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NEUTROSOPHIC $\mathcal{N}$−STRUCTURES ON SHEFFER STROKE BE-ALGEBRAS

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ABSTRACT. In this study, a neutrosophic $\mathcal{N}$−subalgebra, a (implicative) neutrosophic $\mathcal{N}$− filter, level sets of these neutrosophic $\mathcal{N}$−structures and their properties are introduced on a Sheffer stroke BE-algebras (briefly, SBE-algebras). It is proved that the level set of neutrosophic $\mathcal{N}$− subalgebras ((implicative) neutrosophic $\mathcal{N}$−filter) of this algebra is the SBE-subalgebra ((implicative) SBE-filter) and vice versa. Then we present relationships between upper sets and neutrosophic $\mathcal{N}$−filters of this algebra. Also, it is given that every neutrosophic $\mathcal{N}$−filter of a SBE-algebra is its neutrosophic $\mathcal{N}$−subalgebra but the inverse is generally not true. We study on neutrosophic $\mathcal{N}$−filters of SBE-algebras by means of SBE-homomorphisms, and present relationships between mentioned structures on a SBE-algebra in detail. Finally, certain subsets of a SBE-algebra are determined by means of $\mathcal{N}$−functions and some properties are examined.

Keywords: SBE-algebra, (implicative) SBE-filter, neutrosophic $\mathcal{N}$− subalgebra, (implicative) neutrosophic $\mathcal{N}$−filter.
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1. Introduction

Sheffer stroke which is one of the two operators that can be used by itself, without any other logical operators, was originally introduced by H. M. Sheffer to build a logical formal system [19]. Since it provides new, basic and easily applicable axiom systems for many algebraic structures owing to its commutative property, this operation has many applications in algebraic structures such as orthoimplication algebras [1], ortholattices [4], Boolean algebras [11], strong Sheffer stroke non-associative MV-algebras [5], filters [14] and neutrosophic N-structures [15], Sheffer Stroke Hilbert Algebras [12], fuzzy filters [13] and neutrosophic N-structures [16]. On the other hand, H S. Kim and Y. H. Kim introduced BE-algebras as a generalization of a dual BCK-algebra and defined filters and upper sets on this algebra [10]. Also, some types of filters in BE-algebras [3] and some results in BE-algebras [17]. Recently, Katican et

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al. introduced BE-algebras with Sheffer stroke and investigated upper sets, SBE-filters and SBE-homomorphisms [8].

The fuzzy set theory which has the truth (t) (membership) function and positive meaning of information was introduced by L. Zadeh [23]. Therefore, researchers deal with negative meaning of information. For this propose, Atanassov introduced the intuitionistic fuzzy set theory [2] which is a type of fuzzy sets and has truth (t) (membership) and the falsehood (f) (nonmembership) functions. Then, Smarandache introduced the neutrosophic set theory which is the intuitionistic fuzzy set theory and has the indeteminacy/neutrality (i) function with membership and nonmembership functions [20, 21]. Hence, neutrosophic sets are defined on three components (t, i, f) [24]. In recent times, neutrospohic sets are applied to the algebraic structures such as BCK/BCI-algebras and BE-algebras ([6], [7], [9], [18], [22]).

In the second section, basic definitions and notions on Sheffer stroke BE-algebras, neutrosophic $\mathcal{N}-$functions and neutrosophic $\mathcal{N}-$structures are presented (briefly, SBE-algebra). In third section, a neutrosophic $\mathcal{N}-$subalgebra and a level set on neutrosophic $\mathcal{N}-$structures are defined on SBE-algebras. Then it is shown that the level set of a neutrosophic $\mathcal{N}-$subalgebra on a SBE-algebra is its SBE-subalgebra and vice versa, and that the family of all neutrosophic $\mathcal{N}-$subalgebras of the algebraic structure forms a complete distributive modular lattice. In fourth section, a (implicative) neutrosophic $\mathcal{N}-$filter and a (implicative) neutrosophic $\mathcal{N}-$filter of a SBE-algebra are defined, and some properties are investigated. Also, a neutrosophic $\mathcal{N}-$filter of a SBE-algebra is restated by means of upper sets. We demonstrate that every implicative neutrosophic $\mathcal{N}-$filter of a SBE-algebra is the neutrosophic $\mathcal{N}-$filter but the inverse is generally not true. It is propounded that level set of a (implicative) neutrosophic $\mathcal{N}-$filter of a SBE-algebra is its (implicative) SBE-filter and the inverse always holds. Indeed, it is proved that a neutrosophic $\mathcal{N}-$structure on a SBE-algebra defined by means of a (implicative) neutrosophic $\mathcal{N}-$filter of another SBE-algebra and a surjective SBE-homomorphism is a (implicative) neutrosophic $\mathcal{N}-$filter. We illustrate that every neutrosophic $\mathcal{N}-$filter of a SBE-algebra is the neutrosophic $\mathcal{N}-$subalgebra but the inverse is mostly not valid. Besides, the cases which a neutrosophic $\mathcal{N}-$filter of a SBE-algebra is an implicative neutrosophic $\mathcal{N}-$filter are analyzed. Special subsets of a SBE-algebra are described by $\mathcal{N}-$functions and it is shown that these subsets are (implicative) SBE-filters of SBE-algebra for its (implicative) $\mathcal{N}-$filter. Finally, we determine new subsets by means of the $\mathcal{N}-$functions and some elements of a SBE-algebra and prove that these subsets are (implicative) SBE-filters of a SBE-algebra for a (implicative) neutrosophic $\mathcal{N}-$filter of this algebraic structure but the inverse does not hold in general.

2. Preliminaries

In this section, basic definitions and notions about Sheffer stroke BE-algebras (for short, SBE-algebras) and neutrosophic $\mathcal{N}-$structures on crispy sets are given.

Definition 2.1. [4] Let $\mathcal{S} = \langle S, \circ\rangle$ be a groupoid. The operation $\circ$ on S is said to be a *Sheffer operation (or Sheffer stroke)* if it satisfies the following conditions for all $x, y, z \in S$:
$(S1)$ $x \circ y = y \circ x$,
$(S2)$ $(x \circ x) \circ (x \circ y) = x$,
$(S3)$ $x \circ ((y \circ z) \circ (y \circ z)) = ((x \circ y) \circ (x \circ y)) \circ z$,
$(S4)$ $(x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y))) = x$.

Definition 2.2. [8] A Sheffer stroke BE-algebra (shortly, SBE-algebra) is a structure $\langle S; \circ, 1\rangle$ of type $(2, 0)$ such that 1 is the constant in S, $\circ$ is a Sheffer operation on S and the following axioms are satisfied for all $x, y, z \in S$:
$(SBE-1)$ $x \circ (x \circ x) = 1$,
$(SBE-2)$ $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$.

Lemma 2.3. [8] *Let $\langle S; \circ, 1\rangle$ be a SBE-algebra. Then the following hold for all $x, y \in S$:*

(i) $x \circ (1 \circ 1) = 1$,
(ii) $1 \circ (x \circ x) = x$,
(iii) $x \circ ((y \circ (x \circ x)) \circ (y \circ (x \circ x))) = 1$,
(iv) $x \circ (((x \circ (y \circ y)) \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (y \circ y))) = 1$,
(v) $(x \circ 1) \circ (x \circ 1) = x$,
(vi) (vi) $((x \circ y) \circ (x \circ y)) \circ (x \circ x) = 1$ *and* $((x \circ y) \circ (x \circ y)) \circ (y \circ y) = 1$,
(vii) $x \circ ((x \circ y) \circ (x \circ y)) = x \circ y = ((x \circ y) \circ (x \circ y)) \circ y$.

