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THE CONVEXITY OF CHEBYSHEV SETS IN NORMED SPACES

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Abstract. In this paper, we consider **"Nearest points"** and **"Farthest points"** in inner product spaces and Hilbert spaces. The convexity of Chebyshev sets in Hilbert spacse is an open problem. In this paper we define sun sets and sunrise sets in normed spaces.

Keywords: Chebyshev centers, Uniquely remotal centers, Nearest points, Farthest points, Sun sets, Sunrise sets.
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1. Introduction

Approximation theory, which mainly consists of theory of nearest points (best approximation) and theory of farthest points (worst approximation), is an old and rich branch of analysis. The theory is as old as Mathematics itself. Starting in 1853, a Russian mathematician P.L. Chebyshev made significant contributions in the theory of best approximation. The Weierstrass approximation theorem of 1885 by K. Weierstrass is well known.The study was followed in the first half of the 20th Century by L.N.H. Bunt (1934) , T.S . Motzkin (1935) and B. Jessen (1940). B. Jessen was the first to make significant contributions in the theory of farthest points. This theory is less developed as compared to the theory of best approximation .

Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X. A point $y_0 \in W$ is said to be a best approximation point (nearest point) for $x \in X$, if

$$\|x - y_0\| \leq \|x - y\|,$$

for each $y \in W$.
For each $x \in X$, put

$$P_W(x) = \{y_0 \in W : \ \|x - y_0\| = d(W, x) = \inf_{y \in W} \|x - y\|\}.$$

For each $x \in X$, if $P_W(x)$ is non-empty (a singleton), we say that W is proximinal (Chebyshev). For each $x \in X \backslash W$, if $P_W(x) = \emptyset$, we say that W is anti-proximinal (see [2 , 5 , 10, 17]).

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Let X be a normed linear space and W a bounded non-empty subset of X. A point $q(x) \in W$ is said to be a farthest point for $x \in X$, if

$$\|x - q(x)\| \geq \|x - y\|,$$

for each $y \in W$.
For each $x \in X$, put

$$F_W(x) = \{y_0 \in W : \ \|x - y_0\| = \delta(W, x) = \sup_{y \in W} \|x - y\|\}.$$

For each $x \in X$, if $F_W(x)$ is non-empty (a singleton), we say that W is remotal (uniquely remotal). For each $x \in X$, if $F_W(x) = \emptyset$, we say that W is anti-remotal. (see [4,5,7,9,10,11,14,15,16]).

Definition 1.1. Let $(X, \|.\|)$ be a normed space, W a subset in X and $x \in X$. W is called sun set if $x \notin W$, $y = P_W(x)$, then

$$P_W(\lambda x + (1 - \lambda)g) = g, \ for\ every\ \lambda \geq 0.$$

Definition 1.2. Let $(X, \|.\|)$ be a normed space, W is a bounded subset in X and $x \in X$. W is called sunrise set if $x \notin W$, $g = F_W(x)$, then

$$F_W(\lambda x + (1 - \lambda)g) = g, \text{ for every } \lambda \geq 0.$$

Example 1.3. *Let $(X, \|.\|)$ be a normed space, $W = \{x \in X : \|x\| = 1\}$ a subset of X and $x \in X \backslash W$. Then*
1) $d(W, x) = |1 - \|x\||$,
2) nearest point $(x) = \frac{x}{\|x\|}$,
3) farthest point $(x) = \frac{-x}{\|x\|}$,
4) $\delta(W, x) = |1 + \|x\||$,

Definition 1.4. Let X be a vector space and C be a subset of X. We shall say that C is convex if, for any $a, b \in C$ and any $\lambda \in [0, 1]$, we have $[a, b] \subset C$. We say that C is concave if $X \backslash C$ is convex. Furthermore, we say that C is midpoint convex if for any $a, b \in C$, $\frac{a+b}{2} \in C$.

We know that a closed subset of a normed linear space $(X, \|\|)$ is convex if and only if it is midpoint convex We know that every non-empty closed convex set in a Hilbert spaces is Chebyshev (See [19]).

