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A PRECONDITIONED JACOBI-TYPE METHOD FOR SOLVING MULTI-LINEAR SYSTEMS

Mehdi Najafi-Kalyani* and Fatemeh P. A. Beik

Dedicated to sincere professor Mehdi Radjabalipour on turning 75

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Abstract. Recently, Zhang et al. [Applied Mathematics Letters 104 (2020) 106287] proposed a preconditioner to improve the convergence speed of three types of Jacobi iterative methods for solving multi-linear systems. In this paper, we consider the Jacobi-type method which works better than the other two ones and apply a new preconditioner. The convergence of proposed preconditioned iterative method is studied. It is shown that the new approach is superior to the recently examined one in the literature. Numerical experiments illustrate the validity of theoretical results and the efficiency of the proposed preconditioner.

Keywords: Iterative method, multi-linear system, strong $\mathcal{M}$-tensor, preconditioning.
2020 MSC: Primary 15A69, 65F10, 65H10.

1. Introduction

We consider the following multi-linear system

$$\mathscr{A}x^{m-1} = b, \tag{1}$$

where $\mathcal{A} \in \mathbb{R}^{\overbrace{n \times \ldots \times n}^{m}}$ is a given n–dimension tensor of m–order, the right-hand side $b \in \mathbb{R}^n$ is also available and x is the unknown tensor to be determined. Similar to [19], we consider the case that the coefficient tensor $\mathcal{A}$ is a strong $\mathcal{M}$-tensor whose definition is recalled later in this section. The entries of vector $\mathscr{A}x^{m-1}$ are defined as follows:

$$(\mathscr{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m}, \quad i = 1, 2, \ldots, n.$$

We focus on the case that b is a nonnegative vector and as pointed out above $\mathscr{A}$ is also assumed to be strong $\mathcal{M}$-tensor. These assumptions ensure the existence of a unique solution for $\mathscr{A}x^{m-1} = b$; see [16] for details. In practice, the multi-linear systems in the above form appear in several areas such as engineering and scientific computing; see [3, 7–11, 17, 18] for further details.

*Corresponding author, ORCID: 0000-0003-2474-6096,
e-mail: m.najafi.uk@gmail.com
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In the sequel, the set of all n–dimension real tensor of m–order is denoted by $\mathbb{R}^{[m,n]}$ for notational simplicity. The notation $\mathscr{I}_m$ represents the unit tensor in $\mathbb{R}^{[m,n]}$ where $\mathscr{I}_m = (\delta_{i_1 \dots i_m})$ such that

$$\delta_{i_1 \dots i_m} = \begin{cases} 1, & i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, without loss of generality, we assume that $a_{i\dots i} = 1$ for $i = 1, 2, \dots, n$ and consider the decomposition $\mathscr{A} = \mathscr{I}_m - \mathscr{L} - \mathscr{F}$ where $\mathscr{L} = L\mathscr{I}_m$ in which $-L$ is the strictly lower triangle part of $M(\mathscr{A})$.

The remainder of this paper is organized as follows: Before ending this section, we review some basic concepts, definitions and properties which are required for obtaining our main results. In section 2, we give a brief overview on the results recently established in [19] and propose a new preconditioner for accelerating the convergence speed of a Jacobi-type iterative method. We study the properties of presented preconditioned Jacobi-type method and establish some comparison results in section 3. Test problems are experimentally examined in section 4 to numerically demonstrate the validity of presented theoretical results. We finish the paper with a brief conclusion in section 5.

We recall some definitions and preliminaries in the rest of this section. The following definitions are mostly taken from [16] and the references therein.

Definition 1.1. Corresponding to a given tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$, the majorization matrix $M(\mathscr{A})$ is the $n \times n$ matrix whose entries are given by $M(\mathscr{A})_{ij} = a_{ij\dots j}$ for $i, j = 1, 2, \dots, n$.

Definition 1.2. Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. If $M(\mathscr{A})$ is a nonsingular matrix and $\mathscr{A} = M(\mathscr{A})\mathscr{I}_m$, the matrix $M(\mathscr{A})^{-1}$ is called the order 2 left-inverse of $\mathscr{A}$.

Definition 1.3. Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. If $\mathscr{A}$ has an order 2 left-inverse, $\mathscr{A}$ is called a left-invertible tensor or a left-nonsingular tensor.

Definition 1.4. For a given tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$, the decomposition $\mathscr{A} = \mathscr{E} - \mathscr{F}$ is called a (tensor) splitting, if $\mathscr{E}$ is left-nonsingular.

Definition 1.5. Given a tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$, the splitting $\mathscr{A} = \mathscr{E} - \mathscr{F}$ is said to be

- a regular splitting, if $M(\mathscr{E})^{-1} \geq 0$ and $\mathscr{F} \geq 0$;
- a weak regular splitting, if $M(\mathscr{E})^{-1} \geq 0$ and $M(\mathscr{E})^{-1}\mathscr{F} \geq 0$;
- a convergent splitting, if $\rho(M(\mathscr{E})^{-1}\mathscr{F}) < 1$.

Definition 1.6. [12, 13] Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is called an eigenpair of $\mathscr{A}$, if they satisfy the equation $\mathscr{A}x^{m-1} = \lambda x^{[m-1]}$ where $x^{[m-1]} = (x_1^{m-1}, \dots, x_n^{m-1})^T$. The eigenpair (λ, x) is called an H-eigenpair, if both λ and x are real.

