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FINDING FUZZY INVERSE MATRIX USING WU'S METHOD

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ABSTRACT. In this study, the concept of an inverse matrix including fuzzy number elements is extended. Such a concept may be performed in the modeling of uncertain and imprecise real-world problems. The problem of finding a fuzzy inverse matrix is converted to a problem to solve a system of fuzzy polynomial equations. Here, a fuzzy system is transformed to an equivalent system of crisp polynomial equations. The solution to the system of crisp polynomial equations is calculated using Wus method and a criterion is introduced for invertibility of a fuzzy matrix (FM). In addition, an algorithm is proposed to calculate the fuzzy inverse matrix. The most important advantage of the presented method is that it achieves whole inverse entries of an FM simultaneously. In the end, we provide some illustrative examples to show the efficiency and proficiency of our proposed algorithm.

Keywords: Wu's algorithm, Fuzzy number, Fuzzy matrix, Fuzzy identity matrix, Fuzzy linear equation system (FLES).
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1. Introduction

In the situation of happening fuzzy uncertainty in a real-world problem, we see those fuzzy matrices are effectively implemented. We have been witness of the popularity of fuzzy matrices in the recent decades [14,25–27,39]. In matrix theory, the generalized theory of fuzzy inverse matrix has an outstanding position [6,7]. Thomasan worked on the convergence of powers of fuzzy matrices in 1977 [32]. Kim and Roush [29] proposed a systematic expansion of the theory of FM. They also presented algorithms to calculate the inverse of an FM and generalized inverse of an FM. The principal concept of the present paper is the term fuzzy matrix having at least two different meanings in the literature. If $a_{ij} \in [0,1], (i = 1,2,...,m; j = 1,2,...,n)$, matrix $A = (a_{ij})_{m\times n}$ in the first class is known as an FM. The details of them first expressed in [29] and appeared by fuzzy relations. Afterwards, this theme has been of much interest [9,14,23,31]. For instance, Hashimoto [23] used the operator of Gödel-implication and shown some important features of the fuzzy matrices sub-inverses of the first class. Also, their regularity properties was introduced by Cho [9] in 1999. The authors in [5,13,24] called a matrix including fuzzy number entries as an FM. Since the arithmetic structure is complicated, the investigation of another class is ignored. To find a fuzzy inverse matrix consisting of fuzzy numbers of the type LR,

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Basaran introduced a method in 2012 [1]. In a fuzzy case, computation of inverse of a matrix needs to solve the $n \times n$ by $n \times n$ equation system(ES) in which all the values of right-hand side, unknowns, and coefficients are fuzzy numbers. During solving such an ES, the author introduced concepts of the fuzzy one and the fuzzy zero numbers. According to those, the author also defined the fuzzy identity matrix and he used it to calculate the fuzzy inverse matrix approximation. In the computing of the inverse approximation of an FM by his method, should be pay attention to one thing. Generally, the solution of a built FLES depends on the parameter unless the extension values of the fuzzy identity matrix are determined before. In addition, one can directly find the inverse of an FM via fixing the extension values of the fuzzy identity matrix. On one hand, the decision-maker can choose "the best solution" among them when the system is parametrically solved. On the other hand, in the case of not being interested in multiple solutions, one can fix the extensions to be for example 0.5. Next, the researcher decides to find the inverse of an FM. Mosleh and Otadi [30] showed that the inverse approximation of an FM proposed by Basaran in [1] is not correct. In addition, we proposed an eigenvalue technique to obtain the fuzzy inverse matrix [16]. In this paper, our main focus of attention will be on this class of fuzzy matrices. In the recent research works, calculating inverse and invertibility investigating of square interval matrices have been more interesting topics. In the current work, an approach on the base of Wu's method introduced to obtain the fuzzy inverse matrix. Wu's method proposed by Wen-Tsun Wu who is a Chinese mathematician in the late 1970s to solve multivariate polynomial equations [35]. This technique is on the base of mathematical concept of characteristic set (CS) which J.F. Ritt first introduced in the late 1940s. Some smooth algorithms have been developed for zero decomposition of arbitrary systems of polynomials by Wu Wen-Tsun whose Ritts theory has been notably improved since 1980 [34, 36]. Many problems in engineering, economics and science have successfully used the Ritt-Wu's method [38]. This method is completely independent of the method of Gröbner basis which was proposed in 1965 by Bruno Buchberger, even if bases of the Gröbner may be implemented to calculate the CSs. It is also more widely used than the method of Gröbner basis in practice since it is commonly more efficient [8, 20, 28]. Using the algorithm of Wu for solving the systems of polynomial equations leads to solving sets of characteristic. When these kind of sets have the structure of triangle, one can easily calculate the variety of them. Since the first equation of a system with triangular structure has only one variable, its solving is easy. Therefore, a common method may be used to obtain the root of this polynomial of one variable. Firstly, we find the root of the first equation. Then, we substitute it into the second polynomial equation of two variables which lead to compute its solution. This procedure will be continued till achieving all solutions of the system like the forward substitution.

In this way, a bridge between finding fuzzy inverse matrix and the CSs variety is made. The main idea of our method for finding fuzzy inverse matrix is transforming an FM into a crisp system of polynomial equations. The numerical approaches to solve a polynomial equations system have some disadvantages as follows:

- Knowing being positive or negative of the solutions is necessary in the methods. Unless, the methods cannot be used.
- It is not easy to determine an appropriate initial point for these methods.
- Only some approximate solutions can be found in these methods.
- There is not any necessary and sufficient conditions or criteria to distinguish whether the solution of the fuzzy systems exists in these methods.
- We do not know the number of solutions for the fuzzy systems by these methods.
- If there is not any solutions for the fuzzy systems, then these methods lead to misleading.

Nevertheless, the existing methods have the aforementioned disadvantages. To tackle such disadvantages, we are interested in presenting a new technique on the base of the Wu's Method. By using the algorithm of Wu, the crisp system variety is obtained. Therefore, all solutions of the crisp system can be found as elements of the inverse of the FM. The significant merit of our method is that it achieves total entries of the inverse of an FM, simultaneously. Moreover, we propose a criteria according to Wu's technique for FM invertibility.

The remaining of the paper is structured as follows. Section 2 has two subsections. In the first subsection, we present some necessary definitions and results of fuzzy numbers. In the next subsection, the Wu's algorithm is introduced. Our proposed method to calculate the inverse of an FM is given in Section 3. In addition, a criteria is presented for when an FM has inverse. Moreover, an efficient algorithm is suggested to calculate the inverse of an FM. Section 4 contains some illustrative examples to demonstrate the proficiency and efficiency of the proposed algorithm. In the end, Section 5 concludes the paper.

2. Preliminaries

We divide this section into two subsections. The first one contains an introduction of preliminaries on fuzzy matrices, fuzzy numbers, and fuzzy arithmetic. The next subsection also introduces the main concepts regarding Wu's algorithm and polynomials.

2.1. Fuzzy background.

We review some required notation and background of the theory of fuzzy sets in this subsection.

The class of fuzzy numbers is denoted by E, i.e. upper semicontinuous, convex, compactly supported, and normal fuzzy subsets of the real numbers. For the following definitions, consider $\tilde{u}$ as a fuzzy number.

Definition 2.1. [30] $\tilde{u}$ is called an LR fuzzy number if

$$\tilde{u}(x) = \begin{cases} L(\frac{u-x}{\alpha}) & x \leq u, \alpha > 0, \\ \\ R(\frac{x-u}{\beta}) & x \geq u, \beta > 0, \end{cases}$$

in which α denotes the left spread, β presents the right spread, u denotes the mean value of $\tilde{u}$, and the function $L(.)$, which is said to be left shape function, satisfies

(1) $L(x)$ is non increasing on $[0, \infty)$.
(2) $L(0) = 1$ and $L(1) = 0$.
(3) $L(x) = L(-x)$.

The definitions of $L(.)$ and right shape function $R(.)$ are usually similar.

The $\tilde{u} = (u, \alpha, \beta)_{LR}$ is symbolically shown for the shape functions, left spread and right spread, and the mean value of an LR fuzzy number $\tilde{u}$. In LR representation, the L and R as reference functions are linear, and the fuzzy numbers of triangular are fuzzy numbers. The fuzzy number $\tilde{u}$ is said to be a symmetric fuzzy number when α and β are the spreads [13].

Definition 2.2. The $\tilde{u}$ is said to be negative (positive), represented by $\tilde{u} < 0$ ($\tilde{u} > 0$), if $u(x) = 0, \forall x > 0 (\forall x < 0)$ satisfied with its membership function $u(x)$

Definition 2.3. Consider $\tilde{v} = (v, \gamma, \delta)$ and $\tilde{u} = (u, \alpha, \beta)$ as two fuzzy numbers of LR type then

(1) $-\tilde{u} = -(u, \alpha, \beta)_{LR} = (-u, \beta, \alpha)_{LR}$.
(2) $\tilde{u} \ominus \tilde{v} = (u, \alpha, \beta)_{LR} \ominus (v, \gamma, \delta)_{LR} = (u - v, \alpha + \delta, \beta + \gamma)_{LR}$.
(3) $\tilde{u} \oplus \tilde{v} = (u, \alpha, \beta)_{LR} \oplus (v, \gamma, \delta)_{LR} = (u + v, \alpha + \gamma, \beta + \delta)_{LR}$.

Definition 2.4. For the fuzzy numbers $\tilde{v}$ and $\tilde{u}$ as given in Definition 2.3, the multiplication of them is defined as follows:

$$\tilde{u} \otimes \tilde{v} = (u, \alpha, \beta)_{LR} \otimes (v, \gamma, \delta)_{LR} = (uv, m\gamma + n\alpha, m\delta + n\beta)_{LR}$$

for $\tilde{u}, \tilde{v}$ positive;

$$\tilde{u} \otimes \tilde{v} = (u, \alpha, \beta)_{LR} \otimes (v, \gamma, \delta)_{LR} = (uv, -v\beta - u\delta, -v\alpha - u\gamma)_{LR}$$

for $\tilde{u}, \tilde{v}$ negative, and

$$\tilde{u} \otimes \tilde{v} = (u, \alpha, \beta)_{LR} \otimes (v, \gamma, \delta)_{LR} = (uv, v\alpha - u\delta, v\beta - u\gamma)_{LR}$$

for $\tilde{v}$ positive, $\tilde{u}$ negative.

Remark 2.5. The resulting fuzzy number is an approximated result.

Definition 2.6. The Scalar multiplication of two fuzzy numbers $\tilde{u}$ and $\tilde{v}$ given in Definition 2.3 is defined as follows:

$$\lambda \otimes \tilde{u} = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta)_{LR} & \lambda > 0, \\ (\lambda m, -\lambda\beta, -\lambda\alpha)_{LR} & \lambda < 0, \end{cases}$$

Definition 2.7. When $[a_1, a_2]$ be the support of a fuzzy number, then a fuzzy number is said to be positive if $0 \leq a_1 \leq a_2$. Similarly, a fuzzy number is said to be negative if $a_1 \leq a_2 < 0$. Finally, a fuzzy number is said to be zero if $a_1 \leq 0 \leq a_2$.

The authors in [12,13] introduced an FM as a rectangular array of fuzzy numbers. The authors in [11] defined a formal definition of FM as below:

Definition 2.8. If each element of $\tilde{A} = (\tilde{a}_{ij})$ is a fuzzy number, we say that matrix $\tilde{A}$ is an FM.

Also, let $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{B} = (\tilde{b}_{ij})$ be two fuzzy matrices of orders $m \times n$ and $n \times p$, respectively. The product order of two fuzzy matrices is $m \times p$ and is given as below:

$$\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij}),$$

where $\tilde{c}_{ij} = \sum_{k=1,\dots n}^{\oplus} \tilde{a}_{ik} \otimes \tilde{b}_{kj}$ in which approximated multiplication denoted by $\oplus$.

In the same manner, an FM A consists of the right spread, the left spread, and center parts only as a fuzzy number. We can write its parametric form as below:

$$\tilde{A} = (A, L, R),$$

in which L denotes the left spread, R represents the right spread, and A denotes the center which are crisp matrices. Also, their sizes are the same [11]. For more details of basic properties of Fuzzy matrices one can refer to [39].

2.2. **Wu's Algorithm and Varieties.** In this subsection, we introduce the algorithm of Wu and its relation with varieties. Consider $\Gamma = \mathbb{K}[x_1, \dots, x_n]$ as the polynomial ring over a field $\mathbb{K}$ of characteristic zero in n variables. We assume the variables $x_1, \dots, x_n$ ordered so that $x_i < x_j$ for $i < j$. If we select the variable x_m, then one can write a polynomial $f \in \Gamma$ as a univariate polynomial in variable x_m as follows:

$$f = I_t x_m^t + I_{t-1} x_m^{t-1} + \cdots + I_0,$$

where t is the degree of f regarding x_m and is represented by $\deg_{x_m}(f)$, we have

$$I_i \in \mathbb{K}[x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_n],$$

for $0 \le i \le t$. The leading coefficient I_t is represented by $lc(f, x_m)$. The $class(f)$ is the class of f and is known as the greatest subscript c of x appearing in f. The class of a constant function is zero. The x_c is leading variable and $lc(f, x_c)$ is called the initial of f, which are represented by $lv(f)$ and $ini(f)$, respectively.

Definition 2.9. Let $g \in \Gamma$ be a polynomial. g is said to be reduced regarding f if $deg_{x_c}(g) < deg_{x_c}(f)$ where $class(f) = c \neq 0$.

Definition 2.10. If a polynomial g is reduced regarding any $f \in F$, then g is reduced regarding $F \subset \Gamma$.

A partial order of polynomials can be defined as below:

Definition 2.11. For given $f,\ g \in \Gamma$, when one of the below statements holds:

(1) $deg_{x_c}(f) < deg_{x_c}(g)$,
(2) $class(f) < class(g)$ and
(3) $c = class(f) = class(g)$,

the polynomial g has a higher rank than f and represented by $g > f$

Definition 2.12. If $deg_{x_c}(f) = deg_{x_c}(g)$ and $class(f) = class(g) = c$ or both polynomials are constant, then we have the equivalence of two polynomials f and g and we denote it by $f \sim g$.

Definition 2.13. A triangular set is an ordered polynomial set $F = \{f_1, f_2, \ldots, f_r\}$ such that either $class(f_1) < class(f_2) < \cdots < class(f_r)$ or $r = 1$.

Definition 2.14. F as a triangular set is said to be a set of ascending if f_j is reduced regarding f_i for $i < j$.

Here, we extend the partial order of polynomials to create a partial order for the sets of ascending.

Definition 2.15. Let two ascending sets $F = \{f_1, \ldots, f_r\}$ and $G = \{g_1, \ldots, g_k\}$, we say $F < G$ if one of the below statements:

(1) There is $j \leqslant \min\{r, k\}$ such that $f_i \sim g_i$ for $i < j$, but $f_j < g_j$,
(2) $r > k$ and $f_i \sim g_i$ for all $i \leqslant k$,

holds.

For sets of ascending and incomparable F and G, we can write $F \sim G$.

Lemma 2.16. *[37] An ascending sequence of sets with a steady lowering order is finite.*

By Lemma 2.16, there exists the set of ascending with the lowest rank which consists of polynomials selected from the F and is said to be a basic set of F. In the polynomial set there are two basic sets with the same cardinality. Using Theorem 4.11 of [15], we can build the following algorithm to compute a basic set of the polynomial set.

Algorithm 1 ALGORITHM OF BASIC SET

Require: F as a non-empty subset of Γ
Ensure: B as a basic set of F which

i. B is an empty set
ii. While F is not an empty set, Do
  $ii.1.$ B be the union of B and $\{b\}$ in which b is a polynomial with the minimal rank in F,
  $ii.2.$ F is the set of $f \in F$ in which f is reduced regarding b,
iii. Return B

Lemma 2.17. *[37] Consider a polynomial set F and B as a basic set of it. If $g \in \Gamma$ is reduced regarding F, so there exists a basic set of the union of F and g having lower rank than F.*

Now, we introduce a division algorithm which is called the pseudo-division for multivariable polynomials. The Wu's algorithm will use this algorithm.

Proposition 2.18. *[10] Consider f and g in Γ and $class(f) = c$. So, the following equation exists*

$$I_c^m g = qf + r,$$

where $m \geqslant 0$, $q, r \in \Gamma$, $I_c = ini(f)$ as well as either $r = 0$ or r is reduced regarding f.

The polynomials q and r in the above proposition are respectively the pseudo-quotient of g on its pseudo-division by f (when m is not minimal, it is not unique) and the pseudo-remainder $prem(g, f)$. By the below algorithm we can compute the the pseudo-quotient and pseudo-remainder $prem(g, f)$:

Algorithm 2 ALGORITHM OF PSEUDO-DIVISION

Require: g and f in Γ

Ensure: q pseudo-quotient and r pseudo-remainder of g on its pseudo-division by f

i. Consider $q := 0$ and $r := g$

ii. While r is not equal to 0 and $deg_{x_c}(r) \geq deg_{x_c}(f)$ where $x_c = lv(f)$ Do

  $ii.1.$ $r := initial(f)r - lc(r, x_c)fx_c^{deg_{x_c}(r)-deg_{x_c}(f)}$

  $ii.2.$ $q := initial(f)q - lc(r, x_c)x_c^{deg_{x_c}(r)-deg_{x_c}(f)}$

iii. Return r, q

Suppose that $g \in \Gamma$ and $F = f_1, ..., f_r$ be an ascending set. The successive pseudo-divisions as below

$$R = prem(...prem(prem(g, f_r), f_{r-1}), ..., f_1),$$

can be used to obtain the below remainder formula:

$$I_1^{s_1} I_2^{s_2} \cdots I_r^{s_r} g = \sum q_i f_i + R, \tag{1}$$

where $I_i = ini(f_i)$, $s_i \geqslant 0$, $q_i \in \Gamma$ and R is reduced regarding F. If each s_i chosen to be small enough, then R is unique and represented by $prem(g, F)$. For a finite subset G from Γ, we consider $prem(G, F)$ to be

$$\{prem(g, F) \mid g \in G\}.$$

Let $\langle F \rangle$ as an ideal generated in Γ by F.

Definition 2.19. Consider $F \subset \Gamma$. Then, the variety defined by F is the set

$$V(F) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n \mid f(a_1, \ldots, a_n) = 0, \text{ for all } f \in F\}.$$

Definition 2.20. Suppose that $G \subset \Gamma$ be a polynomial set, then the quasi-algebraic variety defined as $V(F/G) = V(F) \backslash V(G)$.

Definition 2.21. A CS of $\emptyset \neq F \subset \Gamma$ is an ascending set B in Γ such that $prem(F, B) = \{0\}$ and $B \subset \langle F \rangle$.

Similar to the procedure stated in [34], the below algorithm can help us to compute a CS.

Algorithm 3 ALGORITHM OF CHARACTERISTIC SET

Require: F as a non-empty subset of Γ
Ensure: B as a characteristic subset of F

i. $S := F$
ii. Select B as a basic set of S
iii. In the case $prem(F, B)$ is not equal to $\{0\}$
 $iii.1.$ $S := prem(F, B) \cup S \backslash \{0\}$
 $iii.2.$ Go to step ii
iv. Return B

In the following theorem, the most important properties of the CSs are summed up.

Theorem 2.22. *[34] (Well-ordering Principle of Wu) Consider B as a CS of $F \subset \Gamma$. Then*

$$V(F) = V(B/I_B) \bigcup \cup_{b \in B} V(F \cup B \cup \{ini(b)\})$$

where $I_B = \prod_{b \in B} ini(b)$.

The main key in the Algorithm of Wu for the computation of varieties is the following corollary.

Corollary 2.23. *Using again of the Well-ordering Principle Theorem of Wu, $b \in B$, $B \cup F \cup \{initial(b)\}$, after a finite number of steps the procedure will end. Hence, one can obtain $V(F)$ as a union of several finite numbers of $V(B/I_B)$ as varieties.*

Proof. This can be proved regarding Lemmas 2.16 and 2.17.

The algorithm of Wu is used on the base of Well-ordering Principle Theorem of Wu to provide all needed CSs for calculating $V(F)$.

Algorithm 4 ALGORITHM OF WU

Require: F as a nonempty subset of Γ,
Ensure: Z, as a set of CSs in which $I_B = \prod_{b \in B} ini(b)$ and $V(F) = \bigcup_{B \in Z} V(B/I_B)$,

i. Let $D := \{F\}$ and $Z := \emptyset$,
ii. While D is not an empty set Do
 $ii.a.$ Choose F' as an element of D
 $ii.b.$ Let D as $D \backslash \{F'\}$
 $ii.c.$ Select B as a CS of F'
 $ii.d.$ In the case B is not equal to $\{1\}$ then
  $ii.d.1.$ Let Z as $Z \cup \{B\}$
  $ii.d.2.$ Let D as $D \bigcup \cup_{b \in B} \{F' \cup B \cup \{ini(b) \mid ini(b) \neq 1\}\}$
iv. Return Z

The $V(F)$ can be written as a union of quasi-algebraic variety of CSs using Wu's algorithm. Therefore, $V(F)$ can be found easily since the solving of these sets are easy. Now, by the following example we will illustrate Wu's algorithm to solve a polynomial system.

Example 2.24. Consider $F = \{x+y+xy, x+y+xy^2\}$, for $y < x$ we apply Wu's algorithm to F. Let $F' := F$, therefore $D = \emptyset$. $B = \{y^3 - y^2, x+y+xy\}$ is a CS. Also, $Z := \{B\}$. In addition, $ini(x+y+xy) = y+1$ and $ini(y^2 - y^3) = 1$, hence, $D := \{\{y+1\} \cup F'\}$. Next, let $F' := \{x+y+xy, x+y+xy^2, y+1\}$. Now, $D = \emptyset$ and $\{1\}$ is a CS of F'. So, the output is $Z = \{\{y^3 - y^2, x+y+xy\}\}$ and

$$V(F) = V(\{y^3 - y^2, x+y+xy\}) \backslash V(y+1) = \{(x=0, y=0), (x = -\frac{1}{2}, y=1)\}.$$

3. **Calculating the fuzzy inverse matrix**

In this section, we first present several necessary definitions to assist the computation of the inverse of an FM. Therefore, a novel method is proposed for calculation of the inverse of an FM on the base of Gröbner basis benefits.

Definition 3.1. We say that a fuzzy number is fuzzy one number if its left and right spread value is ϵ where $0 < \epsilon < 1$ and ϵ is sufficiently small and its center value is 1. It is denoted by $\tilde{1} = (1, \epsilon)$.

We can clearly see that the spread value ϵ is the values between 0 and 1. Then, for a fuzzy one number the end points of the right and left change between $1 + \epsilon < 2$ and $0 < 1 - \epsilon$. Moreover, the below definition is similar for fuzzy zero numbers.

Definition 3.2. We say that a fuzzy number is a fuzzy zero number if its left and right spread value is δ where $0 < \delta < 1$ and δ is small enough and the value of its center is 0. It is denoted by $\tilde{0} = (0, \delta)$.

Then, for a fuzzy zero number the end points of the right and left change between $0 + \delta < 1$ and $-1 < 0 - \delta$. By the above definitions, we can define fuzzy identity matrix as below:

Definition 3.3. We say that an FM is a fuzzy identity matrix if the off-diagonal and the diagonal elements of an FM are respectively fuzzy zero and one numbers. It is denoted by $\tilde{I}$ as follows:

$$\tilde{I} = \begin{bmatrix} \tilde{1} & \dots & \tilde{0} \\ \vdots & \vdots & \vdots \\ \tilde{0} & \dots & \tilde{1} \end{bmatrix}$$

Now, we are ready to present a new method on the base of using the Gröbner basis to calculate the fuzzy inverse matrix.

Definition 3.4. Suppose that $\tilde{A}$ and $\tilde{X}$ are fuzzy matrices of LR type with the size of n which are given as below:

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \dots & \tilde{a}_{1n} \\ \vdots & \vdots & \vdots \\ \tilde{a}_{n1} & \dots & \tilde{a}_{nn} \end{bmatrix} \qquad \tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \dots & \tilde{x}_{1n} \\ \vdots & \vdots & \vdots \\ \tilde{x}_{n1} & \dots & \tilde{x}_{nn} \end{bmatrix}$$

where both a_{ij} and x_{ij}, $i,j = 1,2,...,n$ are triangular fuzzy numbers. Therefore, the inverse of $\tilde{A}$ exists which is $\tilde{X}$, if

$$\tilde{A} \otimes \tilde{X} = \tilde{I}. \tag{2}$$

We can rewrite the expression given in equation (2) as follows:

$$\begin{bmatrix} \tilde{a}_{11} & \dots & \tilde{a}_{1n} \\ \vdots & \vdots & \vdots \\ \tilde{a}_{n1} & \dots & \tilde{a}_{nn} \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \dots & \tilde{x}_{1n} \\ \vdots & \vdots & \vdots \\ \tilde{x}_{n1} & \dots & \tilde{x}_{nn} \end{bmatrix} = \begin{bmatrix} \tilde{1} & \dots & \tilde{0} \\ \vdots & \vdots & \vdots \\ \tilde{0} & \dots & \tilde{1} \end{bmatrix},$$

where $\tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij})$, $\tilde{x}_{ij} = (x_{ij}, w_{ij}, z_{ij})$, $i,j = 1,2,...,n$ and $\tilde{1} = (1, \epsilon)$, $\tilde{0} = (0, \delta)$.

Then, the matrix multiplication is rewritten in the form of FLES as follows:

$$\begin{cases} (\tilde{a}_{i1} \otimes \tilde{x}_{1i}) \oplus (\tilde{a}_{i2} \otimes \tilde{x}_{2i}) \oplus ... \oplus (\tilde{a}_{in} \otimes \tilde{x}_{ni}) = \tilde{1} & i = 1,2,...,n, \\ (\tilde{a}_{i1} \otimes \tilde{x}_{1j}) \oplus (\tilde{a}_{i2} \otimes \tilde{x}_{2j}) \oplus ... \oplus (\tilde{a}_{in} \otimes \tilde{x}_{nj}) = \tilde{0} & i \neq j = 1,2,...,n. \end{cases} \tag{3}$$

The ES in (3) can be written as follows:

(4)
$$\begin{cases} ((a_{i1}, b_{i1}, c_{i1}) \otimes (x_{1i}, w_{1i}, z_{1i})) \oplus ... \oplus ((a_{in}, b_{in}, c_{in}) \otimes (x_{ni}, w_{ni}, z_{ni})) = (1, \epsilon, \epsilon) & i = 1,2,...,n, \\ ((a_{i1}, b_{i1}, c_{i1}) \otimes (x_{1j}, w_{1j}, z_{1j})) \oplus ... \oplus ((a_{in}, b_{in}, c_{in}) \otimes (x_{nj}, w_{nj}, z_{nj})) = (0, \delta, \delta) & i \neq j = 1,2,...,n, \end{cases}$$

where we have the sufficiently small values of ϵ and δ.

The system of equations (4) will be split into two separate ESs namely center and spread parts ESs, respectively. We can write the center part as below:

(5)
$$\begin{cases} a_{i1}x_{1i} + a_{i2}x_{2i} + ... + a_{in}x_{ni} = 1 & i = 1,2,...,n, \\ a_{i1}x_{1j} + a_{i2}x_{2j} + ... + a_{in}x_{nj} = 0 & i \neq j = 1,2,...,n. \end{cases}$$

Consider $F = \{f_1, f_2, ..., f_{3n}\}$ as a polynomials set in the system (5) and x_{ji} (for $i,j = 1,2,..,n$) is a variables set appearing in f_i's which ordered as $x_{11} < x_{12} < ... < x_{1n} < x_{21} < x_{22} < ... < x_{2n} < ... < x_{n1} < x_{n2} < ... < x_{nn}$. Therefore, every CS of F in the ring $\boldsymbol{R} = \mathbb{R}[x_{ij}]$ (for $i,j = 1,2,..,n$) has a triangular structure. Then, we can obtain all the solutions of system (5) by using Wu's method. If we solve the center part, then we can obtain the spread part of the ES given in (3) with regard to Definition 2.4, and sing of variables x_{ij} (for $i, j = 1, 2, .., n$).

Theorem 3.5. *Matrix $\tilde{A}$ is invertible if and only if the set of CSs for system (5) is nonempty set.*

Proof. Let F be the polynomials set in System (5). Next, if Z is not an empty set, then $V(F)$ will not be an empty set, therefore every member of $V(F)$ known as a solution of equations system (3). Another side is trivial and the proof of theorem is completed. Now, according to the above discussions the calculation process of the inverse of an FM has been given in the below algorithm:

Algorithm 5 THE CALCULATION PROCESS OF A FUZZY INVERSE MATRIX

Require: FM $\tilde{A}$ of type LR with the size of n
Ensure: Inverse of FM $\tilde{A}$
[i] Calculate the FLES (3) allowing the calculation of inverse of the matrix $\tilde{A}$
[ii] Calculate the center part ES (3) i.e. F
[iii] Calculate a set of CSs Z for F
[iv] If $Z = \emptyset$ then go to 8
[v] Calculate the variety V of F i.e. $V(F) = \bigcup_{B \in Z} V(B/I_B)$, where $I_B = \prod_{b \in B} ini(b)$ as the values of the variables of the center part ES (3)
[vi] Calculate the spread part ES i.e. F'
[vii] Calculate the variety V of F' i.e. $V(F') = \bigcup_{B \in Z} V(B/I_B)$, where $I_B = \prod_{b \in B} ini(b)$ as the values of the variables of the spread part ES
[viii] FM $\tilde{A}$ of type LR is not invertible
[ix] End

4. Numerical examples

The efficiency of our proposed method is shown by two numerical examples in this section. All used fuzzy numbers throughout this section are fuzzy numbers of symmetric triangular form for the sake of simplicity.

Example 4.1. [1] Consider the below 2×2 FM consisting of fuzzy numbers of symmetric triangular form:

$$\tilde{A} = \begin{bmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{bmatrix}.$$

To calculate the fuzzy inverse matrix $\tilde{A}$, the real case is considered then it is expanded to the fuzzy case as below:

$$\begin{bmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{1} & \tilde{0} \\ \tilde{0} & \tilde{1} \end{bmatrix}.$$

where all unknowns $\tilde{x}_{ij}$ are represented by (x_{ij}, w_{ij}) for $i, j = 1, 2$. Then, the matrix multiplication is rewritten in form of an FLES as follows:

$$\begin{cases} \big((10,4)\otimes\tilde{x}_{11}\big)\oplus\big((8,3)\otimes\tilde{x}_{21}\big)=(1,\delta), \\ \big((10,4)\otimes\tilde{x}_{12}\big)\oplus\big((8,3)\otimes\tilde{x}_{22}\big)=(0,\alpha), \\ \big((6,2)\otimes\tilde{x}_{11}\big)\oplus\big((4,3)\otimes\tilde{x}_{21}\big)=(0,\alpha), \\ \big((6,2)\otimes\tilde{x}_{12}\big)\oplus\big((4,3)\otimes\tilde{x}_{22}\big)=(1,\delta). \end{cases}$$

We write the above ES for the given example as below:

$$\begin{cases} \big((10,4)\otimes(x_{11},w_{11})\big)\oplus\big((8,3)\otimes(x_{21},w_{21})\big)=(1,\delta), \\ \big((10,4)\otimes(x_{12},w_{12})\big)\oplus\big((8,3)\otimes(x_{22},w_{22})\big)=(0,\alpha), \\ \big((6,2)\otimes(x_{11},w_{11})\big)\oplus\big((4,3)\otimes(x_{21},w_{21})\big)=(0,\alpha), \\ \big((6,2)\otimes(x_{12},w_{12})\big)\oplus\big((4,3)\otimes(x_{22},w_{22})\big)=(1,\delta). \end{cases}$$

By using the approximate fuzzy multiplication, the center part is written as follows:

$$F:\begin{cases} 10x_{11}+8x_{21}=1, \\ 10x_{12}+8x_{22}=0, \\ 6x_{11}+4x_{21}=0, \\ 6x_{12}+4x_{22}=1, \end{cases}$$

The set of CSs for the above system using Wu's algorithm is

$$Z=\Big\{z_1=\{2x_{11}+1, 4x_{21}-3, x_{12}-1, 4x_{22}+5\}\Big\}.$$

According to Theorem of Well-ordering Principle of Wu, we have

$$V(F)=\big(V(z_1)\backslash V(a)\big)=\big(V(z_1)\backslash V(32)\big).$$

Then, the values of the variables are: $x_{11}=-0.5, x_{21}=0.75, x_{12}=1, x_{22}=-1.25$. Hence, we have the spread part of the ES as below:

$$F':\begin{cases} 10w_{11}-4x_{11}+8w_{21}+3x_{21}=\delta, \\ 10w_{12}+4x_{12}+8w_{22}-3x_{22}=\alpha, \\ 6w_{11}-2x_{11}+4w_{21}+3x_{21}=\alpha, \\ 6w_{12}+2x_{12}+4w_{22}-3x_{22}=\delta. \end{cases}$$

The set of CSs for F' using Wu's algorithm is

$$Z=\Big\{z_1=\{8w_{11}+9+4\delta-8\alpha, -8w_{12}-15-4\alpha+8\delta, 7-8w_{21}-10\alpha+6\delta, -11+8w_{22}+10\delta-6\alpha\}\Big\}.$$

According to Theorem of Well-ordering Principle of Wu, we have

$$V(F')=\big(V(z_1)\backslash V(a)\big)=\big(V(z_1)\backslash V(4096)\big).$$

Therefore, the solution of system F' is
$w_{11}=-9/8+\alpha-1/2\delta$,
$w_{21}=7/8-5/4\alpha+3/4\delta$,
$w_{12}=-15/8+\delta-1/2\alpha$,

$w_{22} = 11/8 - 5/4\delta + 3/4\alpha$.

It should be noted that the values of w_{ij} are obtained on the base of δ and α parameters unless they are specified beforehand. Also, because of the fuzzy number definition of LR type, the LR type fuzzy numbers' spread values cannot be negative. In the case of being negativity of the spread values, we should take the absolute values of them.

ϵ, δ	w_{11}	w_{21}	w_{12}	w_{22}
0.01	-1.365	1.2375	-1.3800	0.7570
0.1	-1.275	1.1250	-1.4250	0.8250
0.2	-1.175	1.0000	-1.4750	0.9000
0.3	-1.075	0.8750	-1.5250	0.9750
0.4	-0.975	0.7500	-1.5750	1.0500
0.5	-0.875	0.6250	-1.6250	1.1250
0.6	-0.775	0.5000	-1.6750	1.2000
0.7	-0.675	0.3750	-1.7250	1.2750
0.8	-0.575	0.2500	-1.7750	1.3500
0.9	-0.475	0.1250	-1.8250	1.4250

The computations associated to Example 4.1.

For example, if $\alpha = \delta = 0.5$ then we write the inverse of $\tilde{A}$ as below:

$$\tilde{A}^{-1} = \begin{bmatrix} (-0.5, |-0.8750|) & (1, |-1.6250|) \\ (0.75, 0.6250) & (-1.25, 1.1250) \end{bmatrix} = \begin{bmatrix} (-0.5, 0.8750) & (1, 1.6250) \\ (0.75, 0.6250) & (-1.25, 1.1250) \end{bmatrix}.$$

Unfortunately, all elements of the $\tilde{X}$ were considered as positive in [1] and the inverse of $\tilde{A}$ was obtained, obviously all calculations as well as this inverse are incorrect. Since, being negativity of elements of the $\tilde{X}$ have not been considered. But Basaran in [1], assert that the inverse of $\tilde{A}$ at $\alpha = \delta = 0.5$, is given by

$$\tilde{A}^{-1} = \begin{bmatrix} (-0.5, 0.88) & (1, 2.13) \\ (0.75, 1.63) & (-1.25, 2.57) \end{bmatrix}.$$

Example 4.2. Consider the following 2×2 FM consisting of fuzzy numbers of symmetric triangular form:

$$\tilde{A} = \begin{bmatrix} (1,2) & (3,4) \\ (1,6) & (3,7) \end{bmatrix}.$$

The system of equations which allows computing the inverse of an FM $\tilde{A}$ is given as below:

$$\begin{bmatrix} (1,2) & (3,4) \\ (1,6) & (3,7) \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{1} & \tilde{0} \\ \tilde{0} & \tilde{1} \end{bmatrix}.$$

Then, the matrix multiplication is rewritten in the form of FLESs as follows:

$$\begin{cases} \big((1,2)\otimes\tilde{x}_{11}\big)\oplus\big((3,4)\otimes\tilde{x}_{21}\big)=(1,\delta), \\ \big((1,2)\otimes\tilde{x}_{12}\big)\oplus\big((3,4)\otimes\tilde{x}_{22}\big)=(0,\alpha), \\ \big((1,6)\otimes\tilde{x}_{11}\big)\oplus\big((3,7)\otimes\tilde{x}_{21}\big)=(0,\alpha), \\ \big((1,6)\otimes\tilde{x}_{12}\big)\oplus\big((3,7)\otimes\tilde{x}_{22}\big)=(1,\delta). \end{cases}$$

The above ES can be written for the given example as follows:

$$\begin{cases} \big((1,2)\otimes(x_{11},w_{11})\big)\oplus\big((3,4)\otimes(x_{21},w_{21})\big)=(1,\delta), \\ \big((1,2)\otimes(x_{12},w_{12})\big)\oplus\big((3,4)\otimes(x_{22},w_{22})\big)=(0,\alpha), \\ \big((1,6)\otimes(x_{11},w_{11})\big)\oplus\big((3,7)\otimes(x_{21},w_{21})\big)=(0,\alpha), \\ \big((1,6)\otimes(x_{12},w_{12})\big)\oplus\big((3,7)\otimes(x_{22},w_{22})\big)=(1,\delta). \end{cases}$$

By using the approximate fuzzy multiplication, the above ES is equivalent to the following system:

$$\begin{cases} x_{11}+3x_{21}=1, \\ x_{12}+3x_{22}=0, \\ x_{11}+3x_{21}=0, \\ x_{12}+3x_{22}=1, \end{cases}$$

Then, $Z=\emptyset$ is the Wu's algorithm output. Thus, there is not any solution for the system. Therefore, the FM is not invertible due to the Theorem 3.5.

5. Conclusion

The present paper proposed a new approach based on Wu's method to find the inverse of an FM including the LR type fuzzy numbers. Instead of calculating the inverse matrix problem, we solved a polynomial ES in fuzzy case for which all unknown variables, right-hand side values and coefficients are fuzzy numbers. The elegance of the given technique lies in its independence to a suitable starting point. We discussed the sufficient and necessary condition for invertibility of the FM. Numerical examples support the competencies of the proposed algorithm to calculate the inverse of an FM. For the future work, we will find inverse of an FM with other efficient algebraic methods as well as numerical methods such as artificial neural networks.

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