APPLICATION OF HAAR WAVELETS IN SOLVING NONLINEAR FRACTIONAL FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. *A novel and effective method based on Haar wavelets and Block Pulse Functions (BPFs) is proposed to solve nonlinear Fredholm integro-differential equations of fractional order. The operational matrix of Haar wavelets via BPFs is derived and together with Haar wavelet operational matrix of fractional integration are used to transform the mentioned equation to a system of algebraic equations. Our new method is based on this matrix and the vector forms for representation of Haar wavelets. In addition, an error and convergence analysis of the Haar-approximation is discussed. Since this approach does not need any integration, all calculations would be easily implemented, and it has several advantages in reducing the computational burden. Some examples are included to demonstrate the validity and applicability of the technique.*

Keywords: Fredholm integro-differential equations; Haar wavelets; Operational matrix; Fractional calculus; Block Pulse Functions.

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1. INTRODUCTION

The aim of this work is to present an operational method (operational Haar wavelet method) for approximating the solution of a nonlinear fractional integro- differential equation of second kind:

$$(1) \qquad D_*^\alpha f(x) - \lambda \int_0^1 k(x,t)F(f(t))dt = g(x),$$

with these supplementary conditions:

$$(2) \qquad f^{(i)}(0) = \delta_i, \ i = 0, 1, ..., s-1, \ \ni \ s-1 < \alpha \le s, \ s \in \mathbb{N},$$

where $g \in L^2([0,1))$, $k \in L^2([0,1)^2)$ are known functions, $f(x)$ is the unknown function, D_*^α is the Caputo fractional differentiation operator of order α and $F(f(x))$ is a polynomial of $f(x)$ with constant coefficients. With out lost of generality, it can be assumed that $F(f(x)) = [f(x)]^q$ such that $q \in \mathbb{N}$. Such kind of equations arise in the mathematical modeling of various physical phenomena, such as heat conduction in materials, moreover, these equations are encountered in combined conduction, convection and radiation problems [7, 14, 19]. Local and global existence and uniqueness of the solution of the integro-differential equations given by (1) and (2), are given in [1, 15]. In recent years, fractional integro- differential equations have been investigated by many authors [2, 4, 5, 6, 16, 17, 22, 27]. Most of the methods have been utilized in linear problems and a few number of works have considered nonlinear problems [24, 25].

Our technique is an operational method, which is based on reducing the main equation to a system of algebraic equations by expanding the solution of equations (1) and (2) as Haar wavelets with unknown coefficients. The main characteristic of the operational method is to convert a differential equation into an algebraic one. It not only simplifies the problem, but also speeds up the computations. It should be also mentioned that the interest to the wavelet treatment of various integral equations has recently increased due to promising applications of this method in computational chemistry [9, 10, 11].

It is considerable that, for $\alpha \in \mathbb{N}$, Eqs. (1) and (2) are ordinary Fredholm integro- differential equations and one can easily apply the method for them.

1.1. **Fractional Calculus.** Riemann- Liouville fractional integration of order α is defined as:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \ x > 0, \ J^0 f(x) = f(x),$$

and Caputo fractional derivatives of order α is defined as $D_*^\alpha f(x) = I^{m-\alpha} D^m f(x)$, where D^m is the usual integer differential operator of order m and $I^{m-\alpha}$ is Riemann- Liouville integral operator of order $m-\alpha$ and $m-1 < \alpha \leq m$. The relation between Riemann- Liouville operator and Caputo operator is given by the following lemma [20]:

Lemma 1.1. If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, then $D_*^\alpha I^\alpha f(x) = f(x)$, and:

$$I^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \ x > 0.$$

2. FUNCTION APPROXIMATION

2.1. **Haar Wavelets.** The orthogonal set of Haar wavelets $h_n(x)$ is a group of square waves, defined as follows:

(3)
$$h_0(x) = \begin{cases} 1, & 0 \leq x <1; \\ 0, & \text{elsewhere.} \end{cases}, \quad h_1(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}; \\ -1, & \frac{1}{2} \leq x < 1; \\ 0, & \text{elsewhere.} \end{cases}$$

$$h_n(x) = h_1(2^j x - k), \ n = 2^j + k, \ j, k \in \mathbb{N} \cup \{0\}; \ 0 \leq k < 2^j,$$

such that:

$$\int_0^1 h_n(x) h_m(x) dx = 2^{-j} \delta_{nm},$$

where δ_{nm} is the Kronecker delta. For more details see [3, 28, 26].

