Journal of Mahani Mathematical Research Center
Shahid Bahonar University of Kerman
JMMRC 2012
Print ISSN: 2251-7952 Online ISSN: 2645-4505

KLEEN'S THEOREM FOR BL-GENERAL L-FUZZY AUTOMATA

M. Shamsizadeh*, M.M. Zahedi, and Kh. Abolpour

Dedicated to sincere professor Mehdi Radjabalipour on turning 75

Article type: Research Article

(Received: 13 February 2021, Revised: 12 May 2021, Accepted: 24 July 2021)

(Communicated by M. Mashinchi)

Abstract. The contribution of general fuzzy automata to neural networks has been considerable, and dynamical fuzzy systems are becoming more and more popular and useful. Basic logic, or BL for short, has been introduced by Hájek [5] in order to provide a general framework for formalizing statements of fuzzy nature. In this note, some of the closure properties of the BL-general fuzzy automaton based on lattice valued such as union, intersection, connection and a serial connection are considered, after that, the behavior of them are discussed. Moreover, for a given BL-general fuzzy automaton on the basis of lattice valued, a complete BL-general fuzzy automaton on the basis of lattice valued is presented. Afterward, we may test the Pumping Lemma for the BL-general fuzzy automaton based on lattice valued. In particular, a connection between the behavior of BL-general fuzzy automaton based on lattice valued and its language is presented. Also, it is proven that L is a recognizable set if and only if L is rational. Also, it is driven that Kleen's Theorem is valid for the BL-general fuzzy automaton on the basis of lattice valued. Finally, we give some examples to clarify these notions.

Keywords: BL-general fuzzy automata, Closure properties, Behavior of fuzzy automata, Pumping Lemma, Kleen's Theorem.
2020 MSC: Primary 8Q70, 68Q45.

1. Introduction

As the simplest mathematical model in the theory of computation, finite automata not only lay the theoretical foundations of complexity theory [6, 19], but also are closely related to other fields such as neural networks. In 1965, L.A. Zadeh [35] introduced the notion of fuzzy set as a method for representing uncertainty. His ideas have been applied to a wide range of scientific areas. In 1967, Wee [28] first introduced the mathematical formulation of fuzzy automata. As an important application, fuzzy automata have been used to simulate fuzzy discrete event systems. There were many authors such as Santos [20], Lee and Zadeh [7], Peeva and Topencharov [13, 26] who have contributed to this field. The fuzzy finite automaton can be applied in many areas

*Corresponding author, ORCID: 0000-0002-9336-289X
E-mail: shamsizadeh.m@gmail.com
DOI: 10.22103/jmmrc.2021.17171.1134

How to cite: M. Shamsizadeh, M. M. Zahedi and Kh. Abolpour, *Kleen's Theorem for BL-general L-fuzzy automata*, J. Mahani Math. Res. Cent. 2021; 10(2): 125-144.

such as learning systems, the model computing with words, pattern recognition, lattice-valued fuzzy finite automaton, database theory and simulation theory [3, 4, 6, 8, 9, 11, 12, 21–25, 34].

Malik et al. [9, 10] systematically established the theory of algebraic fuzzy finite automata. It is worthy of mention that Doostfatemeh and Kremer [2] introduced a new general definition for fuzzy automata. Their key motivation for introducing the notion of general fuzzy automata was the insufficiency of the current literature to handle the applications which rely on fuzzy automata as a modeling tool and to assign membership values to active states of a fuzzy automaton and to resolve the multi-membership. Another important insufficiency of the current literature is the lack of methodologies that enable us to define and analyze the continuous operation of fuzzy automata.

Basic logic (BL) has been introduced by Hájek [5] in order to provide a general framework for formalizing statements of fuzzy nature. By considering the notions of BL-algebra and residuated lattice, every BL-algebra is a residuated lattice. The supervisory control of fuzzy discrete event systems was established first by Qiu [18].

The idea of studying fuzzy automaton with membership values in some structured abstract set comes back to Wechler [27], and in recent years, researcher's attention has been aimed mostly at fuzzy automaton with membership values in complete residuated lattices, lattice-ordered monoids, and other kinds of lattices. Fuzzy automaton taking membership values in a complete residuated lattice were first studied by Qiu in [14, 15], where some basic concepts were discussed, and later, Qiu and his coworkers have carried out extensive research of these fuzzy automata(cf. [16, 17, 29–33]).

In 2012, Kh. Abolpour and M. M. Zahedi [1] extended the notion of general fuzzy automata and gave the notion of BL-general fuzzy automata (BL-GFA). An interesting problem in this context is the realization problem, which says that given a behavior, we can design a machine which realizes it. Now, by considering the definition of BL-GLFA, we present closure properties of BL-GLFA for example union, intersection, connection and serial connection. Also, we prove the Pumping Lemma for the BL-general L-fuzzy automaton. After that, we show that there is a connection between the behavior of a finite realization and a recognizable language of a BL-general L-fuzzy automaton. Finally, it is driven that Kleen's Theorem is valid for the BL-general fuzzy L-automaton.

2. Preliminaries

In this section, we give some definitions that we need in the sequel.

Definition 2.1. [5] A BL-algebra is an algebra structure $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:

i. $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
ii. $(L, *, 1)$ is a commutative monoid,

iii. $*$ and $\rightarrow$ form an adjoint pair, i.e., $x \leq y \rightarrow z$ if and only if $x * y \leq z$ for all $x, y, z \in L$,
iv. $x \wedge y = x * (x \rightarrow y)$,
v. $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Definition 2.2. [2] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted by $\tilde{F} = (Q, X, \tilde{R}, Z,$
$\tilde{\delta}, \omega, F_1, F_2)$, such that: (i) Q is a finite set of states, $Q = \{q_1, q_2, ..., q_n\}$, (ii) X is a finite set of input symbols, $X = \{a_1, a_2, ..., a_m\}$, (iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, (iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, ..., b_k\}$, (v) $\tilde{\delta} : (Q \times [0, 1]) \times X \times Q \rightarrow [0, 1]$ is the augmented transition function, (vi) $\omega : Q \rightarrow Z$ is the output function, (vii) $F_1 : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called the membership assignment function. The function $F_1(\mu, \delta)$, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the value of a transition.

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

$(viii)$ $F_2 : [0, 1]^* \rightarrow [0, 1]$ is called the multi-membership resolution function.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let the set of all transitions of $\tilde{F}$ is denoted by Δ. Now, suppose that $Q_{act}(t_i)$ is the set of all active states at time t_i, for all $i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) \big| \exists q' \in Q_{act}(t_{i-1}), \exists a \in X, \delta(q', a, q) \in \Delta\}$, for all $i \geq 1$. Since $Q_{act}(t_i)$ is a fuzzy set, to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we write $q \in Domain(Q_{act}(t_i))$. Hereafter, we denote these notations by

$$q \in Q_{act}(t_i) \text{ and } T \subseteq Q_{act}(t_i).$$

Definition 2.3. [22] Let $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton and $\bar{Q} = (P(Q), \subseteq, \cap, \cup, \emptyset, Q)$ be a BL-algebra in Example 2 of [22]. Then the BL-general L-fuzzy automaton (BL-GLFA) as a ten-tuple machine denoted by $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$, such that: (i) $\bar{Q} = P(Q)$, where Q is a finite set and $\bar{Q}$ is the power set of Q, (ii) X is a finite set of input symbols, (iii) $\tilde{R}$ is a set of fuzzy start states, (iv) $\bar{Z}$ is a finite set of output symbols, where $\bar{Z}$ is the power set of Z, (v) $\omega_l : \bar{Q} \rightarrow \bar{Z}$ is the output function defined by: $\omega_l(Q_i) = \{\omega(q) \big| q \in Q_i\}$, (vi) $\delta_l : \bar{Q} \times X \times \bar{Q} \rightarrow L$ is the transition function defined by: $\delta_l(\{p\}, a, \{q\}) = \delta(p, a, q)$ and $\delta_l(Q_i, a, Q_j) = \vee_{q_i \in Q_i, q_j \in Q_j} \delta(q_i, a, q_j)$, for all $Q_i, Q_j \in P(Q)$ and $a \in X$, (vii) $f_l : \bar{Q} \times X \rightarrow \bar{Q}$ is the next state map defined by: $f_l(Q_i, a) = \cup_{q_i \in Q_i} \{q_j \big| \delta(q_i, a, q_j) \in \Delta\}$, $(viii)$ $\tilde{\delta}_l : (\bar{Q} \times L) \times X \times \bar{Q} \rightarrow L$ is the augmented transition function defined $\tilde{\delta}_l((Q_i, \mu^t(Q_i)), a, Q_j) = F_1(\mu^t(Q_i), \delta_l(Q_i, a, Q_j))$, (ix) $F_1 : L \times L \rightarrow L$ is called membership assignment function, (x) $F_2 : L^* \rightarrow L$ is called multi-membership resolution function.

In the next example, at first, for a given general fuzzy automaton, we present the BL-GFA.