The spectral radius of $\mathscr{A}$ is defined by $\rho(\mathscr{A}) = \max\{|\lambda| \mid \lambda \in \sigma(\mathscr{A})\}$ in which $\sigma(\mathscr{A})$ stands for the set of all eigenvalues of $\mathscr{A}$.

Definition 1.7. Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. The tensor $\mathscr{A}$ is called a $\mathcal{Z}$-tensor, if its off-diagonal entries are non-positive. If there exist a nonnegative tensor $\mathscr{B}$ and a positive real number $\eta \geq \rho(\mathscr{B})$ such that $\mathscr{A} = \eta \mathscr{I}_m - \mathscr{B}$, then $\mathscr{A}$ is said to be an $\mathcal{M}$-tensor. In the case that $\eta > \rho(\mathscr{B})$, the tensor $\mathscr{A}$ is called a strong $\mathcal{M}$-tensor.

Definition 1.8. [1,5] If $A \in \mathbb{R}^{[2,n]}$ and $\mathscr{B} = (b_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$, then the tensor $\mathscr{C} = A\mathscr{B}$ belongs to $\mathbb{R}^{[m,n]}$ and its entries are given as follows:

$$c_{j i_2 \dots i_m} = \sum_{j_2=1}^{n} a_{j j_2} b_{j_2 i_2 \dots i_m}, \quad 1 \leq j, i_\tau \leq n \quad (\tau = 2, \dots, m).$$

Lemma 1.9. [3] *If $\mathscr{A}$ is a strong $\mathcal{M}$-tensor, then $M(\mathscr{A})$ is a nonsingular M-matrix.*

We end this section by pointing out that the converse of Lemma 1.9 does not hold in general, see [3, Lemma 3.6] for further details.

2. Proposed preconditioner

Given the tensor splitting $\mathscr{A} = \mathscr{E} - \mathscr{F}$, we consider the following iterative method for solving (1),

$$x_k = [M(\mathscr{E})^{-1} \mathscr{F} x_{k-1}^{m-1} + M(\mathscr{E})^{-1} b]^{[\frac{1}{m-1}]}, \qquad k = 1, 2, \dots, \tag{2}$$

where x_0 is an arbitrary given initial guess. Here, the tensor $M(\mathscr{E})^{-1}\mathscr{F}$ is called the iteration tensor. In [3, Subsection 5.3], it was discussed that the spectral radius of iteration tensor can be regarded as an approximation for the asymptotic convergence rate of iterative method (2).

In order to improve the convergence rate of iterative method (2), it is common to apply left preconditioning technique, e.g., see [3, 8, 17, 19]. Basically, given the preconditioner $P \in \mathbb{R}^{n \times n}$, we employ an iterative method in the form (2) to solve the following preconditioned multi-linear system,

$$P\mathscr{A} x^{m-1} = Pb, \tag{3}$$

instead of $\mathscr{A} x^{m-1} = b$. In particular, more recently, Zhang et al. [19] exploited the preconditioner $P_{(c,1)} = I + S_{(c,1)}$ with

$$S_{(c,1)} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ -\alpha_{21} a_{21\dots 1} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n1} a_{n1\dots 1} & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{4}$$

to accelerate the convergence speed of iterative methods in the form (2) corresponding to the following three Jacobi-type splittings:

$$\begin{aligned} \bar{\mathscr{A}} &= \mathscr{E}_1 - \mathscr{F}_1, \quad \text{where} \quad \mathscr{E}_1 = P_{(c,1)} \mathscr{I}_m, \\ \bar{\mathscr{A}} &= \mathscr{E}_2 - \mathscr{F}_2, \quad \text{where} \quad \mathscr{E}_2 = \mathscr{I}_m, \\ \bar{\mathscr{A}} &= \mathscr{E}_3 - \mathscr{F}_3, \quad \text{where} \quad \mathscr{E}_3 = \mathscr{I}_m - \bar{\mathscr{D}}, \end{aligned} \tag{5}$$

here $\bar{\mathscr{A}} = P_{(c,1)}\mathscr{A}$ and $\bar{\mathscr{D}} = \bar{D}\mathscr{I}_m$ and $\bar{D}$ is a diagonal matrix such that $\bar{D} = \mathrm{diag}(M(S_{(c,1)}\mathscr{F}))$. It was both theoretically and numerically illustrated that the third splitting outperforms other two splittings.

Remark 2.1. Zhang et al. [19] assumed that $\mathscr{A}$ is a strong $\mathcal{M}$-tensor which implies that $\bar{\mathscr{A}} = P_{(c,1)}\mathscr{A}$ is also a strong $\mathcal{M}$-tensor [19, Lemma 2] for $\alpha_{i1} \in [0,1]$ $(2 \le i \le n)$. Then, throughout the paper, the following assumption is set

$$0 < \alpha_{i1}a_{i1\ldots1}a_{1i\ldots i} < 1,$$

where $\alpha_{i1} \in [0,1]$ for $i = 2,\ldots,n$. We comment that the above assumption is a direct conclusion of the fact that $\bar{\mathscr{A}}$ is a strong $\mathcal{M}$-tensor, see [4] for further discussion.