Definition 2.4. [8] A SBE-algebra $\langle S; \circ, 1\rangle$ is called self-distributive if

$$x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = (x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))),$$

for any $x, y, z \in S$.

Definition 2.5. [8] Let $\langle S; \circ, 1\rangle$ be a SBE-algebra. Define a relation $\preceq$ on S by

$$x \preceq y \text{ if and only if } x \circ (y \circ y) = 1,$$

for all $x, y \in S$. The relation is not a partial order on S, since it is only reflexive by $(SBE-1)$.

Lemma 2.6. [8] *Let $\langle S; \circ, 1\rangle$ be a SBE-algebra. Then*

(i) *if $x \preceq y$, then $y \circ y \preceq x \circ x$,*
(ii) $x \preceq y \circ (x \circ x)$,
(iii) $y \preceq (y \circ (x \circ x)) \circ (x \circ x)$,
(iv) *if S is self-distributive, then $x \preceq y$ implies $y \circ z \preceq x \circ z$,*

(v) *if S is self-distributive, then* $y \circ (z \circ z) \preceq (z \circ (x \circ x)) \circ ((y \circ (x \circ x)) \circ (y \circ (x \circ x)))$.

Definition 2.7. [8] Let $\langle S; \circ, 1\rangle$ be a SBE-algebra. Then a nonempty subset $F \subseteq S$ is called a SBE-filter of S if it satisfies the following properties:
$(SBEf-1)$ $1 \in F$,
$(SBEf-2)$ For all $x, y \in S$, $x \circ (y \circ y) \in F$ and $x \in F$ imply $y \in F$.

Lemma 2.8. [8] *Let $\langle S; \circ, 1\rangle$ be a SBE-algebra. Then a nonempty subset $F \subseteq S$ is a SBE-filter of S if and only if for all $x, y \in S$*

(i) *$x \in F$ and $y \in F$ imply $(x \circ y) \circ (x \circ y) \in F$,*
(ii) *$x \in F$ and $x \preceq y$ imply $y \in F$.*

Definition 2.9. [8] Let $\langle S; \circ, 1\rangle$ be a SBE-algebra, $x, y \in S$ and define $U(x, y) = \{z \in S : x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = 1\}$. Then $U(x, y)$ is called an upper set of x and y. For $x, y \in S$, $U(x, y)$ is not a SBE-filter of S in general.

Definition 2.10. [8] A subset T of a SBE-algebra $\langle S; \circ, 1\rangle$ is called a SBE-subalgebra of S if $x \circ (y \circ y) \in T$, for $x, y \in T$. Clearly, S itself and $\{1\}$ are SBE-subalgebras of S.

Definition 2.11. [8] Let $\langle S; \circ_S, 1_S\rangle$ and $\langle P; \circ_P, 1_P\rangle$ be SBE-algebras. A mapping $f : S \longrightarrow P$ is called a SBE-homomorphism if $f(x \circ_S y) = f(x) \circ_P f(y)$, for all $x, y \in S$ and $f(1_S) = 1_P$.

Definition 2.12. [6] $\mathcal{F}(X, [-1, 0])$ denotes the collection of functions from a set X to $[-1, 0]$ and a element of $\mathcal{F}(X, [-1, 0])$ is called a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}-$function on X). An $\mathcal{N}-$structure refers to an ordered pair (X, f) of X and $\mathcal{N}-$function f on X.

Definition 2.13. [9] A neutrosophic $\mathcal{N}-$structure over a nonempty universe X is defined by $X_N := \dfrac{X}{(T_N, I_N, F_N)} = \{\dfrac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\}$ where T_N, I_N and F_N are $\mathcal{N}-$function on X, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}-$structure X_N over X satisfies the condition

$$(\forall x \in X)(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0).$$

Definition 2.14. [7] Let X_N be a neutrosophic $\mathcal{N}-$structure on a set X and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$T_N^\alpha := \{x \in X : T_N(x) \leq \alpha\},$$
$$I_N^\beta := \{x \in X : I_N(x) \geq \beta\}$$

and

$$F_N^\gamma := \{x \in X : F_N(x) \leq \gamma\}.$$

The set $X_N(\alpha, \beta, \gamma) := \{x \in X : T_N(x) \leq \alpha, I_N(x) \geq \beta \text{ and } T_N(x) \leq \gamma\}$ is called the $(\alpha, \beta, \gamma)-$level set of X_N. Moreover, $X_N(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$.

Consider sets

$$X_N^{w_t} := \{x \in X : T_N(x) \leq T_N(w_t)\},$$

$$X_N^{w_i} := \{x \in X : I_N(x) \geq I_N(w_i)\}$$

and

$$X_N^{w_f} := \{x \in X : F_N(x) \leq F_N(w_f)\},$$

for any $w_t, w_i, w_f \in X$. Obviously, $w_t \in X_N^{w_t}, w_i \in X_N^{w_i}$ and $w_f \in X_N^{w_f}$ [7].

3. Neutrosophic $\mathcal{N}-$subalgebras

In this section, we present neutrosophic $\mathcal{N}-$subalgebras of SBE-algebras and some properties. Unless indicated otherwise, S states a SBE-algebra.

Definition 3.1. A neutrosophic $\mathcal{N}-$subalgebra S_N of a SBE-algebra S is a neutrosophic $\mathcal{N}-$structure on S satisfying the condition

$$(1) \qquad \begin{array}{c} \min\{T_N(x), T_N(y)\} \leq T_N(x \circ (y \circ y)), \\ I_N(x \circ (y \circ y)) \leq \max\{I_N(x), I_N(y)\} \\ and \\ F_N(x \circ (y \circ y)) \leq \max\{F_N(x), F_N(y)\}, \end{array}$$

for all $x, y \in S$.

Example 3.2. Consider a SBE-algebra S where the set $S = \{0, u, v, w, t, 1\}$ and the Sheffer operation $\circ$ on S has Table 1 [8]:

Table 1. Cayley table of Sheffer operation $\circ$ on S in Example 3.2

$\circ$	0	u	v	w	t	1
0	1	1	1	1	1	1
u	1	v	1	1	1	v
v	1	1	u	1	1	u
w	1	1	1	t	1	t
t	1	1	1	1	w	w
1	1	v	u	t	w	0

Then a neutrosophic $\mathcal{N}-$structure

$$S_N = \{\frac{x}{(-0.87, -0.1, -0.2)} : x \in S - \{1\}\} \cup \{\frac{1}{(-0.03, -1, -0.78)}\}$$

on S is a neutrosophic $\mathcal{N}-$subalgebra of S.

Definition 3.3. Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. For the sets

$$T_N^{\alpha} := \{x \in S : \alpha \leq T_N(x)\},$$
$$I_N^{\beta} := \{x \in S : I_N(x) \leq \beta\}$$

and

$$F_N^{\gamma} := \{x \in S : F_N(x) \leq \gamma\},$$

the set $S_N(\alpha, \beta, \gamma) := \{x \in S : \alpha \leq T_N(x), I_N(x) \leq \beta \text{ and } F_N(x) \leq \gamma\}$ is called the $(\alpha, \beta, \gamma)-$level set of S_N. Also, $S_N(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$.

