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SOME CONNECTIONS BETWEEN VARIOUS SUBCLASSES OF UNIVALENT FUNCTIONS INVOLVING PASCAL DISTRIBUTION SERIES

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Abstract. The main object of this paper is to define a new class of univalent functions and two subclasses of this class along with the Pascal distribution associated with convolution and subordination structures. We obtained a number of useful properties such as, coefficient bound, convolution preserving and some other geometric properties.

Keywords: Univalent function, Pascal distribution, Subordination.
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1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the type

$$f(z) = z + \sum_{k=2}^{+\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}. \tag{2}$$

Also $\mathcal{N}$ be the main subclass of $\mathcal{A}$ consisting the functions of the type

$$f(z) = z - \sum_{k=2}^{+\infty} a_k z^k, \quad (a_k \geq 0, z \in \mathbb{U}). \tag{3}$$

See [2]. For parameters r, p and $k \in \{0, 1, 2, \ldots\}$ we consider a non-negative discrete random variable X with a Pascal probability generating function

$$P(x = k) = \binom{k+r-1}{r-1} p^k (1-p)^r. \tag{4}$$

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Now we consider a power series whose coefficients are probabilities of the Pascal distribution as follows:

$$(5)\quad \mathcal{P}_p^r(z) = z + \sum_{k=2}^{+\infty} \binom{k+r-2}{r-1} p^{k-1} (1-p)^r z^k, \ (r \geq 1, 0 \leq p \leq 1, z \in \mathbb{C}).$$

It is easy to see that by using ration test, the radius of convergence of the power series given in (5) is infinity. For more details see [1, 4–6].

For f given by (1) and $g(z) = z + \sum_{k=2}^{+\infty} b_z z^k$, the Hadamard product (or convolution) of f and g denote by $f * g$ is defined

$$(6)\qquad (f * g) = z + \sum_{k=2}^{+\infty} a_k b_k z^k = (g * f)(z).$$

Now we consider the function

$$(7)\qquad \mathcal{P}_f(z) = \left[\left(2z - \mathcal{P}_p^r(z)\right) * f\right](z),$$

where $\mathcal{P}_p^r(z)$ and f given by (5) and (3) respectively.

A function $f \in \mathcal{N}$ is a member of $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$ if

$$(8)\qquad \left| \frac{z\left(\mathcal{P}_f(z)\right)^n}{z\lambda\left(\mathcal{P}_f(z)\right)' - \alpha(1+\beta)\lambda} \right| < \gamma,$$

where $\alpha, \beta, \gamma \in [0, 1)$ and $\mathcal{P}_f(z)$ is given by (7).

2. Main Results

In this section we obtain a sharp coefficient bound for functions in the class $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$. Also convolution preserving properties are investigated.

Theorem 2.1. *Let $f \in \mathbb{N}$, then $f \in Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$ if and only if*

$$(9)\sum_{k=2}^{+\infty} k(k-1+2\lambda\gamma) \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k \leq \lambda\gamma\left(2 - \alpha(1+\beta)\right).$$

The result is sharp for the function

$$(10)\qquad F(z) = z - \frac{\lambda\gamma\left(2 - \alpha(1+\beta)\right)}{2r\left(1+2\lambda\gamma\right) p(1-p)^r} z^2.$$

Proof. Let the inequality (9) holds true and suppose $z \in \partial\mathbb{U} = \{z \in \mathbb{C} : \ |z| = 1\}$. Then we obtain

$$\begin{aligned}
&\left|z\left(\mathcal{P}_f(z)\right)'' - \gamma\right| - \gamma\left|2\lambda\left(\mathcal{P}_f(z)\right)' - \alpha(1+\beta)\lambda\right| \\
&= \left|-\sum_{k=2}^{+\infty} k(k-1)\binom{k+r-\lambda}{r-1} p^{k-1}(1-p)^r a_k z^{k-1}\right| \\
&-\gamma\left|2\lambda - 2\lambda\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k z^{k-1} - \alpha(1+\beta)\lambda\right| \\
= \ &\sum_{k=2}^{+\infty} k\left(k-1+2\lambda\gamma\right)\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k - \lambda\gamma\left(2-\alpha(1+\beta)\right) \leq 0.
\end{aligned}$$