In this paper, we consider the Jacobi type method corresponding to splitting of type (5) with the preconditioner $\tilde{P} = I + R$ where

$$R = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_{21}a_{21\ldots1} & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_{31}a_{31\ldots1} & -\alpha_{32}a_{32\ldots2} & 0 & \cdots & 0 & 0 \\ -\alpha_{41}a_{41\ldots1} & 0 & -\alpha_{43}a_{43\ldots3} & \cdots & 0 & 0 \\ -\alpha_{51}a_{51\ldots1} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_{(n-1)1}a_{(n-1)1\ldots1} & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_{n1}a_{n1\ldots1} & 0 & 0 & \cdots & -\alpha_{n(n-1)}a_{n(n-1)\ldots(n-1)} & 0 \end{bmatrix}, \tag{6}$$

where parameters $\alpha_{i1}, \alpha_{jj-1} \in [0,1]$ for $i = 2,\ldots,n$ and $j = 3,\ldots,n$ are given parameters. Basically, we set

$$\tilde{\mathscr{A}} := \tilde{P}\mathscr{A} = \mathscr{E}_{\tilde{P}} - \mathscr{F}_{\tilde{P}}, \quad \text{where} \quad \mathscr{E}_{\tilde{P}} = \mathscr{I}_m - \tilde{\mathscr{D}}, \tag{7}$$

here $\tilde{\mathscr{D}} = \tilde{D}\mathscr{I}_m$ and $\tilde{D}$ is a diagonal matrix such that $\tilde{D} = \mathrm{diag}(M(R\mathscr{F}))$.

We end this section by the definition of semi-positive tensors and a useful theorem.

Definition 2.2. A tensor $\mathscr{A}$ is said to be a semi-positive tensor, if there exists $x > 0$ such that $\mathscr{A}x^{m-1} > 0$.

In [16, Theorem 2], it is proved that $x > 0$ in the above definition can be relaxed into $x \ge 0$ in view of the continuity of the tensor-vector product on the entries of the vector.

Theorem 2.3. [16, Theorem 3] *A $\mathcal{Z}$-tensor is a strong $\mathcal{M}$-tensor if and only if it is semi-positive.*

3. Convergence and comparison analyses

In this section, we study the convergence of the preconditioned Jacobi type method with preconditioner $\tilde{P}$ and prove a comparison result between the proposed method and the one corresponding to splitting (5). Associated with $\tilde{P} = I + R$, we consider the following preconditioned multi-linear system

$\tilde{\mathscr{A}} x^{m-1} = \tilde{b}$ where $\tilde{\mathscr{A}} = (I + R)\mathscr{A}$ and $\tilde{b} = (I + R)b$. It can be observed that

$$\tilde{\mathscr{A}} = \mathscr{I}_m - \mathscr{L} - \mathscr{F} + R\mathscr{I}_m - R\mathscr{L} - R\mathscr{F},$$

recalling that $\mathscr{A} = \mathscr{I}_m - \mathscr{L} - \mathscr{F}$ where $\mathscr{L} = L\mathscr{I}_m$, $-L$ is the strictly lower triangle matrix of $M(\mathscr{A})$. As a natural way, we propose the preconditioned Jacobi-type method as follows:

$$x_k = [M(\mathscr{E}_{\tilde{P}})^{-1}\mathscr{F}_{\tilde{P}} x_{k-1}^{m-1} + M(\mathscr{E}_{\tilde{P}})^{-1}\tilde{b}]^{[\frac{1}{m-1}]}, \quad k = 1, 2, \ldots,$$

where the initial guess x_0 is given and

$$\mathscr{E}_{\tilde{P}} = \mathscr{I}_m - \tilde{\mathscr{D}} \quad \text{and} \quad \mathscr{F}_{\tilde{P}} = \mathscr{E}_{\tilde{P}} - \tilde{\mathscr{A}}, \tag{8}$$

here $\tilde{\mathscr{D}} = \tilde{D}\mathscr{I}_m$ and $\tilde{D}$ is a diagonal matrix such that $\tilde{D} = \text{diag}(M(R\mathscr{F}))$.

The following lemma shows that if $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is assumed to be a strong $\mathcal{M}$-tensor then the preconditioned tensor $\tilde{P}\mathscr{A}$ is also a strong $\mathcal{M}$-tensor.

Lemma 3.1. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathcal{M}$-tensor. Then, $\tilde{\mathscr{A}} = \tilde{P}\mathscr{A}$ is a strong $\mathcal{M}$-tensor where $\tilde{P} = I+R$ and R is defined by* (6) *such that $\alpha_{i1}, \alpha_{jj-1} \in [0, 1]$ for $i = 2, \ldots, n$ and $j = 3, \ldots, n$.*

Proof. We first show that $\tilde{\mathscr{A}} = [\tilde{a}_{i_1 i_2 \ldots i_m}] \in \mathbb{R}^{[m,n]}$ is a $\mathcal{Z}$-tensor, i.e., all off-diagonal of $\tilde{\mathscr{A}}$ are non-positive. In the sequel, we assume that $\delta_{ii_2\ldots i_m} = 0$. Evidently $\tilde{a}_{1i_2\ldots i_m} = a_{1i_2\ldots i_m}$ for $1 \le i_j \le n$ and $j = 2, \ldots, m$. One can observe that for $i = 2$, we have $\tilde{a}_{21\ldots 1} = (1 - \alpha_{21})a_{21\ldots 1} \le 0$ and $\tilde{a}_{2i_2\ldots i_m} = a_{2i_2\ldots i_m} - \alpha_{21}a_{21\ldots 1}a_{1i_2\ldots i_m} \le 0$ when $\delta_{1i_2\ldots i_m} = 0$. Other entries of $\tilde{\mathscr{A}}$ are determined by