Theorem 3.4. *Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S and α, β, γ be any elements of $[-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. If S_N is a neutrosophic $\mathcal{N}-$subalgebra of S, then the nonempty level set $S_N(\alpha, \beta, \gamma)$ of S_N is a SBE-subalgebra of S.*

Proof. Let S_N be a neutrosophic $\mathcal{N}-$subalgebra of S and x, y be any elements of $S_N(\alpha, \beta, \gamma)$, for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $\alpha \leq T_N(x), T_N(y); I_N(x), I_N(y) \leq \beta$ and $F_N(x), F_N(y) \leq \gamma$. Since

$$\alpha \leq \min\{T_N(x), T_N(y)\} \leq T_N(x \circ (y \circ y)),$$
$$I_N(x \circ (y \circ y)) \leq \max\{I_N(x), I_N(y)\} \leq \beta$$

and

$$F_N(x \circ (y \circ y)) \leq \max\{F_N(x), F_N(y)\} \leq \gamma,$$

for all $x, y \in S$, it is obtained that $x \circ (y \circ y) \in T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$, and so, $x \circ (y \circ y) \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma} = S_N(\alpha, \beta, \gamma)$. Hence, $S_N(\alpha, \beta, \gamma)$ is a SBE-subalgebra of S. $\square$

Theorem 3.5. *Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S and $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} be SBE-subalgebras of S, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then S_N is a neutrosophic $\mathcal{N}-$subalgebra of S.*

Proof. Let $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} be SBE-subalgebras of S, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Suppose that $\alpha_1 = T_N(x \circ (y \circ y)) < \min\{T_N(x), T_N(y)\} = \alpha_2$. If $\alpha = \frac{1}{2}(\alpha_1 + \alpha_2) \in [-1, 0)$, then $\alpha_1 < \alpha < \alpha_2$. So, $x, y \in T_N^{\alpha}$ but $x \circ (y \circ y) \notin T_N^{\alpha}$ which is a contradiction. Thus, $\min\{T_N(x), T_N(y)\} \leq T_N(x \circ (y \circ y))$, for all $x, y \in S$. Assume that $\beta_1 = \max\{I_N(x), I_N(y)\} < I_N(x \circ (y \circ y)) = \beta_2$. If $\beta = \frac{1}{2}(\beta_1 + \beta_2) \in [-1, 0)$, then $\beta_1 < \beta < \beta_2$. Hence, $x, y \in I_N^{\beta}$ but $x \circ (y \circ y) \notin I_N^{\beta}$ which is a contradiction. Thus, $I_N(x \circ (y \circ y)) \leq \max\{I_N(x), I_N(y)\}$, for all $x, y \in S$. Moreover, suppose that $\gamma_1 = \max\{F_N(x), F_N(y)\} < F_N(x \circ (y \circ y)) = \gamma_2$. If $\gamma = \frac{1}{2}(\gamma_1 + \gamma_2) \in [-1, 0)$, then $\gamma_1 < \gamma < \gamma_2$. Thus, $x, y \in F_N^{\gamma}$ whereas $x \circ (y \circ y) \notin F_N^{\gamma}$ which is a contradiction. Thereby, $F_N(x \circ (y \circ y)) \leq \max\{F_N(x), F_N(y)\}$, for all $x, y \in S$. Therefore, S_N is a neutrosophic $\mathcal{N}-$subalgebra of S. $\square$

Theorem 3.6. *Let $\{S_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}-$subalgebras of a SBE-algebra S. Then $\{S_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive modular lattice.*

Proof. Let T be a nonempty subset of $\{S_{N_i} : i \in \mathbb{N}\}$. Since S_{N_i} is a neutrosophic $\mathcal{N}-$subalgebra of S, for all $i \in \mathbb{N}$, it satisfies the condition (1) for all $x, y \in S$. Then $\bigcap T$ satisfies the condition (1). Thus, $\bigcap T$ is a neutrosophic $\mathcal{N}-$subalgebra of S. Let P be a family of all neutrosophic $\mathcal{N}-$subalgebras of S containing $\bigcup\{S_{N_i} : i \in \mathbb{N}\}$. So, $\bigcap P$ is a neutrosophic $\mathcal{N}-$subalgebra of S. If $\bigwedge_{i\in\mathbb{N}} S_{N_i} = \bigcap_{i\in\mathbb{N}} S_{N_i}$ and $\bigvee_{i\in\mathbb{N}} S_{N_i} = \bigcap P$, then $(\{S_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ forms a complete lattice. Moreover, it is distibutive by the definitions of $\bigvee$ and $\bigwedge$. Since every distributive lattice is a modular lattice, the lattice is modular. □

Lemma 3.7. *Let S_N be a neutrosophic $\mathcal{N}-$subalgebra of a SBE-algebra S. Then*

$$T_N(x) \leq T_N(1), I_N(1) \leq I_N(x) \text{ and } F_N(1) \leq F_N(x), \tag{2}$$

for all $x \in S$.

Proof. It is clear from (SBE-1). □

The inverse of Lemma 3.7 does not usually hold.

Example 3.8. Consider the SBE-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$S_N = \{\frac{v}{(-0.91, -0.4, -0.5)}\} \cup \{\frac{x}{(0, -0.7, -0.8)} : x \in S - \{v\}\}$$

on S satisfies the condition (2) but it is not a neutrosophic $\mathcal{N}-$subalgebra of S since $\max\{F_N(u), F_N(0)\} = -0.8 < -0.5 = F_N(v) = F_N(u \circ (0 \circ 0))$.

Lemma 3.9. *A neutrosophic $\mathcal{N}-$subalgebra S_N of a SBE-algebra S satisfies $T_N(x) \leq T_N(x \circ (y \circ y))$, $I_N(x \circ (y \circ y)) \leq I_N(x)$ and $F_N(x \circ (y \circ y)) \leq F_N(x)$, for all $x, y \in S$ if and only if T_N, I_N and F_N are constant.*

Proof. The proof is obtained from Lemma 2.3 (ii) and Lemma 3.7. □

4. Neutrosophic $\mathcal{N}-$filters

In this section, implicative SBE-filters and (implicative) neutrosophic $\mathcal{N}-$filters of SBE-algebras are introduced. Also, relationships between aforementioned structures are analyzed.

Definition 4.1. A nonempty subset F of a SBE-algebra S is called a implicative SBE-filter of S if it satisfies
$(SIF-1)$ $1 \in F$,
$(SIF-2)$ $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) \in F$ and $x \circ (y \circ y) \in F$ imply $x \circ (z \circ z) \in F$, for all $x, y, z \in S$.

Table 2. Cayley table of Sheffer operation $\circ$ on S in Example 4.2

$\circ$	0	u	v	w	t	1
0	1	1	1	1	1	1
u	1	t	w	1	1	t
v	1	w	w	1	1	w
w	1	1	1	v	u	v
t	1	1	1	u	u	u
1	1	t	w	v	u	0

Example 4.2. Consider a SBE-algebra S in which the set $S = \{0, u, v, w, t, 1\}$ and the Sheffer operation $\circ$ on S has Table 2 [8]: Then $\{w, t, 1\}$ is an implicative SBE-filter of S while $\{v, 1\}$ is not, since $w \circ (u \circ u) = u \notin \{v, 1\}$ when $w \circ ((v \circ (u \circ u)) \circ (v \circ (u \circ u))) = 1 \in \{v, 1\}$ and $w \circ (v \circ v) = v \in \{v, 1\}$.

Lemma 4.3. *Every implicative SBE-filter of a SBE-algebra S is a SBE-filter of S.*

Proof. It follows from Lemma 2.3 (ii) and $(SIF-2)$. □

The inverse of Lemma 4.3 is not true in general.

Example 4.4. Consider a SBE-algebra S in Example 3.2. Then $\{1\}$ is a SBE-filter of S but it is not implicative since $u \circ (v \circ v) = v \notin \{1\}$ when $u \circ ((w \circ (v \circ v)) \circ (w \circ (v \circ v))) = 1 \in \{1\}$ and $u \circ (w \circ w) = 1 \in \{1\}$.