Hence by maximum modulus theorem, we conclude that $f \in Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$.

conversely, let f be in the class $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta, \gamma)$, so the condition (8) yields

$$\begin{aligned}
&\left|\frac{z\left(\mathcal{P}_f(z)\right)''}{z\lambda\left(\mathcal{P}_f(z)\right)' - \alpha(1+\beta)\lambda}\right| \\
&= \left|\frac{\sum_{k=2}^{+\infty} k(k-1)\binom{k+r-2}{r-1} p^{k-r}(1-p)^r a_k z^{k-1}}{\lambda\left(2-\alpha(1+\beta)\right) - 2\lambda\sum_{k=2}^{+\infty} k\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k z^{k-1}}\right| < \gamma.
\end{aligned}$$

Since for any z, $|\text{Re } z| < |z|$, then

$$\text{Re}\left\{\frac{\sum_{k=2}^{+\infty} k(k-1)\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k z^{k-1}}{\lambda\left(2-\alpha(1+\beta)\right) - 2\lambda\sum_{k=2}^{+\infty} k\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k z^{k-1}}\right\} < \gamma,$$

by letting $z \to 1$ through real values, we have

$$\begin{aligned}
&\sum_{k=2}^{+\infty} k(k-1)\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k \\
&\leq \lambda\gamma\left(2-\alpha(1+\beta)\right) - 2\lambda\gamma\sum_{k=2}^{+\infty} k\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k,
\end{aligned}$$

and this completes the proof. □

Theorem 2.2. *Let* $f(z) = z - \sum_{k=2}^{+\infty} a_k z^k$ *and* $g(z) = z - \sum_{k=2}^{+\infty} b_k z^k$ *belong to* $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta\, \gamma)$. *Then*

(i) $(f * g)(z)$ *belong to* $Y_{\mathcal{P}}^{\lambda}(\alpha, \beta_0, \gamma)$, *where*

$$\beta_0 \leq \frac{2}{a} - \left(1 + \frac{\lambda\gamma\left(2-\alpha(1+\beta)\right)^2}{\alpha k\left(k-1+2\lambda\gamma\right)\binom{k+r-2}{r-1} p^{k-1}(1-p)^r}\right). \tag{11}$$

(ii) $(f*g)(z)$ *belong to* $Y_{\mathcal{P}}^{\lambda}(\alpha_0,\beta,\gamma)$, *where*

$$\alpha_0 \leq \frac{2}{1+\beta} - \frac{\lambda\gamma\left(2-\alpha(1+\beta)\right)^2}{(1+\beta)k\left(k-1+2\lambda\gamma\right)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r}. \tag{12}$$

Proof. (i). It is sufficient to show that

$$\frac{\sum_{k=2}^{+\infty} k(k-1+2\lambda\gamma)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r}{\lambda\gamma\left(2-\alpha(1+\beta)\right)} a_k b_k \leq 1.$$

By using Cauchy-Schwartz inequality from (9), we obtain

$$\sum_{k=2}^{+\infty} \frac{k(k-1+2\lambda\gamma)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r}{\lambda\gamma\left(2-\alpha(1+\beta)\right)} \sqrt{a_k b_k} \leq 1.$$

