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SPIRALLIKENESS PROPERTIES ON SALAGEAN-TYPE HARMONIC UNIVALENT FUNCTIONS

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Abstract. In this paper, we define and investigate a new class of spirallike harmonic functions defined by a Salagean differential operator and we obtain a coefficient inequality for the functions in this class. Following, we investigated convolution and obtain the order of convolution consistence for certain spirallike harmonic univalent functions with negative coefficients

Keywords: Harmonic univalent functions; Spirallike functions; Convolution; Coefficient bound; Salagean differential operator, $\tilde{\circledast}$-convolution consistence.
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1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a unite disk $D = \{z \in \mathbb{C} : |z| < 1\}$ is said to be harmonic in D if both u and v are real harmonic in D. In simply connected domain, we can write

$$f = h + \bar{g}, \tag{1}$$

where the analytic functions h and g by the following power series expansions:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \qquad g(z) = \sum_{m=1}^{\infty} b_m z^m, \quad |b_1| < 1. \tag{2}$$

We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|g'(z)| < |h'(z)|$ in D. The class of harmonic univalent functions f from (1) is denoted by S_H. Duren in [9] investigated some properties of the class of harmonic univalent functions. Let S_H^*, K_H and C_H denote the subclass of S_H consisting of harmonic univalent functions which are, respectively, starlike, convex, and close-to-convex in D. The reader is referred to [10,14,19] for many interesting results and expositions on planar univalent harmonic mappings.

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If f is given by (1), it was shown in [17] that $|a_n| \leq (n+1)(2n+1)/6, |b_n| \leq (n-1)(2n-1)/6$ for every $f \in S_H^*$. It was shown in [8] that $|a_n| \leq (n+1)/2,\ |b_n| \leq (n-1)/2$ for $f \in K_H$. After a appearance of this paper [2–4, 7] investigate a coefficient bounds for a new subclass of harmonic starlike, convex and close-to-convex functions.

If f is given by (1), the modified Salagean operator of f is defined by

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}, \tag{3}$$

where $D^n h(z) = z + \sum_{m=2}^{\infty} m^n a_m z^m$ and $D^n g(z) = \sum_{m=1}^{\infty} m^n b_m z^m$. The operator D^n, i.e. differential Salagean operator, is introduced and investigated by Salagean [16] for the case of analytic functions.

If the co-analytic part of $f = h + \overline{g}$ is identically zero and

$$Re\Big\{e^{i\lambda}\frac{zf'(z)}{f(z)}\Big\} > \alpha\cos\lambda,$$

for $\lambda \in (-\pi/2, \pi/2)$ and $0 \leq \alpha < 1$, then the class of such functions f is denoted by $SP_\alpha(\lambda)$, introduced and investigated by Libra [11] for the analytic case.

If f is given by (1) and

$$F(z) = H + \bar{G} = z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} \bar{B}_m \bar{z}^m,$$

the convolution of two complex-valued harmonic functions f and F is defined by

$$f(z)\tilde{*}F(z) = z + \sum_{m=2}^{\infty} a_m A_m z^m + \sum_{m=1}^{\infty} \bar{b}_m \bar{B}_m \bar{z}^m.$$

Clearly, $f(z)\tilde{*}F(z) = F(z)\tilde{*}f(z)$. In the case of conformal mappings, the literature on convolution theory is exhaustive. For example, we have [15]

$$K_H\tilde{*}K_H \subset K_H, \quad S_H^*\tilde{*}K_H \subset S_H^*, \quad C_H\tilde{*}K_H \subset K_H,$$

For some related containment relations, we refer to [1, 12, 13] and other later works of Ruscheweyh.

We let the subclass NS_H of S_H consist of harmonic functions

$$f_n = h + \bar{g}_n, \tag{4}$$

where h and g_n are of the form

$$h(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad g_n(z) = (-1)^n \sum_{m=1}^{\infty} b_m z^m, \quad a_m \geq 0, \quad b_m \geq 0. \tag{5}$$

Silverman [18] studied the harmonic univalent functions with negative coefficients.

If the co-analytic part of $f = h + \overline{g}$ is identically zero, the Salagean integral operator ([5,16]) $I^s : S_H \longrightarrow S_H, s \in \mathbb{R}$ is defined by

$$I^s f(z) = I^s\Big(z + \sum_{m=2}^{\infty} a_m z^m\Big) = z + \sum_{m=2}^{\infty} \frac{a_m}{m^s} z^m. \tag{6}$$

Bednarz and Sokol in [6], by using the Salagean integral operator, studied the properties of the convolution for a certain class of univalent functions.

In the present paper, we define a special subclass of spirallike harmonic functions by Salagean integral operator. In the following, we study the coefficient bounds and terms of convolution. Following, we define the order of convolution consistence for the functions in this class. Furthermore, we obtain the order of convolution consistence for the subclass of spirallike harmonic functions with negative coefficients.