Definition 4.5. A neutrosophic $\mathcal{N}-$structure S_N on a SBE-algebra S is called a neutrosophic $\mathcal{N}-$filter of S if

$$\begin{array}{c} \min\{T_N(x \circ (y \circ y)), T_N(x)\} \le T_N(y) \le T_N(1), \\ I_N(1) \le I_N(y) \le \max\{I_N(x \circ (y \circ y)), I_N(x)\} \\ \textit{and} \\ F_N(1) \le F_N(y) \le \max\{F_N(x \circ (y \circ y)), F_N(x)\}, \end{array} \tag{3}$$

for all $x, y \in S$.

Example 4.6. Consider the SBE-algebra S in Example 4.2. Then a neutrosophic $\mathcal{N}-$structure

$$S_N = \{\frac{x}{(-1, -0.02, 0)} : x = 0, w, t\} \cup \{\frac{x}{(0, -0.2, -0.5)} : x = u, v, 1\}$$

on S is a neutrosophic $\mathcal{N}-$filter of S.

Lemma 4.7. *Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S. Then S_N is a neutrosophic $\mathcal{N}-$filter of S if and only if*

(i) $x \preceq y$ *implies* $T_N(x) \le T_N(y)$, $I_N(y) \le I_N(x)$ *and* $F_N(y) \le F_N(x)$,
(ii) $\min\{T_N(x), T_N(y)\} \le T_N((x \circ y) \circ (x \circ y))$, $I_N((x \circ y) \circ (x \circ y)) \le \max\{I_N(x), I_N(y)\}$ *and* $F_N((x \circ y) \circ (x \circ y)) \le \max\{F_N(x), F_N(y)\}$,

for all $x, y \in S$.

Proof. Let S_N be a neutrosophic $\mathcal{N}-$filter of S.

(i) Assume that $x \preceq y$. Then $x \circ (y \circ y) = 1$. Thus,

$$T_N(x) = \min\{T_N(1), T_N(x)\} = \min\{T_N(x \circ (y \circ y)), T_N(x)\} \leq T_N(y),$$

$$I_N(y) \leq \max\{I_N(x \circ (y \circ y)), I_N(x)\} = \max\{I_N(1), I_N(x)\} \leq I_N(x)$$

and

$$F_N(y) \leq \max\{F_N(x \circ (y \circ y)), F_N(x)\} = \max\{F_N(1), F_N(x)\} \leq F_N(x),$$

for all $x, y \in S$.

(ii) Since $y \preceq y \circ (((x \circ x) \circ (x \circ x)) \circ ((x \circ x) \circ (x \circ x))) = x \circ (x \circ y)$ from Lemma 2.6 (iii), (S1) and (S2), it is obtained from (i) that $\min\{T_N(x), T_N(y)\} \leq \min\{T_N(x), T_N(x \circ (x \circ y))\} = \min\{T_N(x), T_N(x \circ (((x \circ y) \circ (x \circ y)) \circ ((x \circ y) \circ (x \circ y)))))\} \leq T_N((x \circ y) \circ (x \circ y))$, $I_N((x \circ y) \circ (x \circ y)) \leq \max\{I_N(x), I_N(x \circ (((x \circ y) \circ (x \circ y)) \circ ((x \circ y) \circ (x \circ y)))))\} = \max\{I_N(x), I_N(x \circ (x \circ y))\} \leq \max\{I_N(x), I_N(y)\}$ and $F_N((x \circ y) \circ (x \circ y)) \leq \max\{F_N(x), F_N(x \circ (((x \circ y) \circ (x \circ y)) \circ ((x \circ y) \circ (x \circ y)))))\} = \max\{F_N(x), F_N(x \circ (x \circ y))\} \leq \max\{F_N(x), F_N(y)\}$, for all $x, y \in S$.

Conversely, let S_N be a neutrosophic $\mathcal{N}-$structure on S satisfying (i) and (ii). Since $x \preceq 1$ from Lemma 2.3 (i), we have from (i) that $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ and $F_N(1) \leq F_N(x)$, for all $x \in S$. Also, $((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (y \circ y) = (x \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) = 1$ from (S1), (SBE-1) and (SBE-2), and so, it follows that $(x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))) \preceq y$. Then it follows from (i) and (ii) that $\min\{T_N(x \circ (y \circ y)), T_N(x)\} \leq T_N((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \leq T_N(y)$, $I_N(y) \leq I_N((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \leq \max\{I_N(x \circ (y \circ y)), I_N(x)\}$ and $F_N(y) \leq F_N((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \leq \max\{F_N(x \circ (y \circ y)), F_N(x)\}$, for all $x, y \in S$. Therefore, S_N is a neutrosophic $\mathcal{N}-$filter of S. □

Lemma 4.8. *Let S_N be a neutrosophic $\mathcal{N}-$filter of a SBE-algebra S. Then*

(i) $T_N(x) \leq T_N(y \circ (x \circ x))$, $I_N(y \circ (x \circ x)) \leq I_N(x)$ *and* $F_N(y \circ (x \circ x)) \leq F_N(x)$,

(ii) $\min\{T_N(x), T_N(y)\} \leq T_N(x \circ (y \circ y))$, $I_N(x \circ (y \circ y)) \leq \max\{I_N(x), I_N(y)\}$ *and* $F_N(x \circ (y \circ y)) \leq \max\{F_N(x), F_N(y)\}$,

(iii) $T_N(x) \leq T_N((x \circ (y \circ y)) \circ (y \circ y))$, $I_N((x \circ (y \circ y)) \circ (y \circ y)) \leq I_N(x)$ *and* $F_N((x \circ (y \circ y)) \circ (y \circ y)) \leq F_N(x)$,

(iv) $\min\{T_N(x), T_N(y)\} \leq T_N((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z))$, $I_N((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z)) \leq \max\{I_N(x), I_N(y)\}$ *and* $F_N((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z)) \leq \max\{F_N(x), F_N(y)\}$,

for all $x, y, z \in S$.

Proof. Let S_N be a neutrosophic $\mathcal{N}-$filter of a SBE-algebra S. Then

(i) It is proved from Lemma 2.6 (ii) and Lemma 4.7 (i).

(ii) It follows from (1).
(iii) We get from Lemma 2.6 (iii) and Lemma 4.7 (i).
(vi) It is obtained from (iii) and (SBE-2) that

$$\begin{aligned}\min\{T_N(x), T_N(y)\} &\leq \min\{T_N(x), T_N((y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))))\} \\ &\leq \min\{T_N(x), T_N(x \circ (((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z)) \circ ((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z))))\} \\ &\leq T_N((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z)),\end{aligned}$$

$$\begin{aligned}&I_N((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z)) \\ &\leq \max\{I_N(x), I_N(x \circ (((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z)) \circ ((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z))))\} \\ &= \max\{I_N(x), I_N((y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))))\} \\ &\leq \max\{I_N(x), I_N(y)\}\end{aligned}$$

and

$$\begin{aligned}&F_N((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z)) \\ &\leq \max\{F_N(x), F_N(x \circ (((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z)) \circ ((x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (z \circ z))))\} \\ &= \max\{F_N(x), F_N((y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))))\} \\ &\leq \max\{F_N(x), F_N(y)\},\end{aligned}$$

for all $x, y, z \in S$.

□

Theorem 4.9. *Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S. Then S_N is a neutrosophic $\mathcal{N}-$filter of S if and only if*

(4)
$$z \in U(x, y) \textit{ implies } \min\{T_N(x), T_N(y)\} \leq T_N(z), I_N(z) \leq \max\{I_N(x), I_N(y)\} \textit{ and } F_N(z) \leq \max\{F_N(x), F_N(y)\},$$

for all $x, y, z \in S$.