Each square integrable function $f(x)$ in the interval $[0, 1)$ can be expanded into a Haar series of infinite terms:

(4)
$$f(x) = c_0 h_0(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j - 1} c_{2^j + k} h_{2^j + k}(x), \ x \in [0, \ 1],$$

where the Haar coefficients are determined as:

(5)
$$c_i = 2^j \int_0^1 f(x) h_i(x) dx, \ i = 0, \ 2^j + k, \ j, k \in \mathbb{N} \cup \{0\}; \ 0 \leq k < 2^j,$$

such that the following integral square error ϵ_m is minimized:

$$\epsilon_m = \int_0^1 [f(x) - \sum_{i=0}^{m-1} c_i h_i(x)]^2 dx, \ m = 2^{J+1}, \ J \in \mathbb{N} \cup \{0\}.$$

If $f(x)$ is a piecewise constant or may be approximated as a piecewise constant during each subinterval, the series sum in Eq. (4) can be truncated after m terms ($m = 2^{J+1}$, $J \geq 0$ is a resolution, level of wavelet), that is:

(6)
$$f(x) \cong c_0 h_0(x) + \sum_{j=0}^{J} \sum_{k=0}^{2^j - 1} c_{2^j + k} h_{2^j + k}(x) = \mathbf{c}^T \mathbf{h}(x) = \mathbf{h}^T(x) \mathbf{c} = f_m(x), \ x \in [0, \ 1],$$

where $\mathbf{c} = \mathbf{c}_{m \times 1} = [c_0, \ c_1, \ldots, c_{m-1}]^T$, $\mathbf{h}(x) = \mathbf{h}_{m \times 1}(x) = [h_0(x), \ h_1(x), \ldots, h_{m-1}(x)]^T$.

2.2. **Block Pulse Functions (BPFs).** Another basis set is the set of BPFs. This set over the interval $[0, T)$, is defined as:

$$b_i(x) = \begin{cases} 1, & \text{if } \frac{iT}{m} \leq x < \frac{(i+1)T}{m}; \\ 0, & \text{otherwise,} \end{cases} \quad i = 0, 1, 2, \ldots, m-1, \ m \in \mathbb{N}.$$

In this paper, it is assumed that $T = 1$, so BPFs are defined over $[0, 1)$. BPFs have some useful properties such as disjointness, orthogonality and completeness. From the orthogonality property

of BPFs, it is possible to expand functions into their Block- Pulse series [21], this means that for every $f(x) \in L^2([0,1))$, one can write:

$$(7) \qquad f(x) \cong \sum_{i=0}^{m-1} f_i b_i(x) = \mathbf{f}^T \mathbf{B}_m(x) = \mathbf{B}_m^T(x)\mathbf{f},$$

where:

$$\mathbf{f} = [f_0, f_1, ..., f_{m-1}]^T,$$

$$\mathbf{B}_m(x) = [b_0(x), b_1(x), ..., b_{m-1}(x)],$$

such that f_is for $i = 0, 1, ..., m-1$ are obtained as follows:

$$(8) \qquad f_i = m \int_0^1 b_i(x)f(x)dx.$$

2.3. **Expanding Haar Wavelets via BPFs.** There is a relation between the BPFs and Haar wavelets as the following Lemma:

Lemma 2.1. The vector of Haar functions , $\mathbf{h}(x)$, can be expanded via vector of BPFs , $\mathbf{B}_m(x)$, as the following matrix equation:

$$(9) \qquad \mathbf{h}(x) = \Phi_{m \times m} \mathbf{B}_m(x), \ m = 2^{J+1}, \ J \in \mathbb{N},$$

where $\Phi_{m \times m} = [\phi_{ij}]_{m \times m}$ and:

$$\phi_{(i+1)(j+1)} = h_i(\frac{2j+1}{2m}), \ i, j = 0, 1, \ldots m - 1.$$

Proof. Since Haar functions and BPFs are piecewise functions on the interval $[0, 1)$, according to Eq. (7), Haar wavelets can be expanded into an m-term vector of BPFs as the following equation:

$$h_i(x) = \sum_{j=0}^{m-1} f_j b_j(x) = \Phi_{i+1}\mathbf{B}_m(x), \ i = 0, 1, \ldots m - 1,$$