Example 2.4. *[22] Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice as in Figure 1. Now, consider general fuzzy automaton $\tilde{F} = (Q, X, \tilde{\delta}, \tilde{R}, Z, \omega, F_1, F_2)$ as in Fig-*

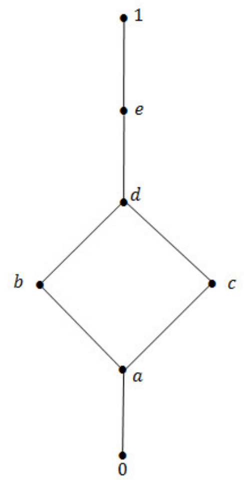

FIGURE 1. The complete lattice L of Example 2.4

ure 2, where $Q = \{q_0, q_1, q_2\}$, $\tilde{R} = \{(q_0, 1)\}, X = \{\sigma\}, Z = \{z_1, z_2\}, \omega(q_0) = z_1, \omega(q_1) = z_1, \omega(q_2) = z_2$ and

$$\delta(q_0, \sigma, q_0) = a, \ \delta(q_0, \sigma, q_1) = b,$$
$$\delta(q_1, \sigma, q_1) = c, \ \delta(q_2, \sigma, q_0) = d,$$
$$\delta(q_2, \sigma, q_2) = e.$$

Also, we have BL-general fuzzy automaton $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$, where

$$\bar{Q} = \{\emptyset, \{q_0\}, \{q_1\}, \{q_2\}, \{q_0, q_1\}, \{q_0, q_2\}, \{q_1, q_2\}, \{q_0, q_1, q_2\}\},$$

$\bar{Z} = \{\emptyset, \{z_1\}, \{z_2\}, \{z_1, z_2\}\}, \omega_l(\{q_0\}) = \omega_l(\{q_1\}) = \omega_l(\{q_0, q_1\}) = \{z_1\}, \omega_l(\{q_2\}) = \{z_2\}, \omega_l(\{q_0, q_2\}) = \omega_l(\{q_1, q_2\}) = \{z_1, z_2\} = \omega_l(\{q_0, q_1, q_2\})$, *and*

$$f_l(\{q_0\}, \sigma) = f_l(\{q_0, q_1\}, \sigma) = \{q_0, q_1\},$$
$$f_l(\{q_1\}, \sigma) = \{q_1\},$$
$$f_l(\{q_2\}, \sigma) = \{q_0, q_2\},$$
$$f_l(\{q_0, q_2\}, \sigma) = f_l(\{q_1, q_2\}, \sigma) = f_l(\{q_0, q_1, q_2\}, \sigma) = \{q_0, q_1, q_2\}$$

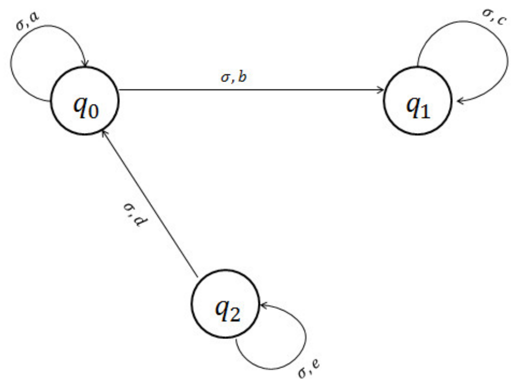

FIGURE 2. general fuzzy automaton $\tilde{F}$ of Example 2.4

and

$$\begin{aligned}
&\delta_l(\{q_0\}, \sigma, \{q_0\}) = a,\\
&\delta_l(\{q_0\}, \sigma, \{q_1\}) = b,\\
&\delta_l(\{q_0\}, \sigma, \{q_0, q_1\}) = b,\\
&\delta_l(\{q_0\}, \sigma, \{q_0, q_2\}) = a,\\
&\delta_l(\{q_0\}, \sigma, \{q_1, q_2\}) = b,\\
&\delta_l(\{q_0\}, \sigma, \{q_0, q_1, q_2\}) = b,\\
&\delta_l(\{q_1\}, \sigma, \{q_1\}) = c,\\
&\delta_l(\{q_1\}, \sigma, \{q_0, q_1\}) = c,\\
&\delta_l(\{q_1\}, \sigma, \{q_1, q_2\}) = c,\\
&\delta_l(\{q_1\}, \sigma, \{q_0, q_1, q_2\}) = c,\\
&\delta_l(\{q_2\}, \sigma, \{q_0\}) = d,\\
&\delta_l(\{q_2\}, \sigma, \{q_2\}) = e,\\
&\delta_l(\{q_2\}, \sigma, \{q_0, q_1\}) = d,\\
&\delta_l(\{q_2\}, \sigma, \{q_0, q_2\}) = e,\\
&\delta_l(\{q_2\}, \sigma, \{q_1, q_2\}) = e,\\
&\delta_l(\{q_2\}, \sigma, \{q_0, q_1, q_2\}) = e,\\
&\delta_l(\{q_0, q_1\}, \sigma, \{q_0\}) = a,\\
&\delta_l(\{q_0, q_1\}, \sigma, \{q_1\}) = d,\\
&\delta_l(\{q_0, q_1\}, \sigma, \{q_0, q_1\}) = d,\\
&\delta_l(\{q_0, q_1\}, \sigma, \{q_0, q_2\}) = a,\\
&\delta_l(\{q_0, q_1\}, \sigma, \{q_1, q_2\}) = d,\\
&\delta_l(\{q_0, q_1\}, \sigma, \{q_0, q_1, q_2\}) = d,
\end{aligned}$$

$$\delta_l(\{q_0,q_1\},\sigma,\{q_0,q_1,q_2\}) = d,$$
$$\delta_l(\{q_1,q_2\},\sigma,\{q_0\}) = d,$$
$$\delta_l(\{q_1,q_2\},\sigma,\{q_1\}) = c,$$
$$\delta_l(\{q_1,q_2\},\sigma,\{q_2\}) = e,$$
$$\delta_l(\{q_1,q_2\},\sigma,\{q_0,q_1\}) = d,$$
$$\delta_l(\{q_1,q_2\},\sigma,\{q_0,q_2\}) = e,$$
$$\delta_l(\{q_1,q_2\},\sigma,\{q_1,q_2\}) = e,$$
$$\delta_l(\{q_1,q_2\},\sigma,\{q_0,q_1,q_2\}) = e,$$
$$\delta_l(\{q_0,q_2\},\sigma,\{q_0\}) = d,$$
$$\delta_l(\{q_0,q_2\},\sigma,\{q_1\}) = b,$$
$$\delta_l(\{q_0,q_2\},\sigma,\{q_2\}) = e,$$
$$\delta_l(\{q_0,q_2\},\sigma,\{q_0,q_1\}) = d,$$
$$\delta_l(\{q_0,q_2\},\sigma,\{q_0,q_2\}) = e,$$
$$\delta_l(\{q_0,q_2\},\sigma,\{q_1,q_2\}) = e,$$
$$\delta_l(\{q_0,q_2\},\sigma,\{q_0,q_1,q_2\}) = e,$$
$$\delta_l(\{q_0,q_1,q_2\},\sigma,\{q_0\}) = d,$$
$$\delta_l(\{q_0,q_1,q_2\},\sigma,\{q_1\}) = d,$$
$$\delta_l(\{q_0,q_1,q_2\},\sigma,\{q_2\}) = e,$$
$$\delta_l(\{q_0,q_1,q_2\},\sigma,\{q_0,q_1\}) = d,$$
$$\delta_l(\{q_0,q_1,q_2\},\sigma,\{q_0,q_2\}) = e,$$
$$\delta_l(\{q_0,q_1,q_2\},\sigma,\{q_1,q_2\}) = e,$$
$$\delta_l(\{q_0,q_1,q_2\},\sigma,\{q_0,q_1,q_2\}) = e.$$

Definition 2.5. [1] Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GLFA. The run map of the BL-GLFA $\tilde{F}_l$ is the map $\rho : X^* \to \bar{Q}$ defined by the following induction: $\rho(\Lambda) = \{q_0\}$ and $\rho(a_1a_2...a_n) = Q_{i_n}, \rho(a_1a_2...a_na_{n+1})$ $= f_l(Q_{i_n}, a_{n+1})$, where $(Q_{i_n}, \mu^{t_0+n}(Q_{i_n})) \in Q_{act}(t_n)$, for every $a_1, ..., a_n \in X$.
The behavior of $\tilde{F}_l$ is the map $\beta = \omega_l \circ \rho : X^* \to \bar{Z}$.

In the rest of this note, L is denoted as a bounded complete lattice.

3. Closure properties for BL-general L-fuzzy automata

In this section, we present some properties of BL-general L-fuzzy automata such as union, intersection, connection and serial connection also study the behavior of them.

Definition 3.1. Let $\beta : X^* \to \bar{Z}$. Then we say that the behavior β is a finite realization if there exists a BL-GLFA $\tilde{F}_l$, where $\beta_{\tilde{F}_l} = \beta$.