2. Coefficient bounds

Definition 2.1. For $0 \leq \alpha < 1$, $n \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$ and $z \in D$, let $SP_\alpha^H(\lambda, n)$, be the class of (λ, n)-spirallike harmonic functions of order α consist of harmonic univalent functions f of the form (1) such that

$$Re\Big\{e^{i\lambda} \frac{D^{n+1} f(z)}{D^n f(z)}\Big\} = \Big\{e^{i\lambda} \frac{D^{n+1} h(z) - (-1)^n \overline{D^{n+1} g(z)}}{D^n h(z) + (-1)^n \overline{D^n g(z)}}\Big\} \geq \alpha \cos \lambda.$$

Note that, for $n = 0$, $SP_\alpha^H(\lambda)$ is the class of λ-spirallke harmonic functions of order α and for $n = 1$, $CSP_\alpha^H(\lambda)$ is the class of convex λ-spirallike harmonic functions of order α .

Definition 2.2. For $0 \leq \alpha < 1$, $n \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$ and $z \in D$, we let $NSP_\alpha^H(\lambda, n)$ consist the harmonic univalent functions $f_n = h + \overline{g_n}$ in $SP_\alpha^H(\lambda, n)$ so that h and g_n are of the form (5).

Note that, for $n = 0$, $NSP_\alpha^H(\lambda)$, is the class of λ-spirallke harmonic functions of order α with negative coefficients and for $n = 1$, $NCSP_\alpha^H(\lambda)$, is the class of convex λ-spirallike harmonic functions of order α with negative coefficients.

Theorem 2.3. *Let $\lambda \in (-\pi/2, \pi/2)$, $0 \leq \alpha < 1$ such that $\sqrt{2} - 2\alpha \cos \lambda > 0$ and f be the function given by (1). If*

$$\sum_{m=2}^{\infty} m^n |a_m| \frac{(2m + 2 + 2\alpha \cos \lambda)}{\sqrt{2} - 2\alpha \cos \lambda} + \sum_{m=1}^{\infty} m^n |b_m| \frac{(2m + 2 + 2\alpha \cos \lambda)}{\sqrt{2} - 2\alpha \cos \lambda} \leq 1, \tag{7}$$

where $a_1 = 1$, then f is the orientation preserving harmonic univalent in D, and $f \in SP_\alpha^H(\lambda, n)$.

Proof. If $z_1 \neq z_2$ and h, g and g given by (2), then

$$\begin{aligned}
\Big|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\Big| &\geq 1-\Big|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\Big| = 1-\left|\frac{\sum_{m=1}^{\infty} b_m(z_1^m - z_2^m)}{(z_1-z_2)+\sum_{m=2}^{\infty} a_m(z_1^m - z_2^m)}\right| \\
&\geq 1-\frac{\sum_{m=1}^{\infty} m|b_m|}{1-\sum_{m=2}^{\infty} m|a_m|} \geq 1-\frac{\sum_{m=1}^{\infty} m^n|b_m|\frac{2m+2+2\alpha\cos\lambda}{\sqrt{2}-2\alpha\cos\lambda}}{1-\sum_{m=2}^{\infty} m^n|a_m|\frac{2m+2+2\alpha\cos\lambda}{\sqrt{2}-2\alpha\cos\lambda}} \geq 1,
\end{aligned}$$

which shows that f is a univalent function. Note that f is orientation preserving in D. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{m=2}^{\infty} m|a_m||z|^{m-1} > 1-\sum_{m=2}^{\infty} m^n \frac{2m+2+2\alpha\cos\lambda}{\sqrt{2}-2\cos\lambda}|a_m| \\
&\geq \sum_{m=1}^{\infty} m^n \frac{2m+2+2\alpha\cos\lambda}{\sqrt{2}-2\alpha\cos\lambda}|b_m| > \sum_{k=1}^{\infty} m^n \frac{2m+2+2\alpha\cos\lambda}{\sqrt{2}-2\alpha\cos\lambda}|b_m||z|^{m-1} \\
&\geq \sum_{m=1}^{\infty} m|b_m||z|^{m-1} \geq |g'(z)|.
\end{aligned}$$

Using the fact that $Re\{e^{i\lambda}w\} > \alpha\cos\lambda$, if and only if $|w+\cos\lambda - i\sin\lambda|+|w| \geq 1+2\alpha\cos\lambda$, it suffices to show that

$$|D^{n+1}f(z)+(\cos\lambda - i\sin\lambda)D^n f(z)| + |D^{n+1}f(z)| \geq (1+2\alpha)\cos\lambda|D^n f(z)|. \tag{8}$$

