is a core compact topological fuzz, $b = \bigvee\{a \in \tau_2 \mid a \ll b\}$. It is enough to show that $a \ll_f \hat{f}(b)$. Let $\{u_i\}_{i\in I}$ be an open covering of $\hat{f}(b)$. Every Hausdorff topological fuzz is T_1, by Theorem 3.1, for every $y \in \lhd(b) \cap M(L_2)$, there exist a finite subset F_y of I and $u_y \in \tau_2$ such that $y \leq u_y \leq \bigvee_{i\in F_y} f(u_i)$. Thus there exists a finite subset S of $\lhd(b) \cap M(L_2)$ such that $a \leq \bigvee_{y\in S} u_y$. Hence $a \leq \bigvee_{y\in S} \bigvee_{i\in F_y} f(u_{yi})$. □

In the following, we give some characterizations of product stable quotient order-homomorphisms by the notion of f-relatively compactness.

Definition 4.7. Let $f : L_1 \to L_2$ be an order-homomorphism. We say that f reflects relative compactness to f-relatively compactness, if $a \ll b$ in L_2 implies $a \ll_f \hat{f}(b)$.

Theorem 4.8. *Let $f : (L_1, \tau_1) \to (L_2, \tau_2)$ be a bi-quotient order-homomorphism. Then f reflects relative compactness to f-relative compactness between arbitrary elements.*

Proof. Let $a \ll b$ in L_2 and $\hat{f}(b) \leq \bigvee_{i\in I} u_i$. For every $y \in \lhd(b) \cap M(L_2)$, there exist a finite subset F_y of I and $u_y \in \tau_2$ such that $y \leq u_y \leq \bigvee_{i\in F_y} f(u_i)$. Thus there exists a finite subset S of $\lhd(b) \cap M(L_2)$ such that $a \leq \bigvee_{y\in S} u_y$. Hence $a \leq \bigvee_{y\in S} \bigvee_{i\in F_y} f(u_{yi})$. □

Theorem 4.9. *Let $f : (L_1, \tau_1) \to (L_2, \tau_2)$ be a surjective order-homomorphism and L_2 be a locally compact topological fuzz. Then the following statements are equivalent:*

(1) *f is core compact.*
(2) *f reflects relative compactness to f-relative compactness between open sets.*

Proof. The implication (1) ⇒ (2) is clear. Conversely, since every locally compact topological fuzz is core compact, it follows that $b = \bigvee\{a \in \tau_2 \mid a \ll b\}$ for every $b \in \tau_2$. By assumption, $b = \bigvee\{a \in \tau_2 \mid a \ll_f \hat{f}(b)\}$. Thus, (1) holds. □

By Theorems 4.9 and 4.6, we have the following main result.

Corollary 4.10. *Let $f : L_1 \to L_2$ be a quotient order-homomorphism and L_2 be a locally compact Hausdorff topological fuzz. Then the following statements are equivalent:*

(1) *f is product stable.*
(2) *f is core compact.*
(3) *f reflects relative compactness to f-relative compactness between open sets.*

5. Conclusion

It is well known that the category **TopFuzz** of topological fuzzes with their homomorphisms is both complete and cocomplete, and some categorical properties of it were introduced by many authors. In this paper, we have presented some characterizations of product stable quotient order-homomorphisms. We have defined the concept of a bi-quotient order-homomorphism and shown that for Hausdorff topological fuzzes, a quotient order-homomorphism $f : L_1 \to L_2$ is product stable if and only if f is bi-quotient and L_2 is a core compact topological fuzz. Also, we have introduced the concepts of core compact order-homomorphisms and locally compact topological fuzzes and shown that for locally compact Hausdorff topological fuzzes, a quotient order-homomorphism f is product stable if and only if it is core compact.

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Narges Nazari
Department of Mathematics
University of Hormozgan
Bandarabbas, Iran
E-mail address: nazarinargesmath@yahoo.com

Ghasem Mirhosseinkhani
Department of Mathematics and Computer Sciences
Sirjan University of Technology
Sirjan, Iran
E-mail address: gh.mirhosseini@yahoo.com, gh.mirhosseini@sirjantech.ac.ir