$$\tilde{a}_{ii_2\ldots i_m} = a_{ii_2\ldots i_m} - \alpha_{i1}a_{i1\ldots 1}a_{1i_2\ldots i_n} - \alpha_{i(i-1)}a_{i(i-1)\ldots(i-1)}a_{(i-1)i_2\ldots i_n},$$

for $i = 3, \ldots, n$ and $1 \le i_j \le n$ for $j = 2, \ldots, m$. It is obvious that if $\delta_{i_2\ldots i_m} = 0$, then $\tilde{a}_{ii_2\ldots i_m} \le 0$. Now we consider the case that $i_2 = i_3 = \ldots = i_m = \eta$. For $\eta = 1$, it is already shown that $\tilde{a}_{21\ldots 1} \le 0$. Furthermore, we have

$$\tilde{a}_{i1\ldots 1} = (1 - \alpha_{i1})a_{i1\ldots 1} - \alpha_{i(i-1)}a_{i(i-1)\ldots(i-1)}a_{(i-1)1\ldots 1} \le 0,$$

for $i = 3, \ldots, n$. Without loss of generality, we assume that $\eta \ne 1$. If $\eta \ne i-1$, then

$$\tilde{a}_{i\eta\ldots\eta} = a_{i\eta\ldots\eta} - \alpha_{i1}a_{i1\ldots 1}a_{1\eta\ldots\eta} - \alpha_{i(i-1)}a_{i(i-1)\ldots(i-1)}a_{(i-1)\eta\ldots\eta} \le 0,$$

noticing that $\eta \ne i$. Otherwise, we have

$$\tilde{a}_{i\eta\ldots\eta} = (1 - \alpha_{i\eta})a_{i\eta\ldots\eta} - \alpha_{i1}a_{i1\ldots 1}a_{1\eta\ldots\eta} \le 0.$$

From the above discussion, we can deduce that $\tilde{\mathscr{A}}$ is a $\mathcal{Z}$-tensor. To complete the proof, by Theorem 2.3, we need to show that $\tilde{\mathscr{A}}$ is semi-positive. By the assumption and Theorem 2.3, the tensor $\mathscr{A}$ is semi-positive. Therefore there exists a nonnegative vector x such that $\mathscr{A}x^{m-1} > 0$. Consequently, since $I+R$ is a nonnegative matrix, we have $\tilde{\mathscr{A}}x^{m-1} = (I + R)\mathscr{A}x^{m-1} > 0$ which shows that $\tilde{\mathscr{A}}$ is a strong $\mathcal{M}$-tensor. □

Lemma 3.2. *Let $\mathscr{A}$ be a strong $\mathcal{M}$-tensor. Then $\tilde{\mathscr{A}} = \mathscr{E}_{\tilde{P}} - \mathscr{F}_{\tilde{P}}$ is a regular splitting where $\alpha_{i1}, \alpha_{jj-1} \in [0,1]$ for $i = 2,3,\ldots,n$ and $j = 3,\ldots,n$.*

Proof. The proof is basically a direct conclusion from Lemma 3.1. More precisely, Lemma 1.9 implies that $M(\tilde{\mathscr{A}})$ is an M-matrix which implies that its diagonal entries are all positive. That is, $\mathscr{E}_{\tilde{P}} = (I - \tilde{D})\mathscr{I}_m$ is a nonnegative (diagonal) tensor where $\tilde{\mathscr{D}} = \tilde{D}\mathscr{I}_m$ and $\tilde{D}$ is a diagonal matrix such that $\tilde{D} = \mathrm{diag}(M(R\mathscr{F}))$. As a result, we conclude that $M(\mathscr{E}_{\tilde{P}})^{-1} = (I - \tilde{D})^{-1} \geq 0$. The tensor $\mathscr{F}_{\tilde{P}}$ in (8) is given by

$$\mathscr{F}_{\tilde{P}} = (\mathscr{L} - R\mathscr{I}_m) + \mathscr{F} + R\mathscr{L} + \tilde{\mathscr{F}},$$

where $\tilde{\mathscr{F}} = R\mathscr{F} - \tilde{D}\mathscr{I}_m$. Evidently, we can observe that $\mathscr{F} + R\mathscr{L} + \tilde{\mathscr{F}} \geq 0$. In addition, one can see that $(\mathscr{L} - R\mathscr{I}_m)_{ii_2\ldots i_m} = 0$, if $\delta_{i_2\ldots i_m} = 0$ and $\delta_{ii_2\ldots i_m} = 1$ for $1 \leq i \leq n$. It can be verify that

$$(\mathscr{L} - R\mathscr{I}_m)_{ij\ldots j} = \begin{cases} (\alpha_{ij} - 1)a_{ij\ldots j}, & j = 1 \text{ or } j = i-1, \\ -a_{ij\ldots j}, & j \leq i \quad \text{and} \quad j \neq 1, i-1, \\ 0, & \text{otherwise}, \end{cases}$$

for $i = 2,\ldots,n$. Now, the assertion can be conclude from the fact that $\mathscr{L} - R\mathscr{I}_m \geq 0$. □

Remark 3.3. Under the assumptions of Lemma 3.2, we deduce that $\tilde{\mathscr{A}} = \mathscr{E}_{\tilde{P}} - \mathscr{F}_{\tilde{P}}$ is a convergent splitting, see [16, Lemma 2.8] for more details.