Proof. Let S_N be a neutrosophic $\mathcal{N}-$filter of S and $z \in U(x, y)$. Since $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = 1$, it is obtained that $x \preceq y \circ (z \circ z)$. Then it follows from Lemma 4.7 (i) that $\min\{T_N(x), T_N(y)\} \leq \min\{T_N(y \circ (z \circ z)), T_N(y)\} \leq T_N(z)$, $I_N(z) \leq \max\{I_N(y \circ (z \circ z)), I_N(y)\} \leq \max\{I_N(x), I_N(y)\}$ and $F_N(z) \leq \max\{F_N(y \circ (z \circ z)), F_N(y)\} \leq \max\{F_N(x), F_N(y)\}$, for all $x, y, z \in S$.

Conversely, let S_N be a neutrosophic $\mathcal{N}-$structure on S satisfying the condition (iv). Since $x \circ ((x \circ (1 \circ 1)) \circ (x \circ (1 \circ 1))) = 1$ from Lemma 2.3 (i), we have that $1 \in U(x, x)$, for all $x \in S$. Then $T_N(x) = \min\{T_N(x), T_N(x)\} \leq T_N(1)$, $I_N(1) \leq \max\{I_N(x), I_N(x)\} = I_N(x)$ and $F_N(1) \leq \max\{F_N(x), F_N(x)\} = F_N(x)$, for all $x \in S$. Since $x \circ (((x \circ (y \circ y)) \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (y \circ y))) = 1$ Lemma 2.3 (iv), we obtain that $y \in U(x, x \circ (y \circ y))$. Thus, it follows from the condition (iv) that $\min\{T_N(x \circ (y \circ y)), T_N(x)\} \leq T_N(y)$, $I_N(y) \leq \max\{I_N(x \circ (y \circ y)), I_N(x)\}$ and $F_N(y) \leq \max\{F_N(x \circ (y \circ y)), F_N(x)\}$, for all $x, y \in S$. Hence, S_N is a neutrosophic $\mathcal{N}-$filter of S. □

Definition 4.10. A neutrosophic $\mathcal{N}-$structure S_N on a SBE-algebra S is called an implicative neutrosophic $\mathcal{N}-$filter of S if it satisfies
(inf-1) $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ and $F_N(1) \leq F_N(x)$,
(inf-2) $\min\{T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))\} \leq T_N(x \circ (z \circ z))$,
$I_N(x \circ (z \circ z)) \leq \max\{I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y))\}$ and
$F_N(x \circ (z \circ z)) \leq \max\{F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y))\}$,
for all $x, y, z \in S$.

Example 4.11. Consider the SBE-algebra S in Example 4.2. Then a neutrosophic $\mathcal{N}-$structure

$$S_N = \{\frac{x}{(-0.87, 0, -0.97)} : \ x = 0, u, v\} \cup \{\frac{x}{(-0.13, -1, -0.99)} : \ x = w, t, 1\}$$

on S is an implicative neutrosophic $\mathcal{N}-$filter of S.

Lemma 4.12. *Every implicative neutrosophic $\mathcal{N}-$filter of a SBE-algebra S is a neutrosophic $\mathcal{N}-$filter of S.*

Proof. The proof is clear from Lemma 2.3 (ii) and Definition 4.10. □

The inverse of Lemma 4.12 is mostly not true.

Example 4.13. Consider the SBE-algebra S in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$S_N = \{\frac{1}{(0, -1, -1)}\} \cup \{\frac{x}{(-1, 0, 0)} : \ x \in S - \{1\}\}$$

on S is a neutrosophic $\mathcal{N}-$filter of S but it is not implicative since $T_N(u \circ (v \circ v)) = T_N(v) = -1 < 0 = T_N(1) = \min\{T_N(u \circ ((w \circ (v \circ v)) \circ (w \circ (v \circ v)))), T_N(u \circ (w \circ w))\}$.

Theorem 4.14. *Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S and α, β, γ be any elements of $[-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. If S_N is a (implicative) neutrosophic $\mathcal{N}-$filter of S, then the nonempty (α, β, γ)-level set $S_N(\alpha, \beta, \gamma)$ of S_N is a (implicative) SBE-filter of S.*

Proof. Let S_N be a neutrosophic $\mathcal{N}-$filter of S and $S_N(\alpha, \beta, \gamma) \neq \emptyset$, for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Since $\alpha \leq T_N(x) \leq T_N(1), I_N(1) \leq I_N(x) \leq \beta$ and $F_N(1) \leq F_N(x) \leq \gamma$, for all $x \in S$, we obtain that $1 \in S_N(\alpha, \beta, \gamma)$. Let $x \circ (y \circ y), x \in S_N(\alpha, \beta, \gamma)$. Since $\alpha \leq T_N(x), T_N(x \circ (y \circ y))$, $I_N(x), I_N(x \circ (y \circ y)) \leq \beta$ and $F_N(x), F_N(x \circ (y \circ y)) \leq \gamma$, it follows that

$$\alpha \leq \min\{T_N(x \circ (y \circ y)), T_N(x)\} \leq T_N(y),$$

$$I_N(y) \leq \max\{I_N(x \circ (y \circ y)), I_N(x)\} \leq \beta$$

and

$$F_N(y) \leq \max\{F_N(x \circ (y \circ y)), F_N(x)\} \leq \gamma,$$

for all $x, y \in S$, which imply that $y \in S_N(\alpha, \beta, \gamma)$. So, $S_N(\alpha, \beta, \gamma)$ is a SBE-filter of S.

Let S_N be an implicative neutrosophic $\mathcal{N}-$filter of S and $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))), x \circ (y \circ y) \in S_N(\alpha, \beta, \gamma)$. Since $\alpha \leq T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))$, $I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y)) \leq \beta$ and $F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y)) \leq \gamma$, it is obtained that $\alpha \leq \min\{T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))\} \leq T_N(x \circ (z \circ z))$, $I_N(x \circ (z \circ z)) \leq \max\{I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y))\} \leq \beta$ and $F_N(x \circ (z \circ z)) \leq \max\{F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y))\} \leq \gamma$. Thus, $x \circ (z \circ z) \in T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$, and so, $x \circ (z \circ z) \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma} = S_N(\alpha, \beta, \gamma)$. Hence, $S_N(\alpha, \beta, \gamma)$ is an implicative SBE-filter of S. □

Example 4.15. Consider the (implicative) neutrosophic $\mathcal{N}-$filter of S in Example 4.11. Then the (α, β, γ)-level set $S_N(\alpha, \beta, \gamma) = \{w, t, 1\}$ of S_N is a (implicative) SBE-filter of S, where the elements $\alpha = -0.17, \beta = -0.41$ and $\gamma = -0.42$ in $[-1, 0]$.