Substituting $D^n f$ and $D^{n+1} f$ in (8), yields

$$\begin{aligned}
&\Big|D^{n+1}f(z)+(\cos\lambda - i\sin\lambda)D^n f(z)| + |D^{n+1}f(z)| - |(1+2\alpha\cos\lambda)D^n f(z)\Big| \\
&= \Big|z+\sum_{m=2}^{\infty} m^{n+1}a_m z^m - (-1)^n\sum_{m=1}^{\infty} m^{n+1}\bar{b}_m \bar{z}^m \\
&+(\cos\lambda - i\sin\lambda)z - \sum_{m=2}^{\infty} m^n a_m(\cos\lambda - i\sin\lambda)z^m - (-1)^n\sum_{m=1}^{\infty} m^n(\cos\lambda - i\sin\lambda)\bar{b}_m\bar{z}^m\Big| \\
&+\Big|z+\sum_{m=2}^{\infty} m^{n+1}a_m z^m - (-1)^n\sum_{m=1}^{\infty} m^{n+1}\bar{b}_m\bar{z}^m\Big| \\
&-\Big|(1+2\alpha\cos\lambda)z+\sum_{m=2}^{\infty} m^n(1+2\alpha\cos\lambda)a_m z^m - (-1)^n\sum_{m=1}^{\infty} m^n(1+2\alpha\cos\lambda)\bar{b}_m\bar{z}^m\Big| \geq 0.
\end{aligned}$$

The above inequality is equivalent to

$$\Big|(1+\cos\lambda - i\sin\lambda)z + \sum_{m=2}^{\infty} m^n a_m (m+\cos\lambda - i\sin\lambda) z^m$$

$$-(-1)^n \sum_{m=1}^{\infty} m^n \overline{b}_m (m+\cos\lambda - i\sin\lambda)\overline{z}^m\Big|$$

$$+\Big|z + \sum_{m=2}^{\infty} m^{n+1} a_m z^m - (-1)^n \sum_{m=1}^{\infty} m^{n+1}\overline{b}_m \overline{z}^m\Big| - \Big|(1+2\alpha\cos\lambda)z$$

$$+\sum_{m=2}^{\infty} m^n (1+2\alpha\cos\lambda) a_m z^m - (-1)^n \sum_{m=1}^{\infty} m^n (1+2\alpha\cos\lambda)\overline{b}_m \overline{z}^m\Big|$$

$$\geq |1+\cos\lambda - i\sin\lambda||z| - \sum_{m=2}^{\infty} m^n |a_m||m+\cos\lambda - i\sin\lambda||z|^m$$

$$-\sum_{m=1}^{\infty} m^n |b_m||m+\cos\lambda - i\sin\lambda||z|^m$$

$$+|z| - \sum_{m=2}^{\infty} m^{n+1}|a_m||z|^m - \sum_{m=1}^{\infty} m^{n+1}|b_m||z|^m - (1+2\alpha\cos\lambda)|z|$$

$$-\sum_{m=2}^{\infty} m^n (1+2\alpha\cos\lambda)|a_m||z|^m - \sum_{m=1}^{\infty} (1+2\alpha\cos\lambda)|b_m||z|^m$$

$$\geq (\sqrt{2} - 2\alpha\cos\lambda)|z| - \sum_{m=2}^{\infty} m^n |a_m|(2m+2+2\alpha\cos\lambda)|z|^m$$

$$-\sum_{m=1}^{\infty} m^n |b_m|(2m+2+2\alpha\cos\lambda)|z|^m$$

$$\geq (\sqrt{2} - 2\alpha\cos\lambda)|z|\Big\{1 - \sum_{m=2}^{\infty} m^n |a_m| \frac{(2m+2+2\alpha\cos\lambda)}{\sqrt{2}-2\alpha\cos\lambda}|z|^{m-1}$$

$$-\sum_{m=1}^{\infty} m^n |b_m| \frac{(2m+2+2\alpha\cos\lambda)}{\sqrt{2}-2\alpha\cos\lambda}|z|^{m-1}\Big\}$$

$$\geq (\sqrt{2} - 2\alpha\cos\lambda)|z|\Big\{1 - \sum_{m=2}^{\infty} m^n |a_m| \frac{(2m+2+2\alpha\cos\lambda)}{\sqrt{2}-2\alpha\cos\lambda}$$

$$-\sum_{m=1}^{\infty} m^n |b_m| \frac{(2m+2+2\alpha\cos\lambda)}{\sqrt{2}-2\alpha\cos\lambda}\Big\}.$$

This last expression is non-negative by (7), and so the proof is complete. □

The (λ, n)-spirallike harmonic function of order α

$$f(z) = z + \sum_{m=2}^{\infty} \frac{\sqrt{2} - 2\alpha\cos\lambda}{m^n(2m + 2 + 2\alpha\cos\lambda)} x_m z^m + \sum_{m=1}^{\infty} \frac{\sqrt{2} - 2\alpha\cos\lambda}{m^n(2m + 2 + 2\alpha\cos\lambda)} \overline{y}_m \overline{z}^m,$$

where $\sum_{m=2}^{\infty} |x_m| + \sum_{m=1}^{\infty} |y_m| = 1$, shows that the coefficient bounds in (7) is sharp.