Consider the splitting $\mathscr{A} = \bar{\mathscr{E}}_3 - \bar{\mathscr{F}}_3$ such that

$$\bar{\mathscr{E}}_3 = P_{(c,1)}^{-1}\mathscr{E}_3 \quad \text{and} \quad \bar{\mathscr{F}}_3 = P_{(c,1)}^{-1}\mathscr{F}_3, \tag{9}$$

where $\mathscr{E}_3$ and $\mathscr{F}_3$ are given by (5). Evidently, we have

$$M(\bar{\mathscr{E}}_3)^{-1}\bar{\mathscr{F}}_3 = M(\mathscr{E}_3)^{-1}\mathscr{F}_3 \geq 0.$$

Let $(\bar{\rho}, \bar{x})$ be the Perron eigenpair of $M(\bar{\mathscr{E}}_3)^{-1}\bar{\mathscr{F}}_3$ and assume that $\bar{\rho} \neq 0$. Therefore, we derive

$$\mathscr{A}\bar{x}^{m-1} = \frac{1-\bar{\rho}}{\bar{\rho}}\bar{\mathscr{F}}_3\bar{x}^{m-1}. \tag{10}$$

It can be verify that

$$\begin{aligned} \bar{\mathscr{F}}_3 &= (I + S_{(c,1)})^{-1}((\mathscr{L} - S_{(c,1)}\mathscr{I}_m) + \mathscr{F} + S_{(c,1)}\mathscr{L} + S_{(c,1)}\mathscr{F} - \bar{D}\mathscr{I}_m) \\ &= (I + S_{(c,1)})^{-1}((I + S_{(c,1)})\mathscr{L} + (I + S_{(c,1)})\mathscr{F} - S_{(c,1)}\mathscr{I}_m - \bar{D}\mathscr{I}_m) \\ &= \mathscr{L} + \mathscr{F} - (I + S_{(c,1)})^{-1}(S_{(c,1)}\mathscr{I}_m + \bar{D}\mathscr{I}_m) \\ &= \mathscr{L} + \mathscr{F} - (I - S_{(c,1)})(S_{(c,1)}\mathscr{I}_m + \bar{D}\mathscr{I}_m) \\ &= (\mathscr{L} - S_{(c,1)}\mathscr{I}_m) + (\mathscr{F} - \bar{D}\mathscr{I}_m) + S_{(c,1)}\bar{D}\mathscr{I}_m \geq 0, \end{aligned}$$

noticing that $S_{(c,1)}S_{(c,1)} = 0$. In view of (10), we can conclude that $\mathscr{A}\bar{x}^{m-1} \geq 0$.

We end this section by a comparison theorem which shows that under certain conditions the considered Jacobi-type iterative method with preconditioner $\tilde{P}$ has better (asymptotic) rate of convergence in comparison with the one corresponding to $P_{(c,1)}$. To this end, we first need to recall the concept of (ir)reducibility for tensors and a useful theorem.

Definition 3.4. [6] A tensor $\mathscr{C} = (c_{i_1\dots i_m}) \in \mathbb{R}^{[n,m]}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1,\dots,n\}$ such that $c_{i_1\dots i_m} = 0$ for all $i_1 \in I$, for all $i_2,\dots,i_m \notin I$. If $\mathscr{C}$ is not reducible, then $\mathscr{C}$ is said to be irreducible.

Theorem 3.5. [15, Theorem 5.2] *Let $\mathscr{A} \in \mathbb{R}^{[n,m]}$, and $\mathscr{A} \geq 0$. Furthermore, suppose that $\mathscr{A}$ has a positive eigenvector corresponding to some eigenvalue. Then,*

$$\rho(\mathscr{A}) = \min_{x>0} \max_{x_i>0} \frac{\left(\mathscr{A}x^{m-1}\right)_i}{x_i^{m-1}}.$$

Theorem 3.6. *Let $\mathscr{A} \in \mathbb{R}^{[n,m]}$ be a strong $\mathcal{M}$-tensor and $M(\mathscr{E}_3)^{-1}\mathscr{F}_3$ be irreducible. Assume that $S_{(c,1)} \leq R$ where $S_{(c,1)}$ and R are respectively given by* (4) *and* (6)*. Then, $\rho(M(\mathscr{E}_{\tilde{P}})^{-1}\mathscr{F}_{\tilde{P}}) \leq \rho(M(\mathscr{E}_3)^{-1}\mathscr{F}_3) < 1$.*

Proof. Consider the splittings $\mathscr{A} = \bar{\mathscr{E}}_3 - \bar{\mathscr{F}}_3 = \hat{\mathscr{E}}_{\tilde{P}} - \hat{\mathscr{F}}_{\tilde{P}}$ where $\bar{\mathscr{E}}_3$ and $\bar{\mathscr{F}}_3$ are given by (9) and

$$\hat{\mathscr{E}}_{\tilde{P}} = \tilde{P}^{-1}\tilde{\mathscr{E}}_{\tilde{P}} \quad \text{and} \quad \hat{\mathscr{F}}_{\tilde{P}} = \tilde{P}^{-1}\tilde{\mathscr{F}}_{\tilde{P}}.$$