Theorem 4.16. *Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S and $T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$ be (implicative) SBE-filters of S, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then S_N is a (implicative) neutrosophic $\mathcal{N}-$filter of S.*

Proof. Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S and $T_N^{\alpha}, I_N^{\beta}$, F_N^{γ} be SBE-filters of S, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha+\beta+\gamma \leq 0$. Suppose that $T_N(1) < T_N(x)$, $I_N(x) < I_N(1)$ and $F_N(x) < F_N(1)$, for some $x \in S$. If $\alpha = \dfrac{1}{2}(T_N(1) + T_N(x))$, $\beta = \dfrac{1}{2}(I_N(1) + I_N(x))$ and $\gamma = \dfrac{1}{2}(F_N(1) + F_N(x))$ in $[-1, 0)$, then $T_N(1) < \alpha < T_N(x)$, $I_N(x) < \beta < I_N(1)$ and $F_N(x) < \gamma < F_N(1)$, which imply that $1 \notin T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$. This contradicts with (SBEf-1). So, $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ and $F_N(1) \leq F_N(x)$, for all $x \in S$. Assume that

$$\alpha_1 = T_N(y) < \min\{T_N(x \circ (y \circ y)), T_N(x)\} = \alpha_2,$$

$$\beta_1 = \max\{I_N(x \circ (y \circ y)), I_N(x)\} < I_N(y) = \beta_2,$$

and

$$\gamma_1 = \max\{F_N(x \circ (y \circ y)), F_N(x)\} < F_N(y) = \gamma_2.$$

If $\alpha^{'} = \dfrac{1}{2}(\alpha_1 + \alpha_2), \beta^{'} = \dfrac{1}{2}(\beta_1 + \beta_2)$ *and* $\gamma^{'} = \dfrac{1}{2}(\gamma_1 + \gamma_2)$ in $[-1, 0)$, then $\alpha_1 < \alpha^{'} < \alpha_2$, $\beta_1 < \beta^{'} < \beta_2$ and $\gamma_1 < \gamma^{'} < \gamma_2$. Thus, $x \circ (y \circ y), x \in T_N^{\alpha^{'}}, I_N^{\beta^{'}}, F_N^{\gamma^{'}}$ but $y \notin T_N^{\alpha^{'}}, I_N^{\beta^{'}}, F_N^{\gamma^{'}}$, which contradicts with (SBEf-2). Hence,

$$\min\{T_N(x \circ (y \circ y)), T_N(x)\} \leq T_N(y),$$

$$I_N(y) \leq \max\{I_N(x \circ (y \circ y)), I_N(x)\}$$

and

$$F_N(y) \leq \max\{F_N(x \circ (y \circ y)), F_N(x)\},$$

for all $x, y \in S$. Therefore, S_N is a neutrosophic $\mathcal{N}-$filter of S.

Let $T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$ be implicative SBE-filters of S, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Suppose that $a_1 = T_N(x \circ (z \circ z)) < \min\{T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))\} = a_2$, $b_1 = \max\{I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y))\} < I_N(x \circ (z \circ z)) = b_2$ and $c_1 = \max\{F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y))\} < F_N(x \circ (z \circ z)) = c_2$, for some $x, y, z \in S$. If $\alpha_0 = \frac{1}{2}(a_1 + a_2)$, $\beta_0 = \frac{1}{2}(b_1 + b_2)$ and $\gamma_0 = \frac{1}{2}(c_1 + c_2)$ in $[-1, 0)$, then $a_1 < \alpha_0 < a_2$, $b_1 < \beta_0 < b_2$ and $c_1 < \gamma_0 < c_2$. So, $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))), x \circ (y \circ y) \in T_N^{\alpha_0}, I_N^{\beta_0}, F_N^{\gamma_0}$ but $x \circ (z \circ z) \notin T_N^{\alpha_0}, I_N^{\beta_0}, F_N^{\gamma_0}$, which is a contradiction. Thus, $\min\{T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))\} \leq T_N(x \circ (z \circ z))$, $I_N(x \circ (z \circ z)) \leq \max\{I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y))\}$ and $F_N(x \circ (z \circ z)) \leq \max\{F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y))\}$, for all $x, y, z \in S$. □

Theorem 4.17. *Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S. Then S_N is a neutrosophic $\mathcal{N}-$filter of S if and only if*

$$x, y \in S_N(\alpha, \beta, \gamma) \quad \Leftrightarrow \quad U(x, y) \subseteq S_N(\alpha, \beta, \gamma), \tag{5}$$

for all $x, y \in S$ and $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$.

Proof. Let S_N be a neutrosophic $\mathcal{N}-$filter of S. Assume that $x, y \in S_N(\alpha, \beta, \gamma)$, for any $x, y \in S$ and $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$, and $z \in U(x, y)$. Since $\alpha \leq T_N(x), T_N(y)$, $I_N(x), I_N(y) \leq \beta$ and $F_N(x), F_N(y) \leq \gamma$, it follows from Theorem 4.9 that $\alpha \leq \min\{T_N(x), T_N(y)\} \leq T_N(z)$, $I_N(z) \leq \max\{I_N(x), I_N(y)\} \leq \beta$ and $F_N(z) \leq \max\{F_N(x), F_N(y)\} \leq \gamma$. Thus, $z \in T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$, and so, $z \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma} = S_N(\alpha, \beta, \gamma)$. Hence, $U(x, y) \subseteq S_N(\alpha, \beta, \gamma)$. Suppose that $U(x, y) \subseteq S_N(\alpha, \beta, \gamma)$. Since $x \circ ((y \circ (x \circ x)) \circ (y \circ (x \circ x))) = 1$ and $x \circ ((y \circ (y \circ y)) \circ (y \circ (y \circ y))) = x \circ (1 \circ 1))1$ from (SBE-1), Lemma 2.3 (i) and (iii), we have that $x, y \in U(x, y) \subseteq S_N(\alpha, \beta, \gamma)$.

Conversely, Let S_N be a neutrosophic $\mathcal{N}-$structure on S satisfying the condition (5). Then it is obtained from Lemma 2.3 (i) and the condition (5) that $1 \in U(x, y) \subseteq S_N(\alpha, \beta, \gamma)$. Assume that $x, x \circ (y \circ y) \in S_N(\alpha, \beta, \gamma)$. Thus, $U(x, x \circ (y \circ y)) \subseteq S_N(\alpha, \beta, \gamma)$. Since $x \circ (((x \circ (y \circ y)) \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (y \circ y))) = 1$ from Lemma 2.3 (iv), it follows that $y \in U(x, x \circ (y \circ y)) \subseteq S_N(\alpha, \beta, \gamma)$. Thereby, $S_N(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$ is a SBE-filter of S, and so, S_N be a neutrosophic $\mathcal{N}-$filter of S by Theorem 4.16. □

Corollary 4.18. *Let S_N be a neutrosophic $\mathcal{N}-$structure on a SBE-algebra S. Then S_N is a neutrosophic $\mathcal{N}-$filter of S if and only if*

$$\emptyset \neq S_N(\alpha, \beta, \gamma) = \bigcup_{x, y \in S_N(\alpha, \beta, \gamma)} U(x, y), \tag{6}$$

for all $x, y \in S$ and $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$.

Proof. Let S_N be a neutrosophic $\mathcal{N}-$filter of S. By Theorem 4.17, it is obvious that $\bigcup_{x,y\in S_N(\alpha,\beta,\gamma)} U(x,y) \subseteq S_N(\alpha,\beta,\gamma)$, for all $x,y \in S$ and $\alpha,\beta,\gamma \in [-1,0]$ with $-3 \leq \alpha+\beta+\gamma \leq 0$. Then it is sufficient to show that $S_N(\alpha,\beta,\gamma) \subseteq \bigcup_{x,y\in S_N(\alpha,\beta,\gamma)} U(x,y)$. Since S_N is a neutrosophic $\mathcal{N}-$filter of S, it follows from Theorem 4.14 that $S_N(\alpha,\beta,\gamma)$ is a SBE-filter of S. Assume that $x \in S_N(\alpha,\beta,\gamma)$. Since $x\circ((1\circ(x\circ x))\circ(1\circ(x\circ x))) = x\circ(x\circ x) = 1$ from Lemma 2.3 (ii) and (SBE-1), we have that $x \in U(x,1)$. Thus,

$$S_N(\alpha,\beta,\gamma) \subseteq U(x,1) \subseteq \bigcup_{x\in S_N(\alpha,\beta,\gamma)} U(x,1) \subseteq \bigcup_{x,y\in S_N(\alpha,\beta,\gamma)} U(x,y).$$