Corollary 2.4. *Let $\lambda \in (-\pi/2, \pi/2)$, $0 \leq \alpha < 1$ such that $\sqrt{2} - 2\alpha\cos\lambda > 0$ and f given by* (1)*. If*

$$\sum_{m=2}^{\infty} |a_m| \frac{(2m + 2 + 2\alpha\cos\lambda)}{\sqrt{2} - 2\alpha\cos\lambda} + \sum_{m=1}^{\infty} |b_m| \frac{(2m + 2 + 2\alpha\cos\lambda)}{\sqrt{2} - 2\alpha\cos\lambda} \leq 1,$$

where $a_1 = 1$, then f is the orientation preserving harmonic univalent in D and $f \in SP_\alpha^H(\lambda)$.

Corollary 2.5. *Let $\lambda \in (-\pi/2, \pi/2)$, $0 \leq \alpha < 1$ such that $\sqrt{2} - 2\alpha\cos\lambda > 0$ and f be the function given by* (1)*. If*

$$\sum_{m=2}^{\infty} m|a_m| \frac{(2m + 2 + 2\alpha\cos\lambda)}{\sqrt{2} - 2\alpha\cos\lambda} + \sum_{m=1}^{\infty} m|b_m| \frac{(2m + 2 + 2\alpha\cos\lambda)}{\sqrt{2} - 2\alpha\cos\lambda} \leq 1,$$

where $a_1 = 1$, then f is the orientation preserving harmonic univalent in D and $f \in CSP_\alpha^H(\lambda)$.

Theorem 2.6. *For $0 \leq \alpha < 1$, $n \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$ such that $\sqrt{2} - 2\alpha\cos\lambda > 0$ and $f_n = h + \bar{g}_n$ be the function given by* (4)*, then $f_n \in NSP_\alpha^H(\lambda, n)$, if and only if*

$$\sum_{m=2}^{\infty} m^n a_m \frac{(2m + 2 + 2\alpha\cos\lambda)}{\sqrt{2} - 2\alpha\cos\lambda} + \sum_{m=1}^{\infty} m^n b_m \frac{(2m + 2 + 2\alpha\cos\lambda)}{\sqrt{2} - 2\alpha\cos\lambda} \leq 1. \tag{9}$$

Proof. Since $NSP_\alpha^H(\lambda, n) \subset SP_\alpha^H(\lambda, n)$, we need to prove the sufficient part of the theorem. To prove this, for the function f_n of the form (4), we notice that the condition

$$Re\left\{e^{i\lambda} \frac{D^{n+1} f_n(z)}{D^n f_n(z)}\right\} > \alpha\cos\lambda$$

is equivalent to

$$Re\left\{\frac{(e^{i\lambda} - \alpha\cos\lambda)z - \sum_{m=2}^{\infty} m^n(me^{i\lambda} - \alpha\cos\lambda)a_m z^m - (-1)^{2n} \sum_{m=1}^{\infty} m^n(me^{i\lambda} + \alpha\cos\lambda)b_m \overline{z}^m}{z - \sum_{m=2}^{\infty} m^n a_m z^m + (-1)^{2n} \sum_{m=1}^{\infty} m^n b_m \overline{z}^m}\right\} \geq 0. \tag{10}$$

The above condition (10) must hold for all values of z in D. Upon choosing the value of z on the positive real axis where $0 \leq z = r < 1$, we most have

$$Re\left\{\frac{(e^{i\lambda} - \alpha\cos\lambda) - \sum_{m=2}^{\infty} m^n(me^{i\lambda} - \alpha\cos\lambda)a_m r^{m-1} - (-1)^{2n} \sum_{m=1}^{\infty} m^n(me^{i\lambda} + \alpha\cos\lambda)b_m r^{m-1}}{1 - \sum_{m=2}^{\infty} m^n a_m r^{m-1} + (-1)^{2n} \sum_{m=1}^{\infty} m^n b_m r^{m-1}}\right\} \geq 0.$$

The above equation is equivalent to

(11)
$$\frac{\cos\lambda\Bigg((1-\alpha)-\sum_{m=2}^{\infty}m^n(m-\alpha)a_m r^{m-1}-\sum_{m=1}^{\infty}m^n(m+\alpha)b_m r^{m-1}\Bigg)}{1-\sum_{m=2}^{\infty}m^n a_m r^{m-1}+\sum_{m=1}^{\infty}m^n b_m r^{m-1}}\geq 0.$$

If the condition (11) does not hold, we get

$$\sum_{m=2}^{\infty}m^n(m-\alpha)a_m r^{m-1}+\sum_{m=1}^{\infty}m^n(m+\alpha)b_m r^{m-1}>1-\alpha. \tag{12}$$

By applying the condition (12), we concluded that the numerator in (9) is negative for r sufficiently close to 1. This contradicts the required condition for $f\in NSP_{\alpha}^{H}(\lambda,n)$, and so the proof is complete. □

By simple computation, we shown that the extremal function

$$f(z)=z-\frac{\sqrt{2}-2\alpha\cos\lambda}{2^{n+2}(3+\alpha\cos\lambda)}z^2-(-1)^n\frac{\sqrt{2}-2\alpha\cos\lambda}{2^{n+2}(3+\alpha\cos\lambda)}\bar{z}^2,$$

satisfying the condition of Theorem 2.6, hence $f\in NSP_{\alpha}^{H}(\lambda,n)$.