where Φ_i is the i-th row of matrix $\Phi_{m \times m}$. Therefore we have $\phi_{(i+1)(j+1)} = f_j$, for $i, \ j = 0, 1, \ldots, m-1$, and:

$$\begin{aligned}
\phi_{(i+1)(j+1)} &= m \int_0^1 h_i(x)b_j(x)dx \\
&= m \int_{\frac{j}{m}}^{\frac{j+1}{m}} h_i(x)dx \\
&= m(\frac{j+1}{m} - \frac{j}{m})h_i(\eta_j), \ \eta_j \in [\frac{j}{m}, \frac{j+1}{m}),
\end{aligned}$$

where the last equality is obtained by using mean value theorem for integrals. According to constancy of $h_i(x)$ on $[\frac{j}{m}, \frac{j+1}{m})$, one can put:

$$\eta_j = \frac{2j+1}{2m}, \ j = 0, 1, \ldots m - 1,$$

so we have:

$$\phi_{(i+1)(j+1)} = h_i(\frac{2j+1}{2m}).$$

$\square$

2.4. **Operational Matrix of Fractional Integration.** The fractional integration of order α of $\mathbf{h}(x)$ can be expanded into Haar series with Haar coefficient matrix $\mathbf{P}_m^\alpha$ as follows:

(10)
$$I^\alpha \mathbf{h}(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \mathbf{h}(t) dt \cong \mathbf{P}_m^\alpha \mathbf{h}(x).$$

Such $m \times m$ square matrix $\mathbf{P}_m^\alpha$, is called the operational matrix of fractional integration. The matrix $\mathbf{P}_m^\alpha$ has been obtained in a pretty in [23].

Also, the operational matrix $\mathbf{P}_m^\alpha$ can be derived as the following (see [12, 18]):

(11)
$$\mathbf{P}_{m \times m}^\alpha = \mathbf{\Phi}_{m \times m} \mathbf{F}^\alpha \mathbf{\Phi}_{m \times m}^{-1},$$

where:

$$\mathbf{F}^\alpha = \frac{1}{m^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

and $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

Remark 2.1. To obtain $\mathbf{P}_m^\alpha$ by (11), we should calculate the inverse of matrix $\mathbf{\Phi}_{m \times m}$, so using the direct method in [23] is suggested.

3. APPLICATION OF METHOD

Consider Eq. (1), the two variable function $k(x,t) \in L^2([0,1))^2$ and the right hand side of Eq. (1) can be approximated as:

$$k(x,t) \cong \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} k_{ij} h_i(x) h_j(t), \ g(x) \cong \sum_{i=0}^{m-1} g_i h_i(x),$$

or in the matrix form:

(12)
$$k(x, \ t) \cong \mathbf{h}^T(x) \mathbf{K} \mathbf{h}(t), \ g(x) \cong \mathbf{g}^T \mathbf{h}(x)$$

where $\mathbf{K} = [k_{i,j}]_{m \times m}$ and $k_{ij} = 2^{i_1+j_1} \int_0^1 \int_0^1 k(x,t) h_i(x) h_j(t) dt dx, \ i = 2^{i_1} + k_1, \ j = 2^{i_2} + k_2$ such that $i_1, i_2, k_1, k_2 \in \mathbb{N} \cup \{0\}$ and $0 \le k_1 < 2^{i_1}, \ 0 \le k_2 < 2^{i_2}$ and $g_i = 2^{i_1} \int_0^1 g(x) h_i(x)$ for

$i, j = 0, 1, \ldots, m-1.$

Now, let:

$$D_*^\alpha f(x) \cong \mathbf{c}^T \mathbf{h}(x). \tag{13}$$

Using lemma 1.1 and Eqs. (13) and (10), we have:

$$f(x) \cong \mathbf{c}^T \mathbf{P}_{m \times m}^\alpha \mathbf{h}(x) + \sum_{k=0}^{s-1} f^{(k)}(0^+) \frac{x^k}{k!}. \tag{14}$$

Hence, by substituting the supplementary conditions (2) in the above equation and approximate it via Haar wavelets, we have:

$$f(x) \cong (\mathbf{c}^T \mathbf{P}_{m \times m}^\alpha + \mathbf{c_1}^T) \mathbf{h}(x), \tag{15}$$

where $\mathbf{c_1}$ is an m-vector. According to Eq. (9) we get:

$$f(x) \cong (\mathbf{c}^T \mathbf{P}_{m \times m}^\alpha + \mathbf{c_1}^T) \mathbf{\Phi}_{m \times m} \mathbf{B}_m(x).$$