Conversely, let S_N be a neutrosophic $\mathcal{N}-$structure on S satisfying the condition (6). Since $1 \in U(x,y)$ from Lemma 2.3 (i), we get that

$$1 \in \bigcup_{x,y\in S_N(\alpha,\beta,\gamma)} U(x,y) = S_N(\alpha,\beta,\gamma).$$

Assume that $x, x\circ(y\circ y) \in S_N(\alpha,\beta,\gamma)$. Since $\bigcup_{x,x\circ(y\circ y)\in S_N(\alpha,\beta,\gamma)} U(x,x\circ(y\circ y)) = S_N(\alpha,\beta,\gamma)$ and $x\circ(((x\circ(y\circ y))\circ(y\circ y))\circ((x\circ(y\circ y))\circ(y\circ y))) = 1$ from the condition (6) and Lemma 2.3 (iv), it is obtained that $y \in \bigcup_{x,x\circ(y\circ y)\in S_N(\alpha,\beta,\gamma)} U(x,x\circ(y\circ y)) = S_N(\alpha,\beta,\gamma)$. Hence, $S_N(\alpha,\beta,\gamma)$ is a SBE-filter of S, and so, S_N is a neutrosophic $\mathcal{N}-$filter of S from Theorem 4.16. □

Example 4.19. Consider the SBE-algebra S in Example 4.2. For a neutrosophic $\mathcal{N}-$filter $S_N = \{\frac{x}{(-0.9,0,-0.04)} : x = 0,u,v\} \cup \{\frac{x}{(-0.031,-1,-0.91)} : x = w,t,1\}$ of S, and the elements $\alpha = -0.5, \beta = -0.2$ and $\gamma = -0.3$ in $[-1,0]$, we have $U(x,y) = \{w,t,1\} \subseteq S_N(\alpha,\beta,\gamma) = \{w,t,1\}$, for all $x,y \in S_N(\alpha,\beta,\gamma)$. Also, $S_N(\alpha,\beta,\gamma) = \bigcup_{x,y\in S_N(\alpha,\beta,\gamma)} U(x,y)$.

Theorem 4.20. *Let $\langle S;\circ_S,1_S\rangle$ and $\langle P;\circ_P,1_P\rangle$ be SBE-algebras, $f : S \longrightarrow P$ be a surjective SBE-homomorphism and $P_N = \frac{P}{(T_N,I_N,F_N)}$ be a neutrosophic $\mathcal{N}-$structure on P. Then P_N is a (implicative) neutrosophic $\mathcal{N}-$filter of P if and only if $P_N^f = \frac{S}{(T_N^f,I_N^f,F_N^f)}$ is a (implicative) neutrosophic $\mathcal{N}-$filter of S where the $\mathcal{N}-$functions $T_N^f, I_N^f, F_N^f : S \longrightarrow [-1,0]$ on S are defined by $T_N^f(x) = T_N(f(x)), I_N^f(x) = I_N(f(x))$ and $F_N^f(x) = F_N(f(x))$, for all $x \in S$, respectively.*

Proof. Let $\langle S;\circ_S,1_S\rangle$ and $\langle P;\circ_P,1_P\rangle$ be SBE-algebras, $f : S \longrightarrow P$ be a surjective SBE-homomorphism and $P_N = \frac{P}{(T_N,I_N,F_N)}$ be a neutrosophic $\mathcal{N}-$filter of P. Then $T_N^f(x) = T_N(f(x)) \leq T_N(1_P) = T_N(f(1_S)) = T_N^f(1_S)$, $I_N^f(1_S) = I_N(f(1_S)) = I_N(1_P) \leq I_N(f(x)) = I_N^f(x)$ and $F_N^f(1_S) = F_N(f(1_S) = F_N(1_P)$

$\leq F_N(f(x)) = F_N^f(x)$, for all $x \in S$. Also,

$$\begin{aligned}\min\{T_N^f(x), T_N^f(x \circ_S (y \circ_S y))\} &= \min\{T_N(f(x)), T_N(f(x \circ_S (y \circ_S y)))\} \\ &= \min\{T_N(f(x)), T_N(f(x) \\ &\quad \circ_P (f(y) \circ_P f(y)))\} \\ &\leq T_N(f(y)) \\ &= T_N^f(y),\end{aligned}$$

$$\begin{aligned}I_N^f(y) &= I_N(f(y)) \\ &\leq \max\{I_N(f(x)), I_N(f(x) \circ_P (f(y) \circ_P f(y)))\} \\ &= \max\{I_N(f(x)), I_N(f(x \circ_S (y \circ_S y)))\} \\ &= \max\{I_N^f(x), I_N^f(x \circ_S (y \circ_S y))\}\end{aligned}$$

and

$$\begin{aligned}F_N^f(y) &= F_N(f(y)) \\ &\leq \max\{F_N(f(x)), F_N(f(x) \circ_P (f(y) \circ_P f(y)))\} \\ &= \max\{F_N(f(x)), F_N(f(x \circ_S (y \circ_S y)))\} \\ &= \max\{F_N^f(x), F_N^f(x \circ_S (y \circ_S y))\},\end{aligned}$$

for all $x, y \in S$. Thus, $P_N^f = \dfrac{S}{(T_N^f, I_N^f, F_N^f)}$ is a neutrosophic $\mathcal{N}-$filter of S. Suppose that P_N is an implicative neutrosophic $\mathcal{N}-$filter of P. Since

$$\begin{aligned}&\min\{T_N^f(x \circ_S ((y \circ_S (z \circ_S z)) \circ_S (y \circ_S (z \circ_S z)))), T_N^f(x \circ_S (y \circ_S y))\} \\ &= \min\{T_N(f(x \circ_S ((y \circ_S (z \circ_S z)) \circ_S (y \circ_S (z \circ_S z))))), T_N(f(x \circ_S (y \circ_S y)))\} \\ &= \min\{T_N(f(x) \circ_P ((f(y) \circ_P (f(z) \circ_P f(z))) \circ_P (f(y) \\ &\quad \circ_P (f(z) \circ_P f(z))))), T_N(f(x) \circ_P (f(y) \circ_P f(y)))\} \\ &\leq T_N(f(x) \circ_P (f(z) \circ_P f(z))) \\ &= T_N(f(x \circ_S (z \circ_S z))) \\ &= T_N^f(x \circ_S (z \circ_S z)),\end{aligned}$$

$$\begin{aligned}&I_N^f(x \circ_S (z \circ_S z)) \\ &= I_N(f(x \circ_S (z \circ_S z))) \\ &= I_N(f(x) \circ_P (f(z) \circ_P f(z))) \\ &\leq \max\{I_N(f(x) \circ_P ((f(y) \circ_P (f(z) \circ_P f(z))) \circ_P (f(y) \\ &\quad \circ_P (f(z) \circ_P f(z))))), I_N(f(x) \circ_P (f(y) \circ_P f(y)))\} \\ &= \max\{I_N(f(x \circ_S ((y \circ_S (z \circ_S z)) \circ_S (y \circ_S (z \circ_S z))))), I_N(f(x \circ_S (y \circ_S y)))\} \\ &= \max\{I_N^f(x \circ_S ((y \circ_S (z \circ_S z)) \circ_S (y \circ_S (z \circ_S z)))), I_N^f(x \circ_S (y \circ_S y))\}\end{aligned}$$

and

$$\begin{aligned}&F_N^f(x \circ_S (z \circ_S z)) \\ &= F_N(f(x \circ_S (z \circ_S z))) \\ &= F_N(f(x) \circ_P (f(z) \circ_P f(z))) \\ &\leq \max\{F_N(f(x) \circ_P ((f(y) \circ_P (f(z) \circ_P f(z))) \circ_P (f(y) \\ &\quad \circ_P (f(z) \circ_P f(z))))), F_N(f(x) \circ_P (f(y) \circ_P f(y)))\} \\ &= \max\{F_N(f(x \circ_S ((y \circ_S (z \circ_S z)) \circ_S (y \circ_S (z \circ_S z))))), F_N(f(x \circ_S (y \circ_S y)))\} \\ &= \max\{F_N^f(x \circ_S ((y \circ_S (z \circ_S z)) \circ_S (y \circ_S (z \circ_S z)))), F_N^f(x \circ_S (y \circ_S y))\}\end{aligned}$$

for all $x, y, z \in S$. Hence, P_N^f is an implicative neutrosophic $\mathcal{N}-$filter of S.