3. Convolutoin condition

Definition 3.1. If f given by (1), we expand the Salagean integral operator defined in (6) can be extended for harmonic univalent functions, i.e. $I^s: S_H\longrightarrow S_H$, such that

$$I^s f(z)=I^s\Big(z+\sum_{m=2}^{\infty}a_m z^m+\sum_{m=1}^{\infty}\bar{b}_m\bar{z}^m\Big)=z+\sum_{m=2}^{\infty}\frac{a_m}{m^s}z^m+\sum_{m=1}^{\infty}\frac{\bar{b}_m}{m^s}\bar{z}^m,$$

where $s\in\mathbb{R}$ and $z\in D$.

Definition 3.2. Let $f_n\in NSP_{\alpha}^{H}(\lambda,n)$ given by (4) and $F_n\in NSP_{\alpha}^{H}(\lambda,n)$ given by

$$F_n(z)=z+\sum_{m=2}^{\infty}A_m z^m+(-1)^n\sum_{m=1}^{\infty}\bar{B}_m\bar{z}^m. \tag{13}$$

The modified $\tilde{\circledast}$-convolution of two functions f_n and $F_n\in NS_H$ is the function $(f_n\tilde{\circledast}F_n)$ defined by

$$(f_n\tilde{\circledast}F_n)=z+\sum_{m=2}^{\infty}a_m A_m z^m+\sum_{m=1}^{\infty}\bar{b}_m\bar{B}_m\bar{z}^m.$$

Definition 3.3. The order of $\tilde{\circledast}$-convolution consistence of the triple $(\mathcal{X},\mathcal{Y},\mathcal{Z})$, where $\mathcal{X},\mathcal{Y}$, and $\mathcal{Z}$ are subclasses of NS_H, is denoted by $S_{\tilde{\circledast}}$, where

$$S_{\tilde{\circledast}}(\mathcal{X},\mathcal{Y},\mathcal{Z})=\min\big\{s\in\mathbb{R}: I^s(f\tilde{\circledast}F)\in\mathcal{Z},\ \forall f\in\mathcal{X},\ \forall F\in\mathcal{Y}\big\}.$$

Theorem 3.4. *Let $|z| < R \leq 1$, $n \in \mathbb{N}$, λ real with $|\lambda| < \pi/2$, $0 \leq \beta < 1$ and f given by* (1), *then*

$$Re\Big\{e^{i\lambda}\frac{D^{n+1}f(z)}{D^n f(z)}\Big\} > \beta\cos\lambda, \tag{14}$$

if and only if

$$\frac{1}{z}\Big[h * \frac{z + \Big(\frac{(x+1)e^{-i\lambda}}{2(1-\beta)\cos\lambda} - 1\Big)z^2}{(1-z)^{n+2}}\Big] - (-1)^n\frac{1}{\bar{z}}\overline{\Big[g * \frac{z + \Big(\frac{(\bar{x}+1)e^{i\lambda}}{2(\bar{x}+1)-2(1-\beta)\cos\lambda} - 1\Big)z^2}{(1-z)^{n+2}}\Big]} \neq 0, \tag{15}$$

where $|x| = 1$, $x \neq -1$ and $z \in D$.

Proof. The inequality (14) is equivalent to

$$Re\Big\{\frac{e^{i\lambda}\frac{D^{n+1}f(z)}{D^n f(z)} - \beta\cos\lambda - i\sin\lambda}{(1-\beta)\cos\lambda}\Big\} > 0. \tag{16}$$

Since $\frac{D^{n+1}f(z)}{D^n f(z)} = 1$ at $z = 0$, (16) is equivalent to

$$\frac{e^{i\lambda}\frac{D^{n+1}f(z)}{D^n f(z)} - \beta\cos\lambda - i\sin\lambda}{(1-\beta)\cos\lambda} \neq \frac{x-1}{x+1},$$

where $|x| = 1, x \neq -1$ and $0 < |z| < 1$, which simplifies to

$$e^{i\lambda}(x+1)D^{n+1}f(z) - \Big((x+1)(\beta\cos\lambda - i\sin\lambda) + (x-1)(\cos\lambda - \beta\cos\lambda)\Big)D^n f(z) \neq 0.$$