Remark 1.5. In [1] Aspland show that, if W is a Chebyshev set in Hilbert space H and the proximity map $P_W : X \to W$ is continous. Then W is convex.

Remark 1.6. Let $(X.\|.\|)$ be a normed linear space and W a Chebyshev sun set of X. Then W is midpoint convex and convex.

In here there exists an open problem:

$$Is\ evey\ Chebyshev\ set\ in\ Hilbert\ spaces\ convex.$$

Theorem 1.7 (10). *Let $X = \mathbb{R}^2$ with Euclidean norm, W a Chebyshev sert in X. Then the proximity map is countinous, therefore W is convex.*

2. **Covexity of Chebyshev Sets**

In this section we obtain some results on convexity of Chebyshev sets,.

Example 2.1. *Let $(X, \|.\|)$ be a normed space, $W = \{x \in X : \|x\| = 1\}$ and $x \in X$. We show that W is sun and sunrise set.*

If $g \in W$, put $x = -\lambda g$ for $\lambda \geq 1$. Therefore $g \in F_W(x)$.
If $x \in F_g$, since $q(x) = -\frac{x}{\|x\|} = g$. Therefore $x = -\|x\|g$ and $\|x\| \geq 1$. It follows that $-\lambda g \in F(x)$. Put $x = \lambda g$, for every $\lambda \geq 0$. Therefore nearest point $g \in P_W(x)$
If $g \in P_W(x)$, since nearest point $(x) = \frac{x}{\|x\|} = g$ and $\|x\| \geq 1$. Therefore $x = \|x\|g$ and $\|x\| \geq 1$.

$$\begin{aligned} \lambda x + (1-\lambda)g &= -\lambda\lambda_0 g + g - \lambda g \\ &= -(-1+\lambda+\lambda\lambda_0)g, \end{aligned}$$

Note that $-1 + \lambda + \lambda\lambda_0 \geq 1$. Therefore

$$F_W(\lambda x + (1-\lambda)g) = g.$$

and W is a sunrise set.
Suppose $g = P_W(x)$, then $x \in P_g$. Therefore for some $\lambda_0 \geq 1$ we have $x = \lambda_0 g$. For $\lambda \geq 0$, we have

$$\begin{aligned} \lambda x + (1-\lambda)g &= \lambda\lambda_0 g + g - \lambda g \\ &= (\lambda\lambda_0 - \lambda + 1)g. \end{aligned}$$

Note that $\lambda\lambda_0 - \lambda + 1 \geq 0$. Therefore

$$P_W(\lambda x + (1-\lambda)g) = g.$$

and W is a sun set in X.

Theorem 2.2. *Let $(X, \|.\|)$ be a normed space and W an uniquely remotal convex subset of X. Then W is a sunrise set.*

Proof. Suppose $x \notin W$, $F_W(x) = y$ and $F_W(\lambda x + (1-\lambda)y) = w \neq y$ for every $\lambda \geq 1$. If $z = \lambda x + (1-\lambda)y$, for every $u \in W$. We put

$$w = \frac{1}{\lambda}x + (1 - \frac{1}{\lambda})y$$

since W is convex, $w \in W$ and

$$\begin{aligned}
\|z-y\| &= \|\lambda x+(1-\lambda)y-y\| \\
&= \|\lambda x-\lambda y\| \\
&= \lambda\|x-y\| \\
&\geq \lambda\|x-y\| \\
&= \|\lambda x-\lambda w\| \\
&= \|\lambda(x)-w-(\lambda-1)y\| \\
&= \|\lambda(x)+(1-\lambda)y-u\| \\
&= \|z-u\|.
\end{aligned}$$

Therefore

$$\|z-y\| > \|z-u\|,$$

$$F_W(z) = y.$$

□

Theorem 2.3. *Let $(X, \|.\|)$ be a normed space and W a Chebyshev convex set in X. Then W is a sun set.*