Example 3.2. *Let BL-general fuzzy automaton $\tilde{F}_{l1}$ as in Figure 3, where $\bar{Q}_1 =$*

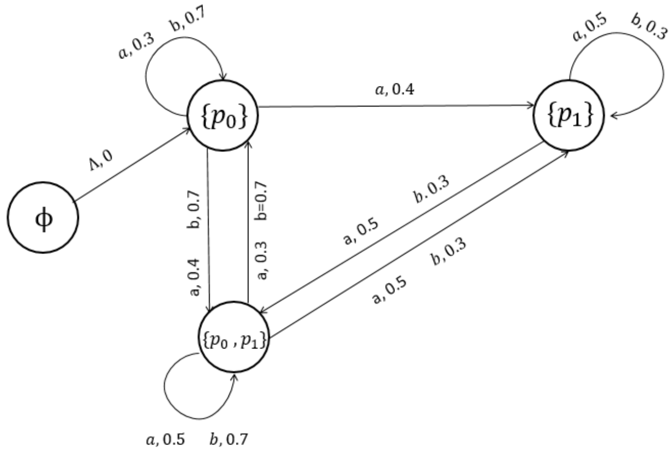

FIGURE 3. The BL-general fuzzy automaton $\tilde{F}_{l1}$ of Example 3.2

$\{\{p_0\},\{p_1\},\{p_0,p_1\},\emptyset\}$, $\tilde{R}_1 = (\{p_0\},1)$, $\bar{Z} = \{\{z_1\},\{z_2\},\{z_1,z_2\},\emptyset\}$,

$$\begin{aligned}
&\delta_{l1}(\{p_0\},a,\{p_0\}) = 0.3, \quad \delta_{l1}(\{p_0\},a,\{p_1\}) = 0.4,\\
&\delta_{l1}(\{p_0\},a,\{p_0,p_1\}) = 0.4, \quad \delta_{l1}(\{p_1\},a,\{p_1\}) = 0.5,\\
&\delta_{l1}(\{p_1\},a,\{p_0,p_1\}) = 0.5, \ \delta_{l1}(\{p_0,p_1\},a,\{p_0\}) = 0.3,\\
&\delta_{l1}(\{p_0,p_1\},a,\{p_1\}) = 0.5, \quad \delta_{l1}(\{p_0,p_1\},a,\{p_0,p_1\}) = 0.5,\\
&\delta_{l1}(\{p_0\},b,\{p_0\}) = 0.7, \quad \delta_{l1}(\{p_0\},b,\{p_0,p_1\}) = 0.7,\\
&\delta_{l1}(\{p_1\},b,\{p_1\}) = 0.3, \quad \delta_{l1}(\{p_1\},b,\{p_0,p_1\}) = 0.3,\\
&\delta_{l1}(\{p_0,p_1\},b,\{p_0\}) = 0.7, \quad \delta_{l1}(\{p_0,p_1\},b,\{p_1\}) = 0.3,\\
&\delta_{l1}(\{p_0,p_1\},b,\{p_0,p_1\}) = 0.7,
\end{aligned}$$

$\omega_{l1}(\{p_0\}) = \{z_1\}, \omega_{l1}(\{p_1\}) = \{z_2\}, \omega_{l1}(\{p_0,p_1\}) = \{z_1,z_2\}$.

$$\begin{aligned}
&\beta(a) = \{z_1,z_2\}, \qquad \beta(b) = \{z_1\},\\
&\beta(ba) = \{z_1,z_2\}.
\end{aligned}$$

Also, we have $\beta(a^+) = \{z_1,z_2\}$.

Definition 3.3. Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GLFA. Then we say that $\tilde{F}_l$ is a complete BL-GLFA if for any $\emptyset \neq Q' \in \bar{Q}$ and $a \in X$ there exists $\emptyset \neq Q'' \in \bar{Q}$ such that $f_l(Q',a) = Q''$.

Example 3.4. *Let BL-general fuzzy automaton $\tilde{F}_{l1}$ as in Example 3.2, Since for every $\emptyset \neq Q' \in \bar{Q}$ and $a \in X$ there exists $\emptyset \neq Q'' \in \bar{Q}$ such that $f_l(Q',a) =$*

Q'', so $\tilde{F}_{l1}$ is complete. For example for $\{p_0\} \in Q$ and $a, b \in X$, we have $f_l(\{p_0\}, a) = \{p_0, p_1\}$ and $f_l(\{p_0\}, b) = \{p_0, p_1\}$.

Theorem 3.5. *Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GLFA. Then there exists a complete BL-GLFA $\tilde{F}_l^c$ such that $\beta_{\tilde{F}_l} = \beta_{\tilde{F}_l^c}$.*

Proof. Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ does not be a complete BL-GLFA. Consider

$$\tilde{F}_l^c = (\bar{Q}^c, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l^c, \delta_l^c, f_l^c, \tilde{\delta}_l^c, F_1, F_2),$$

where $\bar{Q}^c = P(Q \cup t)$, t is an element such that $t \notin Q$.

If $f_l(Q', a) = \emptyset$, then $\delta_l^c(Q', a, P') = d$, where $\emptyset \neq Q' \in \bar{Q}, t \in P' \in \bar{Q}^c$. If $f_l(Q', a) \neq Q'$, then $\delta_l^c(Q', a, Q'') = \delta_l(Q', a, Q'')$, where $t \notin Q', Q'' \in \bar{Q}$. Also, let $\delta_l^c(\{t\}, a, Q') = d$, where $t \in Q'$, and consider $\delta_l^c(Q', a, Q'') = \delta_l(Q', a, P'')$, where $t \notin Q', Q'' = P'' \cup \{t\}$ and $P'' \neq \emptyset$. If $Q' = P' \cup \{t\}, P' \neq \emptyset$ and $t \notin Q''$, then consider $\delta_l^c(Q', a, Q'') = \delta_l(P', a, Q'')$. If $Q' = P' \cup \{t\}, Q'' = P'' \cup \{t\}$ and $P', P'' \notin \emptyset$, then consider $\delta_l^c(Q', a, Q'') = \delta_l(P', a, P'') \vee d$. Finally, if $Q' = P' \cup \{t\}$ and $P' \neq \emptyset$, then $\delta_l^c(Q', a, \{t\}) = d$. Also, let $\omega_l^c(Q') = \omega_l(Q')$, for every $Q' \in \bar{Q}$.

It is easy to see that the BL-GLFA

$$\tilde{F}_l^c = (\bar{Q}^c, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l^c, \delta_l^c, f_l^c, \tilde{\delta}_l^c, F_1, F_2),$$

is complete and $\beta_{\tilde{F}_l} = \beta_{\tilde{F}_l^c}$. □

Definition 3.6. Let

$$\tilde{F}_{li} = (\bar{Q}_i, X, \tilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2,$$

be two BL-GLFAs. Then

$$\tilde{F}_{l1} \cup \tilde{F}_{l2} = (\bar{Q}_1 \times \bar{Q}_2, X, ((\{q_{01}\}, \{q_{02}\}), \mu^{t_0}((\{q_{01}\}, \{q_{02}\})), \bar{Z}, \omega_{l\cup}, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2),$$

where $\mu^{t_0}((\{q_{01}\}, \{q_{02}\})) = (\mu^{t_0}(\{q_{01}\}), \mu^{t_0}(\{q_{02}\}))$, $\omega_{l\cup}(Q', Q'') = \omega_{l1}(Q') \cup \omega_{l2}(Q'')$,

$$\begin{aligned} \delta_l((Q', Q''), a, (P', P'')) &= (\delta_{l1} \times \delta_{l2})((Q', Q''), a, (P', P'')) \\ &= (\delta_{l1}(Q', a, P'), \delta_{l2}(Q'', a, P'')), \end{aligned}$$

$f_l((Q', Q''), a) = (f_{l1} \times f_{l2})((Q', Q''), a) = (f_{l1}(Q', a), f_{l2}(Q'', a))$ and

$$\begin{aligned} \tilde{\delta}_l(((Q', Q''), \mu^t(Q', Q'')), &a, (P', P'')) \\ &= (\tilde{\delta}_{l1} \times \tilde{\delta}_{l2})(((Q', Q''), \mu^t(Q', Q'')), a, (P', P'')) \\ &= (\tilde{\delta}_{l1}((Q', \mu^t(Q')), a, P'), \tilde{\delta}_{l2}((Q'', \mu^t(Q'')), a, P'')), \end{aligned}$$

for every $Q', P' \in \bar{Q}_1, Q'', P'' \in \bar{Q}_2$ and $a \in X$.

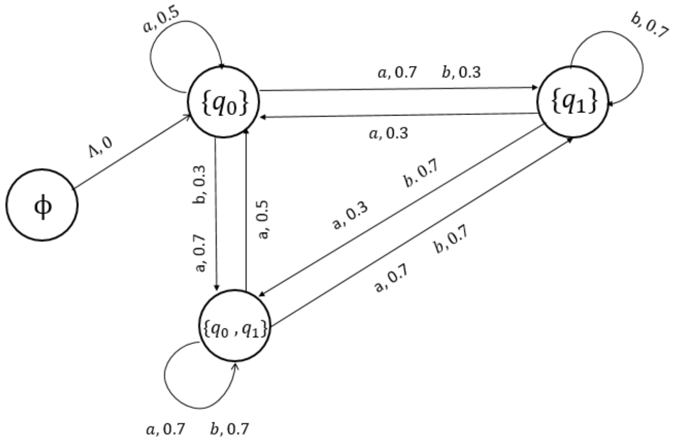

FIGURE 4. The BL-general fuzzy automaton $\tilde{F}_{l2}$ of Example 3.7

Example 3.7. *Let BL-general fuzzy automaton $\tilde{F}_{l1}$ as in Example 3.2. Also, consider BL-general fuzzy automaton $\tilde{F}_{l2}$ as in Figure 4, where* $\bar{Q}_2 = \{\{q_0\}, \{q_1\}, \{q_0, q_1\}, \emptyset\}$, $\tilde{R}_2 = (\{q_0\}, 1)$, $\bar{Z} = \{\{z_1\}, \{z_2\}, \{z_1, z_2\}, \emptyset\}$,

$$\begin{aligned}
&\delta_{l2}(\{q_0\}, a, \{q_0\}) = 0.5,\\
&\delta_{l2}(\{q_0\}, a, \{q_1\}) = 0.7,\\
&\delta_{l2}(\{q_0\}, a, \{q_0, q_1\}) = 0.7,\\
&\delta_{l2}(\{q_1\}, a, \{q_0\}) = 0.3,\\
&\delta_{l2}(\{q_1\}, a, \{q_0, q_1\}) = 0.3,\\
&\delta_{l2}(\{q_0, q_1\}, a, \{q_0\}) = 0.5,\\
&\delta_{l2}(\{q_0, q_1\}, a, \{q_1\}) = 0.7,\\
&\delta_{l2}(\{q_0, q_1\}, a, \{q_0, q_1\}) = 0.7,\\
&\delta_{l2}(\{q_0\}, b, \{q_1\}) = 0.3,\\
&\delta_{l2}(\{q_0\}, b, \{q_0, q_1\}) = 0.3,\\
&\delta_{l2}(\{q_1\}, b, \{q_1\}) = 0.7,\\
&\delta_{l2}(\{q_1\}, b, \{q_0, q_1\}) = 0.7,\\
&\delta_{l2}(\{q_0, q_1\}, b, \{q_1\}) = 0.7,\\
&\delta_{l2}(\{q_0, q_1\}, b, \{q_0, q_1\}) = 0.7,
\end{aligned}$$

$\omega_{l2}(\{q_0\}) = \omega_{l2}(\{q_1\}) = \omega_{l2}(\{q_0, q_1\}) = \{z_2\}$. *It is clear that $\tilde{F}_{li}, i = 1, 2$ are complete. Then by considering Definition 3.6, $\tilde{F}_{l1} \cup \tilde{F}_{l2}$ is as follows:* $\bar{Q}_1 \times \bar{Q}_2 = \{(\{p_0\}, \{q_0\}), (\{p_0\}, \{q_1\}), (\{p_0\}, \{q_0, q_1\}), (\{p_1\}, \{q_0\}),$

$(\{p_1\},\{q_1\}),(\{p_1\},\{q_0,q_1\}),(\{p_0,p_1\},\{q_0\}),(\{p_0,p_1\},\{q_1\}),(\{p_0,p_1\},\{q_0,q_1\})\}$,

$$\delta_l((\{p_0\},\{q_0\}),a,(\{p_0\},\{q_0\}))=(0.3,0.5),$$
$$\delta_l((\{p_0\},\{q_0\}),a,(\{p_0\},\{q_1\}))=(0.3,0.7),$$
$$\delta_l((\{p_0\},\{q_0\}),a,(\{p_1\},\{q_0\}))=(0.4,0.5),$$
$$\delta_l((\{p_0\},\{q_0\}),a,(\{p_1\},\{q_1\}))=(0.4,0.7),$$
$$\delta_l((\{p_0\},\{q_0\}),a,(\{p_1\},\{q_0,q_1\}))=(0.4,0.7),$$
$$\delta_l((\{p_0\},\{q_0\}),a,(\{p_0,p_1\},\{q_0\}))=(0.4,0.5),$$
$$\delta_l((\{p_0\},\{q_0\}),a,(\{p_0,p_1\},\{q_1\}))=(0.4,0.7),$$
$$\delta_l((\{p_0\},\{q_0\}),a,(\{p_0,p_1\},\{q_0,q_1\}))=(0.4,0.7),$$
$$\delta_l((\{p_0\},\{q_0\}),b,(\{p_0\},\{q_1\}))=(0.7,0.3),$$
$$\delta_l((\{p_0\},\{q_0\}),b,(\{p_0\},\{q_0,q_1\}))=(0.7,0.3),$$
$$\delta_l((\{p_0\},\{q_0\}),b,(\{p_0,p_1\},\{q_1\}))=(0.7,0.3),$$
$$\delta_l((\{p_0\},\{q_0\}),b,(\{p_0,p_1\},\{q_0,q_1\}))=(0.7,0.3).$$

In the similar way, we can calculate the other of transitions of δ_l*, also*

$$\omega_{l\cup}((\{p_0\},\{q_0\}))=\{z_1,z_2\},$$
$$\omega_{l\cup}((\{p_0\},\{q_1\}))=\{z_1,z_2\},$$
$$\omega_{l\cup}((\{p_0\},\{q_0,q_1\}))=\{z_1,z_2\},$$
$$\omega_{l\cup}((\{p_1\},\{q_0\}))=\{z_2\},$$
$$\omega_{l\cup}((\{p_1\},\{q_1\}))=\{z_2\},$$
$$\omega_{l\cup}((\{p_1\},\{q_0,q_1\}))=\{z_2\},$$
$$\omega_{l\cup}((\{p_0,p_1\},\{q_0\}))=\{z_1,z_2\},$$
$$\omega_{l\cup}((\{p_0,p_1\},\{q_1\}))=\{z_1,z_2\},$$
$$\omega_{l\cup}((\{p_0,p_1\},\{q_0,q_1\}))=\{z_1,z_2\},$$

Theorem 3.8. *Let*

$$\tilde{F}_{li}=(\bar{Q}_i,X,\tilde{R}_i=(\{q_{0i}\},\mu^{t_0}(\{q_{0i}\})),\bar{Z},\omega_{li},\delta_{li},f_{li},\tilde{\tilde{\delta}}_{li},F_1,F_2),i=1,2,$$

be two complete BL-GLFAs. Then $\beta_{\tilde{F}_{l1}\cup\tilde{F}_{l2}}=\beta_{\tilde{F}_{l1}}\cup\beta_{\tilde{F}_{l2}}$.

Proof. Let ρ_1,ρ_2,ρ be the run relations of $\tilde{F}_{l1},\tilde{F}_{l2}$ and $\tilde{F}_{l1}\cup\tilde{F}_{l2}$, respectively. Then we have

$$\begin{aligned}\beta_{\tilde{F}_{l1}\cup\tilde{F}_{l2}}(x)&=\omega_{l\cup}\circ\rho(x)=\omega_{l\cup}(\rho_1(x),\rho_2(x))\\&=\omega_{l1}(\rho_1(x))\cup\omega_{l2}(\rho_2(x))=\beta_{\tilde{F}_{l1}}(x)\cup\beta_{\tilde{F}_{l2}}(x),\end{aligned}$$

for every $x\in X^*$. □

Definition 3.9. Let

$$\tilde{F}_{li}=(\bar{Q}_i,X,\tilde{R}_i=(\{q_{0i}\},\mu^{t_0}(\{q_{0i}\})),\bar{Z},\omega_{li},\delta_{li},f_{li},\tilde{\tilde{\delta}}_{li},F_1,F_2),i=1,2,$$

be two BL-GLFAs. Then

$\tilde{F}_{l1}\cap\tilde{F}_{l2} = (\bar{Q}_1\times\bar{Q}_2, X, ((\{q_{01}\},\{q_{02}\}), \mu^{t_0}((\{q_{01}\},\{q_{02}\})), \bar{Z}, \omega_{l\cap}, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$,

where $\mu^{t_0}((\{q_{01}\},\{q_{02}\})) = (\mu^{t_0}(\{q_{01}\}), \mu^{t_0}(\{q_{02}\}))$ and $\omega_{l\cap}(Q', Q'') = \omega_{l1}(Q')\cap \omega_{l2}(Q'')$, for every $Q' \in \bar{Q}_1, Q'' \in \bar{Q}_2$. Also, δ_l, f_l and $\tilde{\delta}_l$ are similar to Definition 3.6.

Example 3.10. *Let BL-general fuzzy automata $\tilde{F}_{li}, i = 1, 2$ as in Examples 3.2 and 3.7. Then by considering Definitions 3.6 and 3.9, $\tilde{F}_{l1}\cap\tilde{F}_{l2}$ is as follows:* $\bar{Q}_1\times\bar{Q}_2 = \{(\{p_0\},\{q_0\}), (\{p_0\},\{q_1\}), (\{p_0\},\{q_0,q_1\}), (\{p_1\},\{q_0\}), (\{p_1\},\{q_1\}), (\{p_1\},\{q_0,q_1\}), (\{p_0,p_1\},\{q_0\}), (\{p_0,p_1\},\{q_1\}), (\{p_0,p_1\},\{q_0,q_1\})\}$,

$$\begin{aligned}
&\delta_l((\{p_0\},\{q_0\}), a, (\{p_0\},\{q_0\})) = (0.3, 0.5),\\
&\delta_l((\{p_0\},\{q_0\}), a, (\{p_0\},\{q_1\})) = (0.3, 0.7),\\
&\delta_l((\{p_0\},\{q_0\}), a, (\{p_1\},\{q_0\})) = (0.4, 0.5),\\
&\delta_l((\{p_0\},\{q_0\}), a, (\{p_1\},\{q_1\})) = (0.4, 0.7),\\
&\delta_l((\{p_0\},\{q_0\}), a, (\{p_1\},\{q_0,q_1\})) = (0.4, 0.7),\\
&\delta_l((\{p_0\},\{q_0\}), a, (\{p_0,p_1\},\{q_0\})) = (0.4, 0.5),\\
&\delta_l((\{p_0\},\{q_0\}), a, (\{p_0,p_1\},\{q_1\})) = (0.4, 0.7),\\
&\delta_l((\{p_0\},\{q_0\}), a, (\{p_0,p_1\},\{q_0,q_1\})) = (0.4, 0.7),\\
&\delta_l((\{p_0\},\{q_0\}), b, (\{p_0\},\{q_1\})) = (0.7, 0.3),\\
&\delta_l((\{p_0\},\{q_0\}), b, (\{p_0\},\{q_0,q_1\})) = (0.7, 0.3),\\
&\delta_l((\{p_0\},\{q_0\}), b, (\{p_0,p_1\},\{q_1\})) = (0.7, 0.3),\\
&\delta_l((\{p_0\},\{q_0\}), b, (\{p_0,p_1\},\{q_0,q_1\})) = (0.7, 0.3).
\end{aligned}$$

In the similar way, we can calculate the other of transitions of δ_l, also

$\omega_{l\cap}((\{p_0\},\{q_0\})) = \omega_{l\cap}((\{p_0\},\{q_1\})) = \omega_{l\cap}((\{p_0\},\{q_0,q_1\})) = \emptyset$,
$\omega_{l\cap}((\{p_1\},\{q_0\})) = \omega_{l\cap}((\{p_1\},\{q_1\})) = \omega_{l\cap}((\{p_1\},\{q_0,q_1\})) = \{z_2\}$,
$\omega_{l\cap}((\{p_0,p_1\},\{q_0\})) = \omega_{l\cap}((\{p_0,p_1\},\{q_1\})) = \omega_{l\cap}((\{p_0,p_1\},\{q_0,q_1\})) = \{z_2\}$.

Theorem 3.11. *Let*

$$\tilde{F}_{li} = (\bar{Q}_i, X, \tilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2$$

be two complete BL-GLFAs. Then $\beta_{\tilde{F}_{l1}\cap\tilde{F}_{l2}} = \beta_{\tilde{F}_{l1}}\cap\beta_{\tilde{F}_{l2}}$.

Proof. Let ρ_1, ρ_2, ρ be the run relations of $\tilde{F}_{l1}, \tilde{F}_{l2}$ and $\tilde{F}_{l1}\cap\tilde{F}_{l2}$, respectively. Then we have

$$\begin{aligned}
\beta_{\tilde{F}_{l1}\cap\tilde{F}_{l2}}(x) &= \omega_{l\cap}\circ\rho(x) = \omega_{l\cap}(\rho_1(x), \rho_2(x))\\
&= \omega_{l1}(\rho_1(x))\cap\omega_{l2}(\rho_2(x)) = \beta_{\tilde{F}_{l1}}(x)\cap\beta_{\tilde{F}_{l2}}(x)
\end{aligned}$$

for every $x \in X^*$. □

Definition 3.12. By considering Definition 2.5, we have $\beta : X^* \to \bar{Z}$, i.e., $\beta(x) = Z_l \in \bar{Z}$, for every $x \in X^*$. Define $\bar{\beta} : X^* \to \bar{Z}$ by $\bar{\beta}(x) = Z'_l$ if and only if $\beta(x) = Z_l$, where $Z'_l = Z - Z_l$ and Z is universal set.

Example 3.13. *Let $\tilde{F}_{l1}$ be the BL-GLFA as in Example 3.10. Then $\beta = \beta_{\tilde{F}_{l1}}$ is a finite realization. Now, let*

$$\tilde{F}'_{l1} = (\bar{Q}_1, X, \tilde{R}_1 = (\{p_0\}, \mu^{t_0}(\{p_0\})), \bar{Z}, \omega'_{l1}, \delta_{l1}, f_{l1}, \tilde{\delta}_{l1}, F_1, F_2),$$

where $\omega'_l(\{p_0\}) = \{z_2\}, \omega'_l(\{p_1\}) = \{z_1\}, \omega'_l(\{p_0, p_1\}) = \emptyset$. Then we have

$$\begin{aligned} &\beta(a) = \{z_1, z_2\}, && \beta_{\tilde{F}'_{l1}}(a) = \omega'_1(\{p_0, p_1\}) = \emptyset = \bar{\beta}(a),\\ &\beta(b) = \{z_1\}, && \beta_{\tilde{F}'_{l1}}(b) = \omega'_1(\{p_0\}) = \{z_2\} = \bar{\beta}(a),\\ &\beta(ba) = \{z_1, z_2\}, && \beta_{\tilde{F}'_{l1}}(ba) = \omega'_1(\{p_0, p_1\}) = \emptyset = \bar{\beta}(a). \end{aligned}$$

Theorem 3.14. *Let the behavior $\beta : X^* \to \bar{Z}$ be a finite realization. Then $\bar{\beta}$ is a finite realization.*

Proof. Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GLFA, where $\beta_{\tilde{F}_l} = \beta$. Now, define BL-GLFA $\tilde{F}'_l$ as following: $\tilde{F}'_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega'_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$. Define $\omega'_l : \bar{Q} \to \bar{Z}$ by $\omega'_l(Q') = Z'_l$ if and only if $\omega_l(Q') = Z''_l$, where $Z'_l = Z - Z''_l$. Let ρ be the run relations of $\tilde{F}_l$. Then $\beta_{\tilde{F}'_l}(x) = \omega'_l \circ \rho(x) = \omega'_l(\rho(x)) = Z - Z'_l$, where $\omega_l(\rho(x)) = Z'_l$. Hence $\beta(\tilde{F}'_l) = \bar{\beta}$. □

Lemma 3.15. *(Pumping Lemma) Let β be a finite realization. Then, there exists a positive integer N, such that every word $x \in X^*$ of lengths exceeding that N can be divided into three pieces $x = uvw$, satisfying the conditions:* (1) $|v| > 1$, (2) $|uv| < N$, (3) $\forall m > 0, \beta(uvw) = \beta(uv^m w)$.

Proof. Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GLFA, where $\beta_{\tilde{F}_l} = \beta$ and $|N| = 2^{|Q|}$, $|Q|$ is the cardinality of the state set Q. Now, suppose that $x = a_1 a_2 ... a_k \in X^*$, where $|x| = k > N$. Let $\beta(x) = \omega_l(\rho(x))$, where ρ is the run map of $\tilde{F}_l$ and $\rho(a_1) = Q_1$, $\rho(a_1 a_2) = Q_2$, ..., $\rho(a_1 a_2 ... a_N) = Q_N$, $\rho(a_1 a_2 ... a_N a_{N+1}) = Q_{N+1}$, ..., $\rho(a_1 a_2 ... a_k) = Q_k$. Then there must be at least one repetition among the $N + 1$ states $\{q_0\}, Q_1, Q_2, ..., Q_N$. Let Q_i (with $i \geq 0$) be the first state in the sequence $\{\{q_0\}, Q_1, ..., Q_N\}$ that repeats and let $Q_{i+r} = Q_j (r > 0)$ be its repetition. Then we have $x = uvw$, where

$$u = a_1 a_2 ... a_i, \ v = a_{i+1} a_{i+2} ... a_j, \ w = a_{j+1} a_{j+2} ... a_k.$$

Therefore $\rho(x) = \rho(uvw) = \rho(a_1 ... a_i a_{i+1} ... a_j a_{j+1} ... a_k) = \rho(f_l(Q_i, a_{i+1} ... a_k)) = \rho(f_l(Q_j, a_{j+1} ... a_k)) = \rho(a_1 ... a_i a_{j+1} ... a_k)$. In a similar way $\rho(uv^m w) = \rho(uvw)$, for every $m > 0$. Hence $\beta(uvw) = \beta(uv^m w)$, for every $m > 0$. □

Example 3.16. *Let $\tilde{F}_{l1}$ be as defined in Example 3.10. Clearly, $\beta = \beta_{\tilde{F}_{l1}}$ is a finite realization. Consider $a^n \in X^*$ such that $n > |\bar{Q}_1|$. Obviously, $\rho(a) = \{p_0, p_1\}$ and $\rho(a^2) = \{p_0, p_1\}$. Therefore, we have $a^n = a^p a^q a^r$, where $p = 1, q = 1, r = n-2$. Hence, it is clear that $\beta(a^n) = \beta(a^m)$, where $m \geq n-1$.*

Definition 3.17. A language $\mathcal{L} \subseteq X^*$ is recognizable if there exists a finite BL-general L-fuzzy automaton with the output alphabet $\bar{Z} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$, in which its behavior is a function $\beta : X^* \to \overline{\{0,1\}}$ and $1 \in \beta(x) \Longleftrightarrow x \in \mathcal{L}$.

Theorem 3.18. *A behavior $\beta : X^* \to \bar{Z}$ is a finite realization if and only if*

(i) *the language $\beta^{-1}(Y)$ is recognizable, for each $Y \in \bar{Z}$,*

(ii) *the set $\beta(X^*) \subseteq \bar{Z}$ is finite.*

Proof. $\Rightarrow$) Let β be a finite realization, say $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$.

(i) For every $Y \in \bar{Z}$, define an acceptor $\tilde{F}_Y = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}_Y = \overline{\{0,1\}}, \omega_{lY}^{-1}, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$. We prove that $\beta^{-1}(Y)$ is the language recognized by $\tilde{F}_Y$. Denote by $\rho : X^* \to \bar{Q}$ the run map of $\tilde{F}_l$, then $\beta = \omega_l \circ \rho$. Moreover, ρ is the run map of $\tilde{F}_Y$ as well, its output map $\omega_{lY} : \bar{Q} \to \overline{\{0,1\}}$ is defined by $1 \in \omega_{lY}(Q')$ if and only if $\omega_l(Q') = Y$. Then the behavior of $\tilde{F}_Y, \beta_Y = \omega_{lY} \circ \rho$, for every $x \in X^*$

$$\begin{aligned} 1 \in \beta_Y(x) &\Longleftrightarrow 1 \in \omega_{lY} \circ \rho(x) \\ &\Longleftrightarrow \omega_l(\rho(x)) = Y \\ &\Longleftrightarrow \beta(x) = Y. \end{aligned}$$

Therefore $\mathcal{L}_{\tilde{F}_Y} = \beta^{-1}(Y)$. This proves that $\beta^{-1}(Y)$ is recognizable.

(ii) $\beta(X^*) = \omega_l(\rho(X^*)) = \omega_l(\bar{Q})$ is finite, since Q is finite.

$\Leftarrow$) Let β has properties (i) and (ii) and $\beta(X^*) = \{Y_1, Y_2, ..., Y_n\}$. By hypothesis, for every $i = 1, 2, ..., n$ the language $\mathcal{L}_i = \beta^{-1}(Y_i)$ is a finite realization, say

$$\tilde{F}_{li} = (\bar{Q}_i, X, \tilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \overline{\{0,1\}}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2),\ i = 1, 2, ..., n.$$

Define a finite BL-GLFA $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ as follows: $\bar{Q} = Q_1 \times ... \times Q_n$, $\tilde{R} = ((\{q_{01}\}, ..., \{q_{0n}\}), \mu^{t_0}(\{q_{01}\}, ..., \{q_{0n}\})$, where

$$\mu^{t_0}(\{q_{01}\}, \{q_{02}\}, ..., \{q_{0n}\}) = (\mu^{t_0}(\{q_{01}\}), \mu^{t_0}(\{q_{02}\}), ..., \mu^{t_0}(\{q_{0n}\})),$$

also

$$\begin{aligned} \delta_l((Q'_1, Q'_2, ..., Q'_n), &a, (P'_1, P'_2, ..., P'_n)) \\ &= (\delta_{l1}(Q'_1, a, P'_1), \delta_{l2}(Q'_2, a, P'_2), ..., \delta_{ln}(Q'_n, a, P'_n)), \end{aligned}$$

$$\begin{aligned} \tilde{\delta}_l(((Q'_1, Q'_2, ..., Q'_n), &\mu^t((Q'_1, Q'_2, ..., Q'_n))), a, (P'_1, P'_2, ..., P'_n)) \\ &= (\tilde{\delta}_{l1}((Q'_1, \mu^t(Q'_1)), a, P'_1), \tilde{\delta}_{l2}((Q'_2, \\ &\mu^t(Q'_2)), a, P'_2), ..., \tilde{\delta}_{ln}((Q'_n, \mu^t(Q'_n)), a, P'_n)), \end{aligned}$$

$f_l((Q'_1,...,Q'_n),a) = (f_{l1}(Q'_1,a),...,f_{ln}(Q'_n,a))$ also, $\omega_l((Q'_1,Q'_2,...,Q'_n)) = \omega_{li}(Q'_i)$ if $1 \in \omega_{li}(Q'_i)$ and $\omega_{lj}(Q'_j) = \{0\}$, for every $j \neq i$. Now, let $\rho_i : X^* \to \bar{Q}_i$ be the run relations of $\tilde{F}_{li}, i = 1,...,n$. Then we define a run map of $\tilde{F}_l$ as follow: $\rho : X^* \to \bar{Q}$, where $\rho(x) = (\rho_1(x), \rho_2(x),...,\rho_n(x))$. We have

$$\begin{aligned}\rho(xa) &= (\rho_1(xa), \rho_2(xa),...,\rho_n(xa))\\ &= (f_{l1}(\rho_1(x),a), f_{l2}(\rho_2(x),a),...,f_{ln}(\rho_n(x),a))\\ &= f((\rho_1(x),\rho_2(x),...,\rho_n(x)),a)\\ &= f(\rho(x),a).\end{aligned}$$

Let $\bar{\beta} = \omega_l \circ \rho$. Then for every $x \in X^*$, we have $\beta(x) = Y_i$, i.e., $x \in \mathcal{L}(\tilde{F}_i)$ and $x \notin \mathcal{L}(\tilde{F}_j)$ for $j \neq i$. Therefore

$$\omega_l(\rho(x)) = \omega_l(\rho_1(x), \rho_2(x),...,\rho_n(x)) = Y_i.$$

Then $\beta(x) = Y_i$ implies that $\bar{\beta}(x) = \omega_l(\rho(x)) = Y_i$. Hence $\beta = \bar{\beta}$. □

Definition 3.19. (Connection) Let

$$\tilde{F}_{li} = (\bar{Q}_i, X, \tilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \{0,1\}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1,2,$$

be two BL-GLFAs, where $Q_1 \cap Q_2 = \emptyset$. Define $\tilde{F}_{l1} * \tilde{F}_{l2}$ as follows:

$$\tilde{F}_{l1} * \tilde{F}_{l2} = (P(Q_1 \cup \{q_{02}\}) \cup \bar{Q}_2, X, \tilde{R}_1 = (\{q_{01}\}, \mu^{t_0}(\{q_{01}\})), \{0,1\}, \omega'_l, \bar{\delta}_l, \bar{f}_l, \bar{\tilde{\delta}}_l, F_1, F_2),$$

where

$$\bar{f}_l(Q_i,a) = \begin{cases} f_{l1}(Q_i,a) & if\ Q_i \in \bar{Q}_1, \omega_{l1}(Q_i) = \{0\} \\ f_{l1}(Q_i,a) \cup \{q_{02}\} & if\ Q_i \in \bar{Q}_1, 1 \in \omega_{l1}(Q_i) \end{cases},$$

$\bar{f}_l(Q_i,a) = f_{l2}(Q_i,a)$, for every $Q_i \in \bar{Q}_2$,

$$\bar{\delta}_l(Q_i,a,Q_j) = \begin{cases} \delta_{l1}(Q_i,a,Q_j) & if\ Q_i,Q_j \in \bar{Q}_1 \\ \delta_{l2}(Q_i,a,Q_j) & if\ Q_i,Q_j \in \bar{Q}_2 \\ \delta_{l2}(\{q_{02}\},a,Q_j) & if\ \{q_{02}\} \in Q_i, Q_i - \{q_{02}\} \in \bar{Q}_1, Q_j \in \bar{Q}_2 \\ \delta_{l1}(Q_i - \{q_{02}\},a,Q_j) & if\ \{q_{02}\} \in Q_i, Q_i - \{q_{02}\} \in \bar{Q}_1, Q_j \in \bar{Q}_1 \end{cases},$$

and

$$\bar{\tilde{\delta}}_l((Q_i,\mu^t(Q_i)),a,Q_j) =$$

$$\begin{cases} \tilde{\delta}_{l1}((Q_i,\mu^t(Q_i)),a,Q_j) & if\ Q_i,Q_j \in \bar{Q}_1 \\ \tilde{\delta}_{l2}((Q_i,\mu^t(Q_i)),a,Q_j) & if\ Q_i,Q_j \in \bar{Q}_2 \\ \tilde{\delta}_{l2}((\{q_{02}\},\mu^t(\{q_{02}\})),a,Q_j) & \text{if } \{q_{02}\} \in Q_i, Q_i - \{q_{02}\} \in \bar{Q}_1, Q_j \in \bar{Q}_2 \\ \tilde{\delta}_{l1}((Q_i - \{q_{02}\},\mu^t(Q_i - \{q_{02}\})),a,Q_j) & \text{if } \{q_{02}\} \in Q_i, Q_i - \{q_{02}\} \in \bar{Q}_1, Q_j \in \bar{Q}_1 \end{cases}.$$

Also, we define $\omega'_l(Q') = \omega_{l2}(Q'')$, where $Q' = Q'' \cup Q'''$, $Q' \in \bar{Q}_1 \cup \bar{Q}_2, Q'' \in \bar{Q}_2$ and $Q''' \in \bar{Q}_1$.

Let Δ_1 and Δ_2 be the set of all transitions of BL-GLFAs $\tilde{F}_{l1}$ and $\tilde{F}_{l2}$, respectively. Then we have the set of all transitions of BL-GLFA $\tilde{F}_{l1} * \tilde{F}_{l2}$ as follow: $\Delta = \Delta_1 \cup \Delta_2 \cup \{\delta_{l1}(q_i, a, q_{02}) \big| q_i \in Q_i, 1 \in \omega_{l1}(Q_i)\}$.

Also, the language recognized by $\tilde{F}_{l1} * \tilde{F}_{l2}$ is a subset of X^* defined by:

$$\mathcal{L}(\tilde{F}_{l1} * \tilde{F}_{l2}) = \{x \in X^* \big| 1 \in \beta_{\tilde{F}_{l1} * \tilde{F}_{l2}}(x)\} = \{x = uv \big| 1 \in \beta_{\tilde{F}_1}(u), 1 \in \beta_{\tilde{F}_2}(v)\}.$$

Theorem 3.20. *Let $\tilde{F}_{li} = (\bar{Q}_i, X, \tilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \{0,1\}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1,2$ be two BL-GLFAs, where $Q_1 \cap Q_2 = \emptyset$. Then $\mathcal{L}(\tilde{F}_{l1} * \tilde{F}_{l2}) = \mathcal{L}(\tilde{F}_{l1}).\mathcal{L}(\tilde{F}_{l2})$.*

Proof. Let ρ and ρ' be the run relations of $\tilde{F}_{l1}$ and $\tilde{F}_{l2}$, respectively. The run relation $\bar{\rho}$ of $\tilde{F}_{l1} * \tilde{F}_{l2}$ is defined as follow:

$$\bar{\rho}(a_1...a_n) = \rho_1(a_1...a_n) \cup \bigcup_{1 \in \beta_{\tilde{F}_{l1}}(a_1...a_i)} \rho_2(a_{i+1}...a_n).$$

Now, we have

$$\begin{aligned} a_1a_2...a_n \in \mathcal{L}(\tilde{F}_{l1}.\tilde{F}_{l2}) &\Longleftrightarrow 1 \in \omega'_l \circ \bar{\rho}(a_1a_2...a_n) \\ &\Longleftrightarrow 1 \in \omega_{l2}(Q''), \end{aligned}$$

where $\bar{\rho}(a_1a_2...a_n) = Q' \cup Q'', Q' \in \bar{Q}_1, Q'' \in \bar{Q}_2$. The run in $\tilde{F}_{l1} * \tilde{F}_{l2}$ from $\{q_{01}\}$ to $\omega_{l2}(Q'')$ can enter 1 if and only if $1 \in \beta(u_1)$ for some left factor u_1 of x. Therefore $1 \in \omega_{l2}(Q'')$, where $\bar{\rho}(a_1a_2...a_n) = Q' \cup Q'', Q' \in \bar{Q}_1, Q'' \in \bar{Q}_1$, if and only if there are $a_1a_2...a_i \in \mathcal{L}(\tilde{F}_{l1})$ and $a_{i+1}a_{i+2}...a_n \in \mathcal{L}(\tilde{F}_{l2})$. Thus $a_1a_2...a_n \in \mathcal{L}(\tilde{F}_{l1} * \tilde{F}_{l2})$. □

Example 3.21. *Let $\tilde{F}_{li}, i = 1,2$ be as defined in Example 3.10, where $\omega_{li}, i = 1,2$ is changed as follows: $\omega_{l1}(\{p_0\}) = \{0\}, \omega_{l1}(\{p_1\}) = \{1\}, \omega_{l1}(\{p_0, p_1\}) = \{0,1\}$ and $\omega_{l2}(\{q_0\}) = \omega_{l2}(\{q_1\}) = \omega_{l2}(\{q_0, q_1\}) = \{1\}$. Then we have $\tilde{F}_{l1}.\tilde{F}_{l2}$ as in Figure 5, where*

$$\begin{aligned} &P(Q_1 \cup \{q_0\}) \cup Q_2 \\ &= \{\{q_0\}, \{q_1\}, \{q_0, q_1\}, \{p_0\}, \{p_1\}, \{p_0, p_1\}, \{q_0, p_0\}, \{q_0, p_1\}, \{q_0, p_0, p_1\}\}. \end{aligned}$$

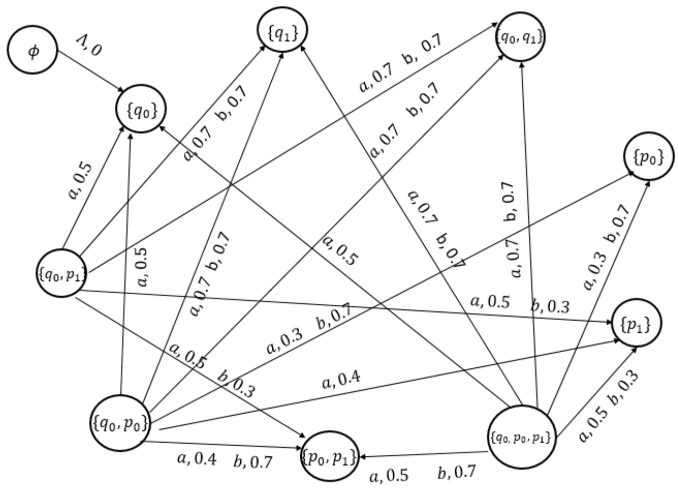

FIGURE 5. The BL general fuzzy automaton $\tilde{F}_{l1} * \tilde{F}_{l2}$ of Example 3.21

$$\begin{aligned}
&\bar{f}_l(\{p_0\}, a) = \{p_0, p_1\},\\
&\bar{f}_l(\{p_0\}, b) = \{p_0\},\\
&\bar{f}_l(\{p_1\}, a) = \{p_1, q_0\},\\
&\bar{f}_l(\{p_1\}, b) = \{p_1, q_0\},\\
&\bar{f}_l(\{p_0, p_1\}, a) = \{p_0, p_1, q_0\},\\
&\bar{f}_l(\{p_0, p_1\}, b) = \{p_0, p_1, q_0\},\\
&\bar{f}_l(\{q_0\}, a) = \{q_0, q_1\},\\
&\bar{f}_l(\{q_0\}, b) = \{q_1\},\\
&\bar{f}_l(\{q_1\}, a) = \{q_0\},\\
&\bar{f}_l(\{q_1\}, b) = \{q_1\},\\
&\bar{f}_l(\{q_0, q_1\}, a) = \{q_0, q_1\},\\
&\bar{f}_l(\{q_0, q_1\}, b) = \{q_1\}.
\end{aligned}$$

Also,

$$\bar{\delta}_l(Q_i, a, Q_j) = \begin{cases} \delta_{l1}(Q_i, a, Q_j) & if\ Q_i, Q_j \in \bar{Q}_1 \\ \delta_{l2}(Q_i, a, Q_j) & if\ Q_i, Q_j \in \bar{Q}_2 \end{cases},$$

$\bar{\delta}_l(\{q_0, p_0\}, a, \{q_0\}) = \bar{\delta}_l(\{q_0, p_1\}, a, \{q_0\}) = \bar{\delta}_l(\{q_0, p_0, p_1\}, a, \{q_0\}) = 0.5,$
$\bar{\delta}_l(\{q_0, p_0\}, a, \{q_1\}) = \bar{\delta}_l(\{q_0, p_1\}, a, \{q_1\}) = \bar{\delta}_l(\{q_0, p_0, p_1\}, a, \{q_1\}) = 0.7,$
$\bar{\delta}_l(\{q_0, p_0\}, a, \{q_0, q_1\}) = \bar{\delta}_l(\{q_0, p_1\}, a, \{q_0, q_1\}) = \bar{\delta}_l(\{q_0, p_0, p_1\}, a, \{q_0, q_1\}) = 0.7,$
$\bar{\delta}_l(\{q_0, p_0\}, b, \{q_1\}) = \bar{\delta}_l(\{q_0, p_1\}, b, \{q_1\}) = \bar{\delta}_l(\{q_0, p_0, p_1\}, b, \{q_1\}) = 0.3,$
$\bar{\delta}_l(\{q_0, p_0\}, b, \{q_0, q_1\}) = \bar{\delta}_l(\{q_0, p_1\}, b, \{q_0, q_1\}) = \bar{\delta}_l(\{q_0, p_0, p_1\}, b, \{q_0, q_1\}) = 0.3,$

$\bar{\delta}_l(\{q_0, p_0\}, a, \{p_0\}) = 0.3,$
$\bar{\delta}_l(\{q_0, p_0\}, a, \{p_1\}) = 0.4,$
$\bar{\delta}_l(\{q_0, p_0\}, a, \{p_0, p_1\}) = 0.4,$
$\bar{\delta}_l(\{q_0, p_1\}, a, \{p_1\}) = 0.5,$
$\bar{\delta}_l(\{q_0, p_1\}, a, \{p_0, p_1\}) = 0.5,$
$\bar{\delta}_l(\{q_0, p_0, p_1\}, a, \{p_0\}) = 0.3,$
$\bar{\delta}_l(\{q_0, p_0, p_1\}, a, \{p_1\}) = 0.5,$
$\bar{\delta}_l(\{q_0, p_0, p_1\}, a, \{p_0, p_1\}) = 0.5,$
$\bar{\delta}_l(\{q_0, p_0\}, b, \{p_0\}) = 0.7,$
$\bar{\delta}_l(\{q_0, p_0\}, b, \{p_0, p_1\}) = 0.7,$
$\bar{\delta}_l(\{q_0, p_1\}, b, \{p_1\}) = 0.3,$
$\bar{\delta}_l(\{q_0, p_1\}, b, \{p_0, p_1\}) = 0.3,$
$\bar{\delta}_l(\{q_0, p_0, p_1\}, b, \{p_0\}) = 0.7,$
$\bar{\delta}_l(\{q_0, p_0, p_1\}, b, \{p_1\}) = 0.3,$
$\bar{\delta}_l(\{q_0, p_0, p_1\}, b, \{p_0, p_1\}) = 0.7,$

and $\omega'_l(\{q_0\}) = \omega'_l(\{q_0, p_0\}) = \omega'_l(\{q_0, p_1\}) = \omega'_l(\{q_0, p_0, p_1\}) = \omega'_l(\{q_1\}) = \omega'_l(\{q_0, q_1\}) = 1.$

Definition 3.22. (Serial Connection) Let

$$\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \{0, 1\}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2),$$

be a BL-GLFA. Define BL-GLFA $\tilde{F}_l^+$ as follows:

$$\tilde{F}_l^+ = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l^+, f_l^+, \tilde{\delta}_l^+, F_1, F_2),$$

where

$$\bar{f}_l^+(Q_i, a) = \begin{cases} f_l(Q_i, a) \cup \{q_0\} & if\ 1 \in \omega_l(f_l(Q_i, a)) \\ f_l(Q_i, a) & if\ \omega_l(f_l(Q_i, a)) = \{0\} \end{cases},$$

$\delta_l^+(Q_i, a, Q_j) = \delta_l(Q_i, a, Q_j)$ and $\tilde{\delta}_l^+((Q_i, \mu^t(Q_i)), a, Q_j) = F_1(\mu^t(Q_i), \tilde{\delta}_l^+(Q_i, a, Q_j))$. Let Δ_1 be the set of all transitions of BL-GLFAs $\tilde{F}_l$. Then the set of all transitions of BL-GLFA $\tilde{F}_l^+$ is equal to $\Delta = \Delta_1 \cup \{\delta_{l1}(q_i, a, q_0) \big| q_i \in Q_i, 1 \in \omega_l(Q_i)\}$.

Theorem 3.23. *Let $\tilde{F}_l^+$ be the above BL-GLFA. Then $\mathcal{L}(\tilde{F}_l^+) = \mathcal{L}^+(\tilde{F}_l)$.*

Proof. Let ρ be the run relation of $\tilde{F}_l$. Then the run relation ρ^+ of $\tilde{F}_l^+$ is defined as follow:

$$\rho_l^+(\Lambda) = \{q_0\},\ \rho_l^+(a_1...a_n) = \bigcup_{i=J_1}^{J_k} \rho_l(a_i...a_n),$$

where $J_1 = 1, J_k \leq n, (K = 0, 1, 2, ...)$ and $1 \in \beta(a_{J_l}...a_{J_{l+1}})$, for every $a_1...a_n \in X^*$ and $1 \leq l \leq k$. Now, let $a_1...a_n \in \mathcal{L}(\tilde{F}_l^+)$. Since $1 \in \omega_l(\rho_l^+(a_1...a_n))$. Then $a_1...a_n \in \mathcal{L}^+(\tilde{F}_l)$.

Now, let $a_1...a_n \in \mathcal{L}^+(\tilde{F}_l)$. Then there exist $i_1, i_2, ..., i_k, (k = n)$, such that

$$1 \in \beta(a_1...a_{i_1}),\ 1 \in \beta(a_{i_1+1}...a_{i_2}),\ ...,\ 1 \in \beta(a_{i_{k-1}+1}...a_{i_k}).$$

Therefore $\rho_l^+(a_1...a_n) = \bigcup_{i_j=0}^{k-1} \rho_l(a_{i_j+1}...a_n)$. So, $1 \in \beta_l^+(a_1...a_n)$ and $\mathcal{L}(\tilde{F}_l^+)$. Hence, the claim holds. □

Definition 3.24. Given a language $\mathcal{L} \subseteq X^*$, the language $\mathcal{L}^* = \emptyset \cup \mathcal{L}^+$ is called its iteration.

Definition 3.25. Let $\mathcal{L}$ be a subset of X^*. Then $\mathcal{L}$ is called rational if it can be obtained from finite subsets of X^* by finitely many applications of union, connection and iteration.

Theorem 3.26. *(Kleen's Theorem) Let $\mathcal{L}$ be a subset of X^*. Then $\mathcal{L}$ is a recognizable set if and only if $\mathcal{L}$ is rational.*

Proof. Let $\mathcal{L}$ be a rational set. Then it can be obtained from finite subsets of X^* by finitely many applications of union, connection and iteration. So, by considering Theorems 3.8, 3.20, and 3.23, $\mathcal{L}$ is a recognizable set. The converse is not difficult. □

4. Conclusion

In 2012, Kh. Abolpour and M. M. Zahedi [1] extended the notion of general fuzzy automata and gave the notion of BL-general fuzzy automata. An interesting problem in this context is the realization problem, which says that given a behavior, we can design a machine that realizes it. This study presents the results about closer operators for the class of BL-general fuzzy automata based on lattice valued. Also, the authors give the Pumping Lemma for the BL-general L-fuzzy automaton. Here we prove that union, intersection and complement are third closed for this class of languages. Moreover, we show that the Pumping lemma is established for BL-general L-fuzzy automata. Also, it is proven that L is a recognizable set if and only if L is rational. This paper shows that there is a connection between the behavior of a finite realization and a recognizable language of a BL-general L-fuzzy automaton. Finally, it is concluded that Kleen's Theorem is valid for the BL-general L-fuzzy automaton. These results extended the previous results presented in [22, 25].

References

[1] K. Abolpour, M.M. Zahedi, Isomorphism between two BL-general fuzzy automata, Soft Computing, 16 (2012), 729-736.

[2] M. Doostfatemeh, S.C. Kremer, New directions in fuzzy automata, International Journal of Approximate Reasoning, 38 (2005), 175-214.

[3] C.L. Giles, C.W. Omlin, K.K. Thornber, Equivalence in knowledge representation: automata, recurrent neural networks, and dynamical fuzzy systems, Proceedings of IEEE, 87 (1999), 1623-1640.
[4] M.M. Gupta, G.N. Saridis, B.R. Gaines, Fuzzy Automata and Decision Processes, North Holland, New York, 1977, 111-175.
[5] P. Hájek, Metamathematics of fuzzy logic, Kluwer, Dordrecht, (1988).
[6] J. E. Hopcroft, R. Motwani, J. D. Ullman, Introduction to Automata Theory, Languages and Computation, seconded., Addison-Wesley, Reading, MA, 2001.
[7] E. T. Lee, L. A. Zadeh, Note on fuzzy languages, Information Sciences 1 (1969) 421-434.
[8] Y. M. Li, W. Pedryez, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoid, Fuzzy Sets and Systems, 156 (2005), 68-92.
[9] D. S. Malik, J. N. Mordeson, Fuzzy Automata and Languages: Theory and Applications, Chapman Hall, CRC Boca Raton, London, New York, Washington DC, 2002.
[10] D. S. Malik, J. N. Mordeson, Fuzzy Discrete Structures, Physica-Verlag, New York, 2000.
[11] C. W. Omlin, K. K. Thornber, C. L. Giles, Fuzzy finite-state automata can be deterministically encoded in recurrent neural networks, IEEE Transactions on Fuzzy Systems, 5 (1998), 76-89.
[12] W. Pedrycz, A. Gacek, Learning of fuzzy automata, International Journal of Computational Intelligence and Applications, 1 (2001), 19-33.
[13] K. Peeva, Equivalence, reduction and minimization of finite automata over semirings, Theoretical Computer Science 88 (1991) 269-285.
[14] D. Qiu, Automata theory based on complete residuated lattice-valued logic, Science in China Series: Information Sciences, 44 (2001), 419-429.
[15] D. Qiu, Automata theory based on complete residuated lattice-valued logic (II), Science in China Series F: Information Sciences, 45 (2002), 442-452.
[16] D. Qiu, Characterizations of fuzzy finite automata, Fuzzy Sets and Systems, 141 (2004), 391-414.
[17] D. Qiu, Pumping lemma in automata theory based on complete residuated lattice-valued logic: A not, Fuzzy Sets and Systems, 157 (2006), 2128-2138.
[18] D. Qiu, Supervisory control of fuzzy discrete event systems: a formal approach, IEEE Transactions on Systems, Man and CyberneticsPart B, 35 (2005), 72-88.
[19] A. Salomaa, Theory of Automata, Pergamon Press, Oxford, 1969.
[20] E. S. Santos, Maximin automata, Information and Control 13 (1968) 363-377.
[21] M. Shamsizadeh, M. M. Zahedi. A note on Quotient structures of intuitionistic fuzzy finite state machines, Journal of Applied Mathematics and Computing, 51(2016), 413-423.
[22] M. Shamsizadeh, M. M. Zahedi, Bisimulation of type 2 for BL-general fuzzy automata, Soft Computing, 23 (2019), 9843-9852.
[23] M. Shamsizadeh, M. M. Zahedi, Intuitionistic general fuzzy automata, Soft Computing, 20 (2015), 3505-3519.
[24] M. Shamsizadeh, M. M. Zahedi, Minimal intutionistic general L-fuzzy automata, Italian Journal of Pure and Applied Mathematics, 35 (2015), 155-186.
[25] M. Shamsizadeh, M. M. Zahedi, Kh. Abolpour, Bisimulation for BL-general fuzzy automata, Iranian Journal of Fuzzy Systems, 13 (2016), 35-50.
[26] V. Topencharov, K. Peeva, Equivalence, reduction and minimization of finite fuzzy automata, Journal of Mathematical Analysis and Applications 84 (1981) 270-281.
[27] W. Wechler, The Concept of Fuzziness in Automata and Language Theory, Akademie-Verlag, Berlin, 1978.
[28] W. G. Wee, On generalizations of a daptive algorithm and application of the fuzzy sets concept to pattern classification, Ph.D. Thesis, Purdue University, June, 1967.

[29] L. Wu and D. Qiu, Automata theory based on complete residuated lattice-valued logic: Reduction and minimization, Fuzzy Sets and Systems, 161 (2010), 1635-1656.
[30] H. Xing and D. W. Qiu, Automata theory based on complete residuated lattice-valued logic: A categorical approach, Fuzzy Sets and Systems, 160 (2009), 2416-2428.
[31] H. Xing and D. W. Qiu, Pumping lemma in context-free grammar theory based on complete residuated lattice-valued logic, Fuzzy Sets and Systems, 160 (2009), 1141 - 1151.
[32] H. Xing, D. W. Qiu and F. C. Liu, Automata theory based on complete residuated lattice-valued logic: Pushdown automata, Fuzzy Sets and Systems, 160 (2009), 1125-1140.
[33] H. Xing, D. W. Qiu, F. C. Liu and Z. J. Fan, Equivalence in automata theory based on complete residuated lattice-valued logic, Fuzzy Sets and Systems, 158 (2007), 1407-1422.
[34] M. S. Ying, A formal model of computing with words, IEEE Transactions on Fuzzy Systems, 10 (2002), 640-652.
[35] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965), 338-353.

Marzieh Shamsizadeh
Orcid number: 0000-0002-9336-289X
Department of Mathematics
Behbahan Khatam Alanbia University of Technology
Khouzestan, Iran
E-mail address: shamsizadeh.m@gmail.com

Mohammad Mehdi Zahedi
Orcid number: 0000-0003-3197-9904
Department of Mathematics
Graduate University of Advanced Technology
Kerman, Iran
E-mail address: zahedi_mm@kgut.ac.ir

Khadijeh Abolpour
Orcid number: 0000-0001-5280-2513
Department of Mathematics
Shiraz Branch, Islamic Azad University
Shiraz, Iran
E-mail address: abolpor_kh@yahoo.com