The above equation is equivalent to

$$e^{i\lambda}(x+1)D^{n+1}f(z) - i(x+1)\sin\lambda D^n f(z) - (x+2\beta-1)\cos\lambda D^n f(z) \neq 0. \tag{17}$$

By applying (3) in (17), we obtain

$$\begin{aligned} &e^{i\lambda}(x+1)\Big(D^{n+1}h(z) - (-1)^n\overline{D^{n+1}g(z)}\Big) - i(x+1)\sin\lambda\Big(D^n h(z) + (-1)^n\overline{D^n g(z)}\Big) \\ &\qquad - (x+2\beta-1)\cos\lambda\Big(D^n h(z) + (-1)^n\overline{D^n g(z)}\Big) \neq 0. \end{aligned} \tag{18}$$

Setting h and g by (2), we have

$$D^{n+1}h(z) = \frac{z}{(1-z)^2} * D^n h(z), \qquad D^n h(z) = D^n h(z) * \frac{z}{1-z},$$
$$D^{n+1}g(z) = \frac{z}{(1-z)^2} * D^n g(z), \qquad D^n g(z) = D^n g(z) * \frac{z}{1-z}.$$

So that the inequality (18) may be expressed as

$$\begin{aligned} &\frac{1}{z}\Big\{D^n h(z) * \frac{z + \frac{(x+2\beta-1)\cos\lambda - i(x+1)\sin\lambda}{2(1-\beta)\cos\lambda}z^2}{(1-z)^2}\Big\} \\ &-(-1)^n\frac{1}{\bar{z}}\Big\{\overline{D^n g(z)} * \overline{\frac{z - \frac{i(\bar{x}+1)\sin\lambda + (\bar{x}+2\beta-1)\cos\lambda}{2(\bar{x}+1)e^{i\lambda} + 2(\beta-1)\cos\lambda}z^2}{(1-z)^2}}\Big\} \neq 0. \end{aligned}$$

The above equation simplifies to

$$\frac{1}{z}\Big\{D^n h(z) * \frac{z+\Big(\frac{(x+1)e^{-i\lambda}}{2(1-\beta)\cos\lambda}-1\Big)z^2}{(1-z)^2}\Big\}$$
$$-(-1)^n\frac{1}{\bar{z}}\Big\{\overline{D^n g(z) * \frac{z+\Big(\frac{(\bar{x}+1)e^{i\lambda}}{2(\bar{x}+1)e^{i\lambda}+2(\beta-1)\cos\lambda}-1\Big)z^2}{(1-z)^2}}\Big\}\neq 0.$$

Since $D^n(f*g)(z)=D^nf(z)*g(z)=f(z)*D^ng(z)$, we obtain that

$$\frac{1}{z}\left\{h(z)*\left(D^n\Big(\frac{z}{(1-z)^2}\Big)+\Big(\frac{(\bar{x}+1)e^{i\lambda}}{2(\beta-1)\cos\lambda}-1\Big)D^n\Big(\frac{z^2}{(1-z)^2}\Big)\right)\right\}$$
$$-(-1)^n\frac{1}{\bar{z}}\left\{\overline{g(z)*\left(D^n\Big(\frac{z}{(1-z)^2}\Big)+\Big(\frac{(\bar{x}+1)e^{i\lambda}}{2(\bar{x}+1)e^{i\lambda}+2(\beta-1)\cos\lambda}-1\Big)D^n\Big(\frac{z^2}{(1-z)^2}\Big)\right)}\right\}\neq 0.$$

Since $D^n\Big(\frac{z}{(1-z)^2}\Big)=\sum_{m=1}^{\infty}m^{n+1}z^m$ and $D^n\Big(\frac{z^2}{(1-z)^2}\Big)=\sum_{m=2}^{\infty}m^{n+1}z^m-\sum_{m=2}^{\infty}m^nz^n$, we get

$$\frac{1}{z}\left\{h*\left(z+\sum_{m=2}^{\infty}m^{n+1}z^m+\Big(\frac{(x+1)e^{-i\lambda}}{2(1-\beta)\cos\lambda}-1\Big)\Big(\sum_{m=2}^{\infty}m^{n+1}z^m-\sum_{m=2}^{\infty}m^nz^m\Big)\right)\right\}$$
$$-(-1)^n\frac{1}{\bar{z}}\left\{\overline{g*\left(z+\sum_{m=2}^{\infty}m^{n+1}z^m+\Big(\frac{(\bar{x}+1)e^{i\lambda}}{2(\bar{x}+1)-2(1-\beta)\cos\lambda}-1\Big)\Big(\sum_{m=2}^{\infty}m^{n+1}z^m-\sum_{m=2}^{\infty}m^nz^m\Big)\right)}\right\}\neq 0,$$