Conversely, let P_N^f be a neutrosophic $\mathcal{N}-$filter of S. So, $T_N(y) = T_N(f(x)) = T_N^f(x) \leq T_N^f(1_S) = T_N(f(1_S)) = T_N(1_P)$, $I_N(1_P) = I_N(f(1_S)) = I_N^f(1_S) \leq I_N^f(x) = I_N(f(x)) = I_N(y)$ and $F_N(1_P) = F_N(f(1_S)) = F_N^f(1_S) \leq F_N^f(x) = F_N(f(x)) = F_N(y)$, for all $y \in P$. Besides,

$$\begin{aligned}
&\min\{T_N(f(x_1)), T_N(f(x_1) \circ_P (f(x_2) \circ_P f(x_2)))\}\\
&= \min\{T_N(f(x_1)), T_N(f(x_1 \circ_S (x_2 \circ_S x_2)))\}\\
&= \min\{T_N^f(x_1), T_N^f(x_1 \circ_S (x_2 \circ_S x_2))\}\\
&\leq T_N^f(x_2)\\
&= T_N(f(x_2)),
\end{aligned}$$

$$\begin{aligned}
I_N(f(x_2)) &= I_N^f(x_2)\\
&\leq \max\{I_N^f(x_1), I_N^f(x_1 \circ_S (x_2 \circ_S x_2))\}\\
&= \max\{I_N(f(x_1)), I_N(f(x_1 \circ_S (x_2 \circ_S x_2)))\}\\
&= \max\{I_N(f(x_1)), I_N(f(x_1) \circ_P (f(x_2) \circ_P f(x_2)))\}
\end{aligned}$$

and

$$\begin{aligned}
F_N(f(x_2)) &= F_N^f(x_2)\\
&\leq \max\{F_N^f(x_1), F_N^f(x_1 \circ_S (x_2 \circ_S x_2))\}\\
&= \max\{F_N(f(x_1)), F_N(f(x_1 \circ_S (x_2 \circ_S x_2)))\}\\
&= \max\{F_N(f(x_1)), F_N(f(x_1) \circ_P (f(x_2) \circ_P f(x_2)))\},
\end{aligned}$$

for all $y_1, y_2 \in P$. Thereby, $P_N = \dfrac{S}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}-$filter of P. Assume that P_N^f is an implicative neutrosophic $\mathcal{N}-$filter of S. Since

$$\begin{aligned}
&\min\{T_N(f(x_1) \circ_P ((f(x_2) \circ_P (f(x_3) \circ_P f(x_3))) \circ_P (f(x_2)\\
&\circ_P (f(x_3) \circ_P f(x_3))))), T_N(f(x_1) \circ_P (f(x_2) \circ_P f(x_2)))\}\\
&= \min\{T_N(f(x_1 \circ_S ((x_2 \circ_S (x_3 \circ_S x_3)) \circ_S (x_2\\
&\quad \circ_S (x_3 \circ_S x_3))))), T_N(f(x_1 \circ_S (x_2 \circ_S x_2)))\}\\
&= \min\{T_N^f(x_1 \circ_S ((x_2 \circ_S (x_3 \circ_S x_3)) \circ_S (x_2\\
&\quad \circ_S (x_3 \circ_S x_3)))), T_N^f(x_1 \circ_S (x_2 \circ_S x_2))\}\\
&\leq T_N^f(x_1 \circ_S (x_3 \circ_S x_3))\}\\
&= T_N(f(x_1 \circ_S (x_3 \circ_S x_3)))\\
&= T_N(f(x_1) \circ_P (f(x_3) \circ_P f(x_3))),
\end{aligned}$$

$$\begin{aligned}
&I_N(f(x_1) \circ_P (f(x_3) \circ_P f(x_3))) = I_N(f(x_1 \circ_S (x_3 \circ_S x_3)))\\
&= I_N^f(x_1 \circ_S (x_3 \circ_S x_3))\}\\
&\leq \max\{I_N^f(x_1 \circ_S ((x_2 \circ_S (x_3 \circ_S x_3)) \circ_S (x_2\\
&\quad \circ_S (x_3 \circ_S x_3)))), I_N^f(x_1 \circ_S (x_2 \circ_S x_2))\}\\
&= \max\{I_N(f(x_1 \circ_S ((x_2 \circ_S (x_3 \circ_S x_3)) \circ_S (x_2\\
&\quad \circ_S (x_3 \circ_S x_3))))), I_N(f(x_1 \circ_S (x_2 \circ_S x_2)))\}\\
&= \max\{I_N(f(x_1) \circ_P ((f(x_2) \circ_P (f(x_3) \circ_P f(x_3))) \circ_P (f(x_2)\\
&\quad \circ_P (f(x_3) \circ_P f(x_3))))), I_N(f(x_1) \circ_P (f(x_2) \circ_P f(x_2)))\}
\end{aligned}$$

and

$$\begin{aligned}
&F_N(f(x_1) \circ_P (f(x_3) \circ_P f(x_3))) \\
&= F_N(f(x_1 \circ_S (x_3 \circ_S x_3))) \\
&= F_N^f(x_1 \circ_S (x_3 \circ_S x_3))\} \\
&\leq \max\{F_N^f(x_1 \circ_S ((x_2 \circ_S (x_3 \circ_S x_3)) \circ_S (x_2 \\
&\quad \circ_S(x_3 \circ_S x_3)))), F_N^f(x_1 \circ_S (x_2 \circ_S x_2))\} \\
&= \max\{F_N(f(x_1 \circ_S ((x_2 \circ_S (x_3 \circ_S x_3)) \circ_S (x_2 \\
&\quad \circ_S(x_3 \circ_S x_3))))), F_N(f(x_1 \circ_S (x_2 \circ_S x_2)))\} \\
&= \max\{F_N(f(x_1) \circ_P ((f(x_2) \circ_P (f(x_3) \circ_P f(x_3))) \circ_P (f(x_2) \\
&\quad \circ_P(f(x_3) \circ_P f(x_3))))), F_N(f(x_1) \circ_P (f(x_2) \circ_P f(x_2)))\},
\end{aligned}$$

for all $x_1, x_2, x_3 \in S$. Therefore, P_N is an implicative neutrosophic $\mathcal{N}-$filter of P. □

Theorem 4.21. *Every neutrosophic $\mathcal{N}-$filter of a SBE-algebra S is a neutrosophic $\mathcal{N}-$ subalgebra of S.*

Proof. Let S_N be a neutrosophic $\mathcal{N}-$filter of S. Since $((x\circ y)\circ(x\circ y))\circ(y\circ y) = 1$ from Lemma 2.3 (vi), we get that $(x \circ y) \circ (x \circ y) \preceq y$, for all $x, y \in S$. Then it follows from Lemma 4.7 and Lemma 2.6 (ii) that $\min\{T_N(x), T_N(y)\} \leq T_N((x \circ y) \circ (x \circ y)) \leq T_N(y) \leq T_N(x \circ (y \circ y))$, $I_N(x \circ (y \circ y)