Hence we find the largest β_0 such that

$$\begin{aligned}
&\sum_{k=2}^{+\infty} \frac{k(k-1+2\lambda\gamma)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r}{\lambda\gamma\left(2-\alpha(1+\beta)\right)} a_k b_k \\
&\leq \sum_{k=2}^{+\infty} \frac{k(k-1+2\lambda\gamma)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r}{\lambda\gamma\left(2-\alpha(1+\beta)\right)} \sqrt{a_k b_k} \leq 1,
\end{aligned}$$

or equivalently

$$\sqrt{a_k b_k} \leq \frac{2-\alpha(1+\beta_0)}{2-\alpha(1+\beta)}.$$

This inequality holds if

$$\frac{\lambda\gamma\left(2-\alpha(1+\beta)\right)}{k(k-1+2\lambda\gamma)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r} \leq \frac{2-\alpha(1+\beta_0)}{2-\alpha(1+\beta)},$$

or equivalently

$$\beta_0 \leq \frac{2}{a} - \left(1 + \frac{\lambda\gamma\left(2-\alpha(1+\beta)\right)^2}{\alpha k\left(k-1+2\lambda\gamma\right)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r}\right).$$

(ii) With a same calculation of (i), we obtain the result, hence the details are omitted. □

3. Geometric properties of subclasses of $Y_{\mathcal{P}}^{\lambda}(\alpha,\beta,\gamma)$

In this section we introduce two subclasses of $Y_{\mathcal{P}}^{\lambda}(\alpha,\beta,\gamma)$ and conclude some geometric properties.

Let f and g be analytic in $\mathbb{U}$. Then f is said to be subordinate to g written $f \prec g$ or $f(z) \prec g(z)$ if there exists a function ω analytic in $\mathbb{U}$, with $\omega(0)=0$ and $|\omega(z)|<1$, such that $f(z)=g\left(\omega(z)\right)$ (see [3]).

If g is univalent, then $f \prec g$ if and only if $f(0)=g(0)$ and $f(U)\subset g(U)$.

Let $U(m,n,s)$ consists of all analytic functions $g(z)$ in $\mathbb{U}$ for which $g(0)=1$ and

$$g(z) \prec \frac{1+(n+(m-n)(1-s))z}{1+nz}, \tag{13}$$

where $-1 \le m < n \le 1$, $0 < n \le 1$ and $0 \le s < 1$.

Let $V(m,n,s)$ denote the class of all functions $f(z) \in Y_{\mathcal{P}}^{\lambda}(\alpha,\beta,\gamma)$ for which

$$\frac{z\left(\mathcal{P}_f(z)\right)'}{\mathcal{P}_f(z)} \in U(m,n,s). \tag{14}$$

Theorem 3.1. *$f(z) \in V(m,n,s)$ if and only if*

$$\sum_{k=2}^{+\infty}\left[1+\frac{(n+1)(k-1)}{(n-m)(1-s)}\right]\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k < 1. \tag{15}$$

Proof. Let $f \in V(m,n,s)$. Then by (8), (13) and (14) we have

$$\left|\frac{\mathcal{P}_f(z) - z\left(\mathcal{P}_f(z)\right)'}{nz\left(\mathcal{P}_f(z)\right)' - (n+(m-n)(1-s))\left(\mathcal{P}_f(z)\right)}\right| < 1,$$

which implies that

$$\left|\frac{\sum\limits_{k=2}^{+\infty}(k-1)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k z^{k-1}}{(n-m)(1-s) - \sum\limits_{k=2}^{+\infty}\left[n(k-1)+(n-m)(1-s)\right]\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k z^{k-1}}\right| < 1.$$

Since Re $z < |z|$ for all $z \in \mathbb{U}$, so we conclude that

$$\operatorname{Re}\left\{\frac{\sum\limits_{k=2}^{+\infty}(k-1)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k z^{k-1}}{(n-m)(1-s) - \sum\limits_{k=2}^{+\infty}\left[n(k-1)+(n-m)(1-s)\right]\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k z^{k-1}}\right\} < 1.$$

Choose the values of z on the real axis and letting $z \to 1^-$, we obtain

$$\frac{\sum\limits_{k=2}^{+\infty}(k-1)\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k}{(n-m)(1-s) - \sum\limits_{k=2}^{+\infty}\left[n(k-1)+(n-m)(1-s)\right]\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k} < 1,$$

after a simple calculation, we obtain the result.

Conversely, assume that the condition (15) holds true. We must show that $f \in V(m,n,s)$, or equivalently

$$\begin{aligned}
\mathcal{L} &= \left| \frac{\mathcal{P}_f(z) - z\left(\mathcal{P}_f(z)\right)'}{nz\left(\mathcal{P}_f(z)\right)' - (n+(n-m)(1-s))\mathcal{P}_f(z)} \right| < 1 \\
&= \left| \frac{\sum\limits_{k=2}^{+\infty} (k-1)\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k z^{k-1}}{(n-m)(1-s) - \sum\limits_{k=2}^{+\infty} \left[n(k-1)+(n-m)(1-s)\right]\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k z^{k-1}} \right| \\
&< \frac{\sum\limits_{k=2}^{+\infty} (k-1)\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k}{(n-m)(1-s) - \sum\limits_{k=2}^{+\infty} \left[n(k-1)+(n-m)(1-s)\right]\binom{k+r-2}{r-1} p^{k-1}(1-p)^r a_k}.
\end{aligned}$$

But by applying (15), we conclude that $\mathcal{L} < 1$, so the proof is complete. □

Theorem 3.2. *Let $n \neq 1$, $f \in V(m,n,s)$ and $W = x + iy = \dfrac{z\left(\mathcal{P}_f(z)\right)'}{\mathcal{P}_f(z)}$. Then the values of W are in the circle*

Proof. By (13) and (14), we have

$$W = x + iy = \frac{1 + (n + (m-n)(1-s))J(z)}{1 + nJ(z)}, \ |J(z)| < 1.$$

Then

$$(x+iy)(1+nJ(z)) = 1 + (n + (m-n)(1-s))J(Z),$$

or

$$(x-1)^2 + y^2 < (n + (m-n)(1-s) - xn)^2 + y^2 n^2.$$

After a simple calculation, we obtain

$$\left(x - \frac{1 - \left(n^2 + n(m-n)(1-s)\right)}{1-n^2}\right)^2 + y^2 < \left(\frac{(n-m)(1-s)}{1-n^2}\right)^2.$$

Hence the values of W lie in the circle with center at $\left(\dfrac{1 - (n^2 + n(m-n)(1-s)}{1-n^2}, 0\right)$ and radius $\dfrac{(n-m)(1-s)}{1-n^2}$. □

Theorem 3.3. *Let $f \in V(m,n,s)$, then*

$$\mathcal{P}_f(z) = \exp\left(\int_0^z \frac{1 - (n + (m-n)(1-s)H(t)}{t(1-nH(t))} dt\right),$$

where $|H(z)| < 1$.

Proof. Set $W = \dfrac{z\left(\mathcal{P}_f(z)\right)'}{\mathcal{P}_f(z)}$, since $f \in V(m,n,s)$, so

$$\left|\frac{W-1}{Wn-(n+(m-n)(1-s))}\right| < 1,$$

therefore

$$\frac{W-1}{Wn-(n+(m-n)(1-s))} = H(z), \qquad |H(z)| < 1.$$

Hence, we can write

$$\frac{\left(\mathcal{P}_f(z)\right)'}{\mathcal{P}_f(z)} = \frac{1-(n+(m-n)(1-s))H(z)}{z(1-nH(z)}.$$

After integration, we conclude the required result. □

Theorem 3.4. *Let* $0 \le s_2 \le s_1 < 1$*, then* $V(m,n,s_1) \subset V(m,n,s_2)$.

Proof. Suppose that $f \in V(m,n,s_1)$. Then by (1) we have

$$\sum_{k=2}^{+\infty}\left[1+\frac{(n+1)(k-1)}{(n-m)(1-s_1)}\right]\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k < 1.$$

We have to prove

$$\sum_{k=2}^{+\infty}\left[1+\frac{(n+1)(k-1)}{(n-m)(1-s_2)}\right]\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k < 1.$$

But the last inequality holds true if

$$1+\frac{(n+1)(k-1)}{(n-m)(1-s_2)} \le 1+\frac{(n+1)(k-1)}{(n-m)(1-s_1)},$$

and this inequality by $0 \le s_2 < s_1 < 1$ definitely holds true. □

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