which simplifies to

$$\frac{1}{z}\left\{h*\left(\frac{(x+1)e^{-i\lambda}}{2(1-\beta)\cos\lambda}\sum_{m=1}^{\infty}m^{n+1}z^m-\Big(\frac{(x+1)e^{-i\lambda}}{2(1-\beta)\cos\lambda}-1\Big)\sum_{m=1}^{\infty}m^nz^m\right)\right\}$$
$$-\frac{1}{\bar{z}}\left\{\overline{g*\left(\frac{(\bar{x}+1)e^{i\lambda}}{2(\bar{x}+1)e^{i\lambda}-2(1-\beta)\cos\lambda}\sum_{m=1}^{\infty}m^{n+1}z^m\right.}\right.$$
$$\left.\left.\overline{-\Big(\frac{(\bar{x}+1)e^{i\lambda}}{2(\bar{x}+1)e^{i\lambda}-2(1-\beta)\cos\lambda}-1\Big)\sum_{m=1}^{\infty}m^nz^m\right)}\right\}\neq 0.$$

Since $\sum_{m=1}^{\infty}m^{n+1}z^m=\frac{z}{(1-z)^{n+2}}$ and $\sum_{m=1}^{\infty}m^nz^m=\frac{z}{(1-z)^{n+1}}$, we have

$$\frac{1}{z}\left\{h*\left(\frac{(x+1)e^{-i\lambda}}{2(1-\beta)\cos\lambda}\frac{z}{(1-z)^{n+2}}+\Big(1-\frac{(x+1)e^{-i\lambda}}{2(1-\beta)\cos\lambda}\Big)\frac{z}{(1-z)^{n+1}}\right)\right\}$$
$$-\frac{1}{\bar{z}}\left\{\overline{g*\left(\frac{(\bar{x}+1)e^{i\lambda}}{2(\bar{x}+1)e^{i\lambda}-2(1-\beta)\cos\lambda}\frac{z}{(1-z)^{n+2}}\right.}\right.$$
$$\left.\left.\overline{+\Big(1-\frac{(\bar{x}+1)e^{i\lambda}}{2(\bar{x}+1)e^{i\lambda}-2(1-\beta)\cos\lambda}-1\Big)\frac{z}{(1-z)^{n+1}}\right)}\right\}\neq 0. \tag{19}$$

Thus the inequality (19) is equivalent to (15), and this completes the proof of theorem. □

Theorem 3.5. *Let* $0 \le \alpha < 1, n \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$ *such that* $\sqrt{2} - 2\alpha\cos\lambda > 0$ *and* $f_n, F_n \in NSP_\alpha^H(\lambda, n)$ *given by* (4) *and* (13), *respectively. Then* $I^s(f_n \tilde{\circledast} F_n) \in NSP_\alpha^H(\lambda, n), s \in \mathbb{R}$ *and*

$$s \ge n + \log_2 \frac{6 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda}. \tag{20}$$

This result is sharp, and we have

$$S_{\tilde{\circledast}}\Big(NSP_\alpha^H(\lambda, n), NSP_\alpha^H(\lambda, n), NSP_\alpha^H(\lambda, n)\Big) = n + \log_2 \frac{6 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda}. \tag{21}$$

Proof. If f_n and F_n are of forms (4) and (13), respectively, we have

$$\sum_{m=2}^{\infty} m^n a_m \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} + \sum_{m=1}^{\infty} m^n b_m \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} \le 1,$$

and

$$\sum_{m=2}^{\infty} m^n A_m \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} + \sum_{m=1}^{\infty} m^n B_m \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} \le 1.$$

By applying the Cauchy-Schwarz inequality, we obtain

$$\sum_{m=2}^{\infty} m^n \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} \sqrt{a_m A_m} + \sum_{m=1}^{\infty} m^n \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} \sqrt{b_m B_m} \le 1. \tag{22}$$

We need to find condition on s such that

$$\sum_{m=2}^{\infty} m^{n-s} \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} a_m A_m + \sum_{m=1}^{\infty} m^{n-s} \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} b_m B_m \le 1.$$

Thus, it is sufficient to show that

$$m^{n-s} \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} a_m A_m \le m^n \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} \sqrt{a_m A_m},$$

for $m = 2, 3, ...$, and

$$m^{n-s} \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} b_m B_m \le m^n \frac{2m + 2 + 2\alpha\cos\lambda}{\sqrt{2} - 2\alpha\cos\lambda} \sqrt{b_m B_m}.$$

for $m = 1, 2, 3, ...$ that is

$$\sqrt{a_m A_m} \le m^s, \qquad \sqrt{b_m B_m} \le m^s. \tag{23}$$

From (22), we know that

$$\sqrt{a_m A_m} \le m^{-n} \frac{\sqrt{2} - 2\alpha\cos\lambda}{2m + 2 + 2\alpha\cos\lambda}, \qquad m = 2, 3, 4, ..., \tag{24}$$

and

$$\sqrt{b_m B_m} \le m^{-n} \frac{\sqrt{2} - 2\alpha\cos\lambda}{2m + 2 + 2\alpha\cos\lambda}, \qquad m = 1, 2, 3, ... \tag{25}$$