Define:

$$\mathbf{a} = [a_0, \ a_1, \ldots, \ a_{m-1}] = (\mathbf{c}^T \mathbf{P}_{m \times m}^\alpha + \mathbf{c_1}^T) \mathbf{\Phi}_{m \times m},$$

so, $f(x) \cong \mathbf{a} \, \mathbf{B}_m(x)$. By induction on $q \in \mathbb{N}$ and from the disjoint property of BPFs, we have:

$$[f(x)]^q \cong [a_0^q, a_1^q, \ldots, a_{m-1}^q] \mathbf{B}_m(x) = \widetilde{\mathbf{a}}_q \mathbf{B}_m(x), \tag{16}$$

where:

$$\widetilde{\mathbf{a}}_q = [a_0^q, a_1^q, \ldots, a_{m-1}^q],$$

for all positive integers q.

Using Eqs. (9), (12) and (16) implies that:

$$\int_0^1 k(x,t)[f(t)]^q dt = \int_0^1 \mathbf{h}^T(x) \mathbf{K} \mathbf{h}(t) \mathbf{B}_m^T(t) \widetilde{\mathbf{a}}_q^{\ T} dt$$

$$= \int_0^1 \mathbf{h}^T(x) \mathbf{K} \mathbf{\Phi}_{m \times m} \mathbf{B}_m(t) \mathbf{B}_m^T(t) \widetilde{\mathbf{a}}_q^T dt$$

$$= \mathbf{h}^T(x) \mathbf{K} \mathbf{\Phi}_{m \times m} \int_0^1 \mathbf{B}_m(t) \mathbf{B}_m^T(t) \widetilde{\mathbf{a}}_q^T dt. \tag{17}$$

By using Eqs. (??) and (??), we can simplify the integral part of (17) as:

$$\int_0^1 \mathbf{B}_m(t)\mathbf{B}_m^T(t)\widetilde{\mathbf{a}}_q^T\,dt = \int_0^1 \{\begin{bmatrix} b_0(t) & & & O \\ & b_1(t) & & \\ & & \ddots & \\ O & & & b_{m-1}(t) \end{bmatrix}\begin{bmatrix} a_0^q \\ a_1^q \\ \vdots \\ a_{m-1}^q \end{bmatrix}\}dt$$

$$= \int_0^1 [a_0^q b_0(t), a_1^q b_1(t), \ldots, a_{m-1}^q b_{m-1}(t)]^T\,dt$$

$$= \frac{1}{m}[a_0^q, a_1^q, \ldots, a_{m-1}^q]^T$$

$$= \frac{1}{m}\widetilde{\mathbf{a}}_q^T.$$

Thus we have:

(18)
$$\int_0^1 k(x,t)[f(t)]^q dt \cong \frac{1}{m}\mathbf{h}^T(x)\mathbf{K}\boldsymbol{\Phi}_{m\times m}\widetilde{\mathbf{a}}_q^T.$$

Substituting the approximations (12), (13) and (18) into (1), we obtain:

(19)
$$\mathbf{h}(x)^T\mathbf{c} - \lambda\frac{1}{m}\mathbf{h}^T(x)\mathbf{K}\boldsymbol{\Phi}_{m\times m}\widetilde{\mathbf{a}}_q \cong \mathbf{h}(x)^T\mathbf{g}.$$

Now, by multiplying two sides of (19) in $\mathbf{h}(x)$ and using the orthogonality property of Haar wavelets we get:

$$\mathbf{c} - \lambda\frac{1}{m}\mathbf{K}\boldsymbol{\Phi}_{m\times m}\widetilde{\mathbf{a}}_q = \mathbf{g},$$

which is a nonlinear system of algebraic equations. Solving this system results in the approximate solution of Eq. (1) according to Eq. (15).