Let $\bar{x}$ be the Perron vector of nonnegative irreducible tensor $M(\bar{\mathscr{E}}_3)^{-1}\bar{\mathscr{F}}_3$, by the Perron–Frobenius theorem for nonnegative irreducible tensors $\bar{x} > 0$ [6, Theorem 1.4]. By some straightforward computation, we derive

$$M(\bar{\mathscr{E}}_3)^{-1}\mathscr{A}\bar{x}^{m-1} = (1 - \rho(M(\bar{\mathscr{E}}_3)^{-1}\bar{\mathscr{F}}_3))x^{[m-1]} > 0. \tag{11}$$

It is not difficult to see that

$$M(\hat{\mathscr{E}}_{\tilde{P}})^{-1} = M(\tilde{\mathscr{E}}_{\tilde{P}})^{-1}\tilde{P} \geq M(\tilde{\mathscr{E}}_{\tilde{P}})^{-1}P_{(c,1)} \geq M(\mathscr{E}_3)^{-1}P_{(c,1)} = M(\bar{\mathscr{E}}_3)^{-1}.$$

As a results, from Eq. (11) and the above relation, we deduce that

$$\begin{aligned}\bar{x}^{[m-1]} - M(\hat{\mathscr{E}}_{\tilde{P}})^{-1}\hat{\mathscr{F}}_{\tilde{P}}\bar{x}^{m-1} = M(\hat{\mathscr{E}}_{\tilde{P}})^{-1}\mathscr{A}\bar{x}^{m-1} &\geq M(\bar{\mathscr{E}}_3)^{-1}\mathscr{A}\bar{x}^{m-1} \\ &= (1 - \rho(M(\bar{\mathscr{E}}_3)^{-1}\bar{\mathscr{F}}_3))\bar{x}^{[m-1]}.\end{aligned}$$

As a result, we have $M(\hat{\mathscr{E}}_{\tilde{P}})^{-1}\hat{\mathscr{F}}_{\tilde{P}}\bar{x}^{m-1} \leq \rho(M(\bar{\mathscr{E}}_3)^{-1}\bar{\mathscr{F}}_3)\bar{x}^{[m-1]}$. Now, using Theorem 3.5, we conclude that

$$\begin{aligned}\rho(M(\hat{\mathscr{E}}_{\tilde{P}})^{-1}\hat{\mathscr{F}}_{\tilde{P}}) &= \min_{x>0}\max_{x_i>0} \frac{(M(\hat{\mathscr{E}}_{\tilde{P}})^{-1}\hat{\mathscr{F}}_{\tilde{P}}x^{m-1})_i}{x_i^{m-1}} \\ &\leq \max_{\bar{x}_i>0} \frac{(M(\hat{\mathscr{E}}_{\tilde{P}})^{-1}\hat{\mathscr{F}}_{\tilde{P}}\bar{x}^{m-1})_i}{\bar{x}_i^{m-1}} \leq \rho(M(\bar{\mathscr{E}}_3)^{-1}\bar{\mathscr{F}}_3),\end{aligned}$$

which completes the proof. □

4. Numerical Results

In this section, we examine the proposed preconditioner numerically and compare its performance with the preconditioner proposed by Zhang et al. [19]. All of numerical experiments were computed using MATLAB version 9.4 (R2018a) running on an Intel Core i5 CPU at 2.50 GHz with 8 GB of memory.

We report the total required number of iterations and consumed CPU-time (in seconds) under "Iter" and "CPU(s)", respectively. The iterations were terminated once the maximum iteration number 1000 reached or $\|\mathscr{A}x_k^{m-1}-b\|_2 \leq \varepsilon$ where x_k denotes the kth approximate solution and $\varepsilon = 10^{-11}$. The the initial vector x_0 is taken to be zero. The spectral radii of nonnegative tensors are computed by the power method given in [14].
Two test examples from the literature were chosen for which $a_{i\ldots i} \geq 1$ for $i = 1,\ldots,n$. Therefore, we consider the multi-linear system $D^{-1}\mathscr{A}x^{m-1} = D^{-1}b$ which is equivalent to (1), where D is a diagonal matrix such that $D = \mathrm{diag}(a_{1\ldots1},\ldots,a_{n\ldots n})$. In the implementation of preconditioners, here, we limit ourselves to the cases that $\alpha_{i1} = \alpha$ for $i = 2,\ldots,n$, and $\alpha_{j(j-1)} = \beta$ for $j = 3,\ldots,n$ in (6). Although, in theory, we only considered the case that parameters are lower than one. In the sequel, we further report the results for the cases that the parameters are larger than one for more details.

Example 4.1 ([2]). Let $\mathscr{B} \in \mathbb{R}^{[3,10]}$ be a nonnegative tensor with majorization matrix $M(\mathscr{B}) = \mathrm{rand}(10)$ where "rand$(\cdot)$" is a MATLAB function. For $2 \leq i \leq 10$, $b_{i,i-1,i} = b_{i,i,i-1} = 1/6$, for $1 \leq i \leq 9$, $b_{i,i+1,i} = b_{i,i,i+1} = 1/6$ and other entries of $\mathscr{B}$ are zero. We set $\mathscr{A} = (5.8225)\mathscr{I} - \mathscr{B}$ commenting that we obtain $\rho(\mathscr{B}) = 4.8225$ using power method. The right-hand side is chosen to be a random vector. More precisely, the matrix $M(\mathscr{B})$ and the right-hand side vector b are given as follows:

$$M(\mathscr{B}) = \begin{bmatrix} 0.7894 & 0.4845 & 0.1123 & 0.1098 & 0.6733 & 0.0924 & 0.0986 & 0.5557 & 0.9879 & 0.1544 \\ 0.3677 & 0.1518 & 0.7844 & 0.9338 & 0.4296 & 0.0078 & 0.1420 & 0.1844 & 0.1704 & 0.3813 \\ 0.2060 & 0.7819 & 0.2916 & 0.1875 & 0.4517 & 0.4231 & 0.1683 & 0.2120 & 0.2578 & 0.1611 \\ 0.0867 & 0.1006 & 0.6035 & 0.2662 & 0.6099 & 0.6556 & 0.1962 & 0.0773 & 0.3968 & 0.7581 \\ 0.7719 & 0.2941 & 0.9644 & 0.7978 & 0.0594 & 0.7229 & 0.3175 & 0.9138 & 0.0740 & 0.8711 \\ 0.2057 & 0.2374 & 0.4325 & 0.4876 & 0.3158 & 0.5312 & 0.3164 & 0.7067 & 0.6841 & 0.3508 \\ 0.3883 & 0.5309 & 0.6948 & 0.7690 & 0.7727 & 0.1088 & 0.2176 & 0.5578 & 0.4024 & 0.6855 \\ 0.5518 & 0.0915 & 0.7581 & 0.3960 & 0.6964 & 0.6318 & 0.2510 & 0.3134 & 0.9828 & 0.2941 \\ 0.2290 & 0.4053 & 0.4326 & 0.2729 & 0.1253 & 0.1265 & 0.8929 & 0.1662 & 0.4022 & 0.5306 \\ 0.6419 & 0.1048 & 0.6555 & 0.0372 & 0.1302 & 0.1343 & 0.7032 & 0.6225 & 0.6207 & 0.8324 \end{bmatrix}, b = \begin{bmatrix} 0.5975 \\ 0.3353 \\ 0.2992 \\ 0.4526 \\ 0.4226 \\ 0.3596 \\ 0.5583 \\ 0.7425 \\ 0.4243 \\ 0.4294 \end{bmatrix}.$$

We report the numerical results in Table 1 which illustrates that the preconditioned Jacobi-type method with preconditioner $\tilde{P}$ converges faster than the corresponding one to preconditioner $P_{(c,1)}$.

Example 4.2 ([17]). Let $\mathscr{A} \in \mathbb{R}^{[3,n]}$ and $b \in \mathbb{R}^{n\times n}$ with

$$\begin{cases} a_{111} = a_{nnn} = 1, & \\ a_{iii} = 2, & i = 2,3,\ldots,n-1, \\ a_{ii-1i} = -1/2, & i = 2,3,\ldots,n-1, \\ a_{ii-1i-1} = -1/2, & i = 2,3,\ldots,n-1, \\ a_{ii+1i+1} = -1/2, & i = 2,3,\ldots,n-1, \end{cases}$$

Table 1. Example 4.1: Comparison results for the preconditioned Jacobi-type method.

α	Preconditioners							
	$P_{(c,1)}$		$\tilde{P}\,(\beta=\alpha)$		$\tilde{P}\,(\beta=2\alpha)$		$\tilde{P}\,(\beta=3\alpha)$	
	Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)
0.4	115	0.1353	111	0.1311	107	0.1267	104	0.1249
0.8	111	0.1321	104	0.1234	97	0.1146	91	0.1102
1.0	110	0.1316	101	0.1176	93	0.1093	85	0.0989
1.2	108	0.1295	97	0.1139	88	0.1066	80	0.0940
1.6	104	0.1258	91	0.1078	79	0.0935	69	0.0814
2.0	101	0.1219	85	0.0995	71	0.0824	58	0.0674

and

$$\begin{cases} b_1 = c_0^2, \\ b_i = \frac{a}{(n-1)^2}, \quad i = 2, 3, \ldots, n-1, \\ b_n = c_1^2. \end{cases}$$

Taking $c_0 = 1/2, c_1 = 1/3$ and $a = 2$, we report the numerical results in Table 2 associated with the experimentally obtained optimal parameter α^* for $\tilde{P}$. For more details, we further plot the convergence histories of (preconditioned) Jacobi-type method with respect to α in Figure 1. It is seen that the preconditioned Jacobi-type method with $\tilde{P}$ requires less number of iterations and CPU-time than the Jacobi-type method with $P_{(c,1)}$.

Table 2. Example 4.2: Comparison results for the Jacobi-type method with optimal parameter α.

n	α^*	No preconditioning		Preconditioned Jacobi-type method			
				Preconditioner $P_{(c,1)}$ $\alpha=\alpha^*$		Preconditioner $\tilde{P}$ $\alpha=\beta=\alpha^*$	
		Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)
20	2.1	73	0.0390	72	0.0397	36	0.0206
50	2.1	74	0.0413	72	0.0411	36	0.0211
100	2.0	75	0.0424	73	0.0418	37	0.0217
300	2.0	76	0.0466	74	0.0444	37	0.0257

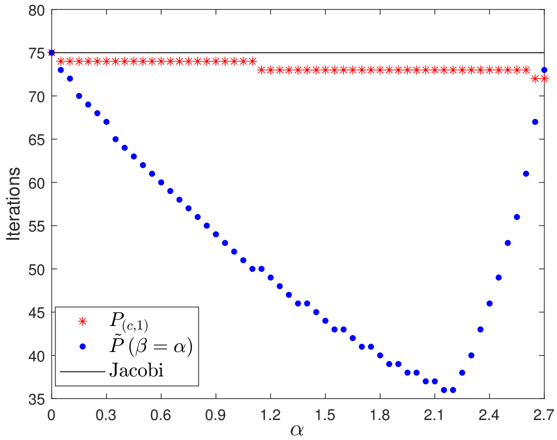

FIGURE 1. Example 4.2: Required number of iterations for the convergence versus parameter α for the Jacobi-type method ($n = 100$).