Clearly, for $m = 1$ in the inequality (25), we obtain $b_1 B_2 \leq 1$. Furthermore, for $m = \{2, 3, 4, ...\}$ in (24) and (25), it is sufficient to have

$$m^{-n} \frac{\sqrt{2} - 2\alpha \cos \lambda}{2m + 2 + 2\alpha \cos \lambda} \leq m^s, \qquad m = 2, 3, 4, ...,$$

or equivalently

$$m^{-n-s} \frac{\sqrt{2} - 2\alpha \cos \lambda}{2m + 2 + 2\alpha \cos \lambda} \leq 1, \qquad m = 2, 3, 4, ... \tag{26}$$

Now, letting $\phi(x) = \frac{x^{-(n+s)}(\sqrt{2}-2\alpha\cos\lambda)}{2x+2+2\alpha\cos\lambda}$ for $x \geq 2$, we obtain

$$\phi(x) = \frac{(\sqrt{2} - 2\alpha \cos \lambda)\Big(- (n+s)x^{-(n+s+1)}(2x + 2 + 2\alpha \cos \lambda) - 2^{-(n+s)}\Big)}{(2x + 2 + 2\alpha \cos \lambda)^2}.$$

Hence, $\phi'(x) \leq 0$ for all $x \geq 2$, or $\phi(x)$ is a decreasing function on x, consequently, from (26) it is sufficient to have

$$2^{-(n+s)} \frac{\sqrt{2} - 2\alpha \cos \lambda}{6 + 2\alpha \cos \lambda} \leq 1. \tag{27}$$

But the inequality (27) holds for s satisfying (20) and this shows that

$$S_{\tilde{\circledast}}\Big(NSP_\alpha^H(\lambda, n), NSP_\alpha^H(\lambda, n), NSP_\alpha^H(\lambda, n)\Big) \leq n + \log_2 \frac{6 + 2\alpha \cos \lambda}{\sqrt{2} - 2\alpha \cos \lambda}. \tag{28}$$

Finally, by using the extremal function

$$f(z) = z - \frac{\sqrt{2} - 2\alpha \cos \lambda}{2^{n+2}(3 - \alpha \cos \lambda)} z^{2+1} - (-1)^n \frac{\sqrt{2} - 2\alpha \cos \lambda}{2^{+2}n(3 - \alpha \cos \lambda)} \bar{z}^2,$$

from (3.2), we obtain that

$$I^s(f \tilde{\circledast} f)(z) = z + \frac{(\sqrt{2} - 2\alpha \cos \lambda)^2}{2^{2(n+2)+s}(3 - \alpha \cos \lambda)^2} z^2 + \frac{(\sqrt{2} - 2\alpha \cos \lambda)^2}{2^{2(n+2)+s}(3 + \alpha \cos \lambda)^2} \bar{z}^2.$$

But from (9) in Theorem 2.6, we deduced

$$I^s(f \tilde{\circledast} f)(z) = z - \frac{\sqrt{2} - 2\alpha \cos \lambda}{2^{n+2}(3 - \alpha \cos \lambda)} z^2 - (-1)^n \frac{\sqrt{2} - 2\alpha \cos \lambda}{2^{n+2}(3 - \alpha \cos \lambda)} \bar{z}^2 \in SP_\alpha^H(\lambda, n), \tag{29}$$

and (29) shows that the inequality (20) is sharp and we have

$$S_{\tilde{\circledast}}\Big(NSP_\alpha^H(\lambda, n), NSP_\alpha^H(\lambda, n), NSP_\alpha^H(\lambda, n)\Big) \geq n + \log_2 \frac{6 + 2\alpha \cos \lambda}{\sqrt{2} - 2\alpha \cos \lambda}. \tag{30}$$

Therefore for from (28) and (30), the relation (21) holds true and the proof of the theorem is complete. □

Corollary 3.6. *Let $0 \leq \alpha < 1$, $\lambda \in (-\pi/2, \pi/2)$ such that $\sqrt{2} - 2\alpha \cos \lambda > 0$. We have the following $\tilde{\circledast}$-convolution consistences*

$$S_{\tilde{\circledast}}\Big(NSP_\alpha^H(\lambda), NSP_\alpha^H(\lambda), NSP_\alpha^H(\lambda)\Big) = \log_2 \frac{6 + 2\alpha \cos \lambda}{\sqrt{2} - 2\alpha \cos \lambda},$$

and

$$S_{\tilde{\circledast}}\Big(NCSP_\alpha^H(\lambda), NCSP_\alpha^H(\lambda), NCSP_\alpha^H(\lambda)\Big) = 1 + \log_2 \frac{6+2\alpha\cos\lambda}{\sqrt{2}-2\alpha\cos\lambda}.$$

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