4. Error Analysis

Definition 4.1. If $f(x)$ and $f_m(x) = \mathbf{c}^T\mathbf{h}(x)$ are the exact and approximate solutions of (1), respectively, the corresponding error is denoted by $e_m(x)$ as the following:

$$e_m(x) = f(x) - f_m(x).$$

Theorem 4.1. ([23]) Suppose that $f(x)$ satisfies in the Lipschitz condition on $[0, 1]$, that is:

(20)
$$\exists M > 0 \; ; \; \forall x, \; y \in [0, \; 1] : \; |f(x) - f(y)| \leq M|x - y|,$$

then the Haar wavelet method will be convergent in the sense that $e_m(x)$ goes to zero as m goes to infinity. Moreover, the convergent is of order unity, that is:

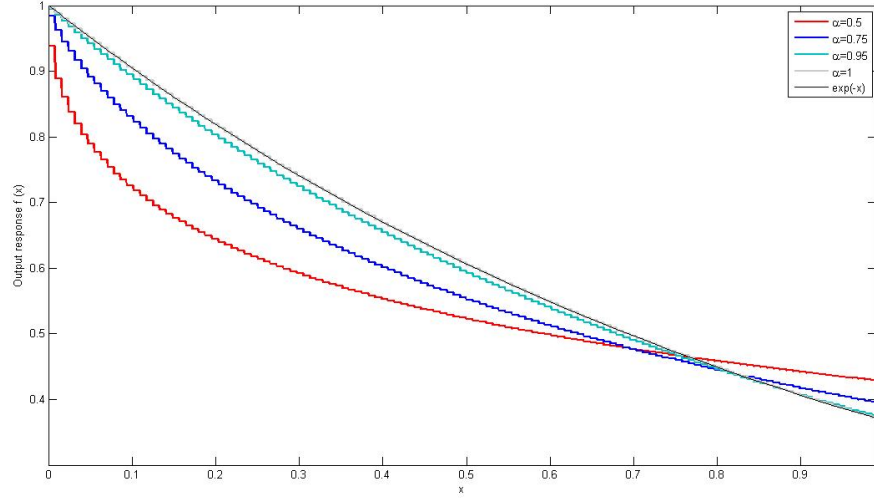$$\|e_m(x)\|_2 = O(\frac{1}{m}).$$

FIGURE 1. The approximate solution of Example 5.1 for $m = 32$ and some $0.5 \leq$ $\alpha \leq 1$.

Remark 4.1. It is easy to see that: if $f(x)$ satisfies in the Holder condition on $[0, 1]$, that is:

$$(21) \qquad \exists M > 0, \ \gamma > 1; \ \forall x, \ y \in [0, \ 1] : \ |f(x) - f(y)| \leq M|x - y|^{\gamma},$$

then the Theorem 4.1 is again correct.

The accuracy of the method can be also checked easily. Since the truncated Haar wavelet series is an approximate solution of Eq. (1), when the approximate functions (13) and (18) are substituted in Eq.(1), the resulting equation, (19), must be satisfied approximately, that is for $x \in [0, 1)$:

$$R_m(x) =| \mathbf{h}(x)^T \mathbf{c} - \lambda \frac{1}{m} \mathbf{h}^T(x) \mathbf{K} \mathbf{\Phi}_{m \times m} \widetilde{\mathbf{a}}_q - \mathbf{h}(x)^T \mathbf{g} | \cong 0.$$

If we set $x = x_i$, then the aim is to have $R_m(x_i) \leq 10^{r_i}$, where r_i is any positive integer. If we prescribe $Max\{r_i\} = r$, then we increase m as long as the following inequality holds at each point x_r:

$$R_m(x_i) \leq 10^r,$$

in other words, by increasing m the error function $R_m(x_i)$ approaches zero. If $R_m(x) \longrightarrow 0$ when m is sufficiently large enough, then the error decreases.
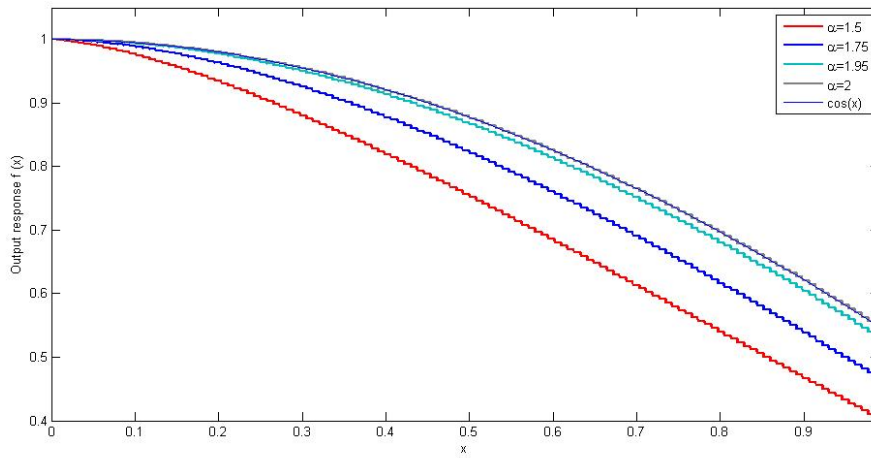
FIGURE 2. The approximate solution of Example 5.1 for $m = 32$ and some $1.5 \leq \alpha \leq 2$.
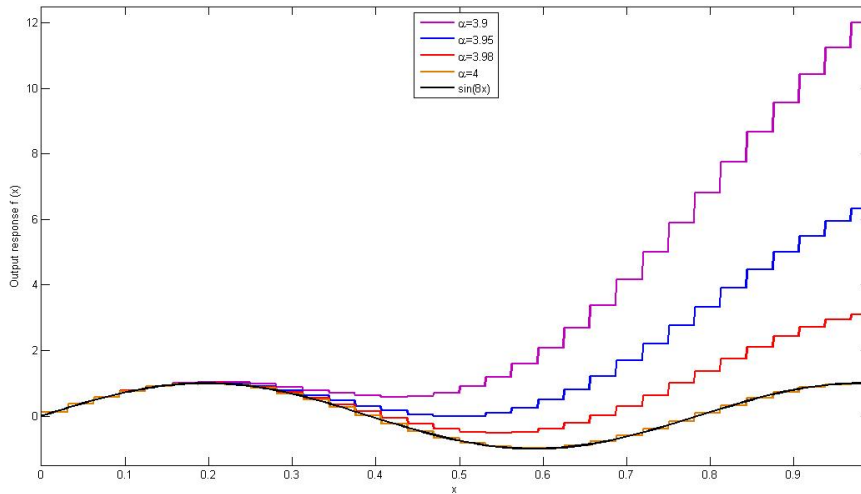

FIGURE 3. The approximate solution of Example 5.2 for $m = 32$ and some $3.9 \leq \alpha \leq 4$.

5. NUMERICAL RESULTS

In this section, the operational Haar wavelet method, presented in this paper, is applied to solve some examples and to show the efficiency of the mentioned method. Note that:

$$\|e_m(x)\|_2 = (\int_0^1 e_m^2(x)dx)^{1/2} \cong (\frac{1}{N}\sum_{i=0}^{N} e_m^2(x_i))^{1/2},$$
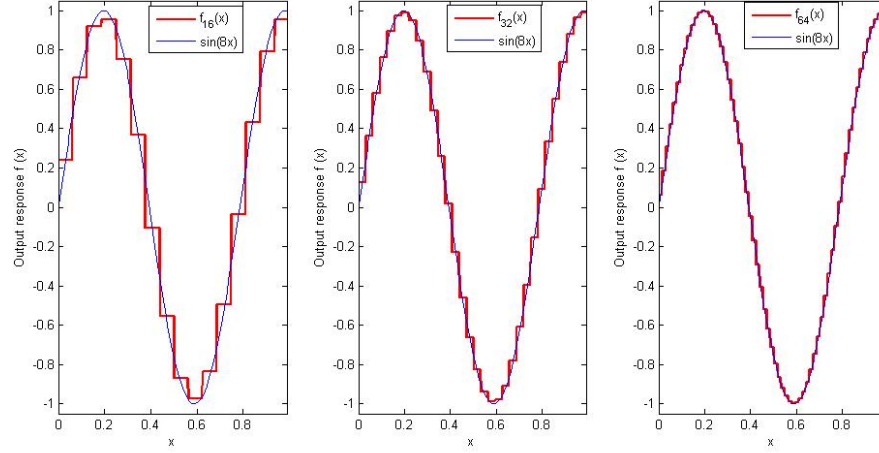
FIGURE 4. A comparison between the approximate and exact solution of Example 5.2.

where $e_m(x_i) = f(x_i) - f_m(x_i)$, $i = 0, 1, ..., N$. $f(x)$ is the exact solution and $f_m(x)$ is the approximate solution which is obtained by Eq. (15).

Consider that all of the computations have been done by MATLAB 7.8.