5. Conclusion

We presented a new preconditioner in conjunction with a Jacobi-type method for solving multi-linear systems. The performance of preconditioner was analyzed and compared with a recently proposed preconditioner in the literature by Zhang et al. [Applied Mathematics Letters. 2020; 104:106287] for accelerating the speed of convergence of Jacobi-type methods. The reported experimental results demonstrate that the new preconditioner outperforms the previously examined one by Zhang et al. and numerically confirm the established theoretical analyses.

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References

[1] C. Bu, X. Zhang, J. Zhou, W. Wang, Y. Wei, *The inverse, rank and product of tensors*, Linear Algebra and Its Applications, vol. 446, (2014) 269–280.
[2] D. Liu, W. Li, S.W. Vong, *A new preconditioned SOR method for solving multi-linear systems with an $\mathcal{M}$-tensor*, Calcolo, vol. 57, no. 2 (2020), DOI: 10.1007/s10092-020-00364-8.
[3] D. Liu, W. Li, S.W. Vong, *The tensor splitting with application to solve multi-linear systems*, Journal of Computational and Applied Mathematics, vol. 330, (2018) 75–94.

[4] F.P.A. Beik, M. Najafi-Kalyani, J. Khalide, *Preconditioned iterative methods for tensor multi-linear systems based on majorization matrix*, Preprint, Available online on ResearchGate.

[5] J.Y. Shao, *A general product of tensors with applications*, Linear Algebra and its applications, vol. 439, no. 8 (2013) 2350–2366.

[6] K.C. Chang, K. Pearson, T. Zhang, *Perron–Frobenius theorem for nonnegative tensors*, Communications in Mathematical Sciences, vol. 6, no. 2 (2008) 507–520.

[7] L.B. Cui, C. Chen, W. Li, *An eigenvalue problem for even order tensors with its applications*, Linear and Multilinear Algebra, vol. 64, no. 4 (2016) 602–621.

[8] L.B. Cui, M.H. Li, Y. Song, *Preconditioned tensor splitting iterations method for solving multi-linear systems*, Applied Mathematics Letters, vol. 96, (2019) 89–94.

[9] L.B. Cui, W. Li, M.K. Ng, *Primitive tensors and directed hypergraphs*, Linear Algebra and its Applications, vol. 471, (2015) 96–108.

[10] L.B. Cui, X.Q. Zhang, S.L. Wu, *A new preconditioner of the tensor splitting iterative method for solving multi-linear systems with $\mathcal{M}$-tensors*, Computational and Applied Mathematics, vol. 39, no. 173 (2020), DOI: 10.1007/s40314-020-01194-8.

[11] L.B. Cui, Y. Song, *On the uniqueness of the positive Z-eigenvector for nonnegative tensors*, Journal of Computational and Applied Mathematics, vol. 352, (2019) 72–78.

[12] L.H. Lim, *Singular values and eigenvalues of tensors: a variational approach*, In: 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing; IEEE; (2005) 129–132.

[13] L. Qi, *Eigenvalues of a real supersymmetric tensor*, Journal of Symbolic Computation, vol. 40, no. 6 (2005) 1302–1324.

[14] M. Ng, L. Qi, G. Zhou, *Finding the largest eigenvalue of a nonnegative tensor*, SIAM Journal on Matrix Analysis and Applications, vol. 31, no. 3 (2010) 1090–1099.

[15] Q. Yang, Y. Yang, *Further results for Perron–Frobenius theorem for nonnegative tensors II*, SIAM Journal on Matrix Analysis and Applications, vol. 32, no. 4 (2011) 1236–1250.

[16] W. Ding, L. Qi, Y. Wei, *$\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors*, Linear Algebra and Its Applications, vol. 439, no. 10 (2013) 3264–3278.

[17] W. Li, D. Liu, S.W. Vong, *Comparison results for splitting iterations for solving multi-linear systems*, Applied Numerical Mathematics, vol. 134, (2018) 105–121.

[18] W. Liu, W. Li, *On the inverse of a tensor*, Linear Algebra and its Applications, vol. 495, (2016) 199–205.

[19] Y. Zhang, Q. Liu, Z. Chen, *Preconditioned Jacobi type method for solving multi-linear systems with $\mathcal{M}$-tensors*, Applied Mathematics Letters, vol. 104, (2020) 106287.

Mehdi Najafi-Kalyani
Orcid number: 0000-0003-2474-6096
Department of Mathematics
Vali-e-Asr University of Rafsanjan
Rafsanjan, Iran
E-mail address: m.najafi.uk@gmail.com

Fatemeh P. A. Beik
Orcid number: 0000-0001-9050-3506
Department of Mathematics
Vali-e-Asr University of Rafsanjan
Rafsanjan, Iran
E-mail address: f.beik@vru.ac.ir