Proof. Suppose $x \in X \setminus W, g \in W$ and $g = P_W(x)$. If $z = \lambda x + (1-\lambda)y$ for $0 \leq \lambda < 1$. Then

$$\begin{aligned}
\|x-z\| &= \|\lambda x+(1-\lambda)x-\lambda x-(1-\lambda)g\| \\
&= \|(1-\lambda)(x-g)\| \\
&= (1-\lambda)\|(x-g)\| \\
&= \|(x-g)\|-\lambda\|(x-g)\| \\
&= \|(x-g)\|-\|(\lambda x-\lambda g)\| \\
&= \|(x-g)\|-\|(\lambda x+(1-\lambda)g-\lambda g-(1-\lambda)g)\| \\
&= \|(x-g)\|-\|(z-g)\|.
\end{aligned}$$

Since $g = P_W(x)$, we have $\|x-g\| \leq \|x-y\|$ for every $g \in W$. Therefore

$$\begin{aligned}
\|z-g\| &= \|x-g\| - \|x-z\| \\
&\leq \|x-g\| \\
&\leq \|x-y\| \ (g \in W).
\end{aligned}$$

It follows that $g = P_W(z)$. If $\lambda > 1$, for every $u \in W$. We put

$$w = \frac{1}{\lambda}u + (1-\frac{1}{\lambda})g.$$

Suppose W is convex, then $w \in W$ and

$$\begin{aligned}
\|z-g\| &= \|\lambda(x)+(1-\lambda)g-g\| \\
&= \|\lambda x-\lambda g\| \\
&= \lambda\|x-g\| \\
&\leq \lambda\|x-y\| \\
&= \|\lambda x-\lambda u\| \\
&= \|\lambda x-u-(\lambda-1)g\| \\
&= \|\lambda x+u-(1-\lambda)g-u\| \\
&= \|z-u\|,\ (u\in W).
\end{aligned}$$

Therefore $P_W(z) = g$. □

Theorem 2.4. *Let $(X, \|.\|)$ be a normed space and W remotal convex set in X, $x \in X$ and $g \in W$. If $g = F_W(x)$ and $z = \lambda x + (1-\lambda)g$. Then*

$$\|z-g\| \geq \|z-(\lambda w+(1-\lambda)g\| \ \forall w \in W.$$

Proof.

$$\begin{aligned}
\|z-g\| &= \|\lambda x+(1-\lambda)g-\lambda g-(1-\lambda)g\| \\
&= \lambda\|x-g\| \\
&\geq \lambda\|x-w\| \ \forall w\in W
\end{aligned}$$

□

Theorem 2.5. *Let K be a subset of an inner product space $(X, <.,.>)$ and let $x \in X$, $y \in K$. Then $y \in P_K(x)$ if and only if*

$$< x-\frac{y+z}{2}, z-y > \leq 0,$$

for all $z \in K$.

Proof.

$$\begin{aligned}
y\in P_K(x) &\Leftrightarrow \|x-y\|\leq\|x-z\| \ z\in K \\
&\Leftrightarrow \|x-y\|^2\leq\|x-z\|^2 \ z\in K \\
&\Leftrightarrow < x-y, x-y > \leq < x-z, x-z > \ \ z\in K \\
&\Leftrightarrow < x-y+z-y, x-y > \leq < x-y+y-z, x-z > \ \ z\in K \\
&\Leftrightarrow < z-y, x-y > - < y-z, x-z > \leq 0 \ z\in K \\
&\Leftrightarrow < z-y, 2x-y-z > \leq 0 \ z\in K \\
&\Leftrightarrow < z-y, x-\frac{y+z}{2} > \ \ z\in K.
\end{aligned}$$

□

Corollary 2.6. *Let K be a midpoint convex subset of an inner product space $(X, <.,.>)$ and $x \in X$, $y \in K$. Then $y \in P_K(x)$ if and only if*

$$< x - z, z - y > \leq 0,$$

for all $z \in K$.

Proof. If $z \in K$, it is sufficient we get in theorem $Z = \frac{y+z}{2}$. □

Corollary 2.7. *Let K be a midpoint convex subset of an inner product space $(X, <.,.>)$. For $x, y \in X$*

$$\|P_W(x) - P_W(y)\| \leq \|x - y\|.$$

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