Example 5.1. Consider the following linear initial value problem [13]:

(22) $$D_*^\alpha f(x) + f(x) = 0, \ 0 < \alpha < 2,$$

$$f(0) = 1, \ f'(0) = 0.$$

The second initial condition is only for $\alpha > 1$. The exact solution of this problem is as the following [8]:

$$f(x) = \sum_{k=0}^{\infty} \frac{(-x^\alpha)^k}{\Gamma(\alpha k + 1)}.$$

Applying the Haar wavelet method, the following system of equations is obtained:

$$\mathbf{c}_m^T \mathbf{h}(x) + (\mathbf{c}_m^T \mathbf{P}_m^\alpha + \mathbf{c_1}^T) \mathbf{h}(x) = \mathbf{0}.$$

The numerical results are shown in figure 1, for $\alpha = 0.5$, 0.75, 0.95, 1. The exact solution in the case $\alpha = 1$ is given as $f(x) = e^{-x}$. Note that as α approaches 1, the numerical solution converges to the analytical solution $f(x) = e^{-x}$, i. e. in the limit case, the solution of the fractional differential equation, 22, approaches to that of the integer-order differential equation, $\alpha = 1$. This also happens when α approaches 2. It is considerable that for $\alpha = 2$, the exact solution is $f(x) = cos(x)$. figures 1 and 2 show the numerical results for some $0.5 \le \alpha \le 1$ and $1.5 \le \alpha \le 2$, respectively. Also, table 1 shows the approximate norm-2 of absolute error for $\alpha = 1$ and $\alpha = 2$.
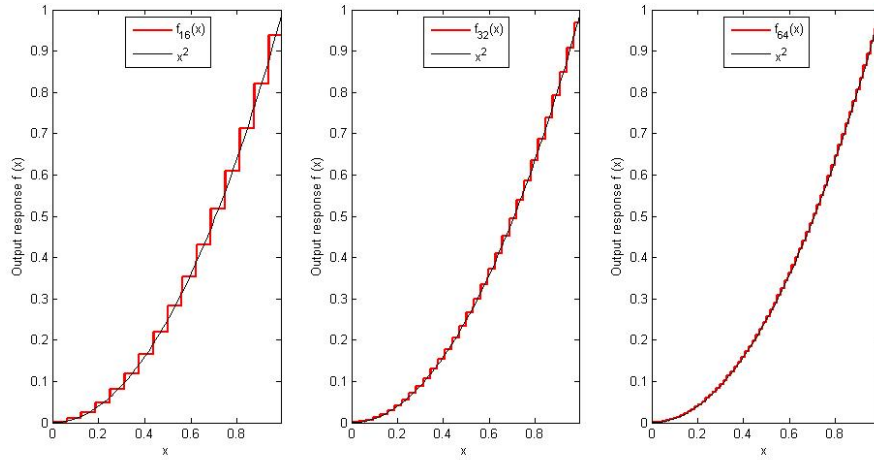
FIGURE 5. A comparison between the approximate and exact solution of Example 5.3.

Example 5.2. Consider the following forth order nonlinear integro-differential equation:

$$(23) \qquad D_*^\alpha f(x) - \int_0^1 xt[f(t)]^2 dt = 4096 sin(8x) - \frac{41}{156}x, \ 0 \leq x < 1, \ 3 < \alpha \leq 4,$$

$$f(0) = 0, \ f'(x) = 8, \ f''(0) = 0, \ f'''(0) = -512.$$

Figure 3 shows the numerical results for $m = 32$ and various $3 < \alpha \leq 4$. The comparisons show that as α approaches 4, the approximate solutions tend to $f(x) = sin(8x)$, which is the exact solution of the equation in the case of $\alpha = 4$. The numerical results and the error in the case $\alpha = 4$, for different values of m, is shown in figure 4 and table 1, respectively.

Example 5.3. Consider the following nonlinear Fredholm integro- differential equation of order $\alpha = \frac{5}{3}$:

$$D_*^{\frac{5}{3}} f(x) - \int_0^1 (x+t)^2 [f(t)]^3 dt = g(x), \ 0 \leq x < 1,$$

$$f(0) = f'(0) = 0,$$

where $g(x) = \frac{6}{\Gamma(1/3)} \sqrt[3]{x} - \frac{x^2}{7} - \frac{x}{4} - \frac{1}{9}$. Figure 4 shows the numerical solutions for various m, with the exact solution $f(x) = x^2$. The error for different values of m, is shown in table 1.

Example 5.4. Consider the equation:

$$D_*^{\frac{1}{2}} f(x) - \int_0^1 xt[f(t)]^4 dt = g(x), \ 0 \leq x < 1,$$

such that $f(0) = 0$ and $g(x) = \frac{1}{\Gamma(1/2)} (\frac{8}{3} \sqrt{x^3} - 2\sqrt{x}) - \frac{x}{1260}$. From figure 5 and table 1, it can be seen that the numerical solutions are in a very good agreement with the exact solution $f(x) = x^2 - x$.
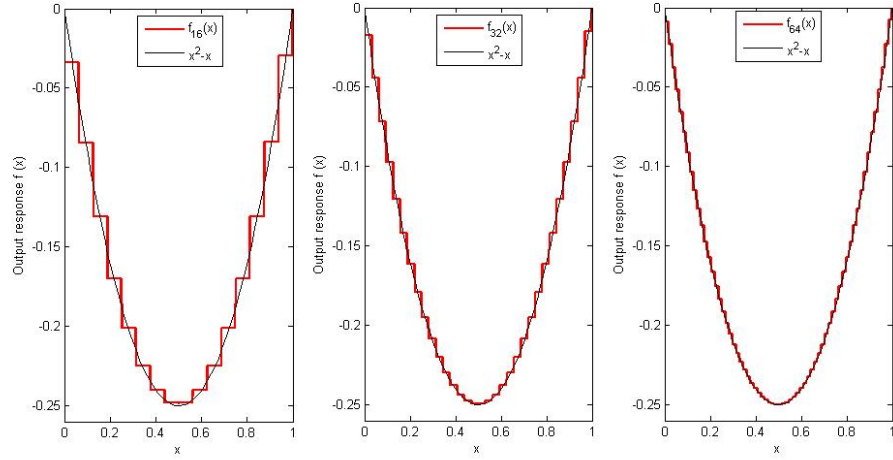
FIGURE 6. A comparison between the approximate and exact solution of Example 5.4.

Examples	$\|e_{16}\|_2,\ (J = 3)$	$\|e_{32}\|_2,\ (J = 4)$	$\|e_{64}\|_2,\ (J = 5)$
Example 5.1 ($\alpha = 1$)	1.41151e-04	3.53741e-05	8.86406e-06
Example 5.1 ($\alpha = 2$)	8.47341e-05	2.15527e-05	5.48756e-06
Example 5.2 ($\alpha = 4$)	1.04785e-02	2.59350e-03	6.46489e-04
Example 5.3 ($\alpha = 5/3$)	4.10383e-04	1.04700e-04	2.67422e-05
Example 5.4 ($\alpha = 1/2$)	1.04389e-04	2.64033e-05	6.71193e-06

Table 1. The approximate norm-2 of absolute error for some m.

As mentioned in section 2, the set of Haar wavelets forms an orthonormal basis for $L^2([0,\ 1))$. Therefore, the solution of Eq. 1 can be expanded as:

$$(24) \qquad f(x) = c_0 h_0(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j - 1} c_{2^j + k} h_{2^j + k}(x).$$

In the above examples, the approximate solution of the equations has been obtained as:

$$(25) \qquad f(x) \cong c_0 h_0(x) + \sum_{j=0}^{J} \sum_{k=0}^{2^j - 1} c_{2^j + k} h_{2^j + k}(x),\ J \in \mathbb{N},$$

which is the truncated series of (24). By substituting the solutions c_n in (25) we have:

$$(26) \qquad e(x) = |f(x) - c_0 h_0(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j - 1} c_{2^j + k} h_{2^j + k}(x)|,$$

where $e(x)$ is defined as an error function. So, as J increases, the series (25) becomes larger and closer to the series (24), namely the error function approaches zero. The numerical results of the above examples confirm this.

6. Conclusion

Here the Haar wavelet operational matrix of fractional integration is derived, by using BPFs, and used to solve a class of nonlinear Fredholm integro- differential equations of fractional order. Several examples are given to demonstrate the powerfulness of the proposed method. It is shown that the solution is convergent, even though the size of increment may be large. Also this method can be used to obtain the numerical solutions of ordinary nonlinear integro- differential equations. Other wavelets can also be applied to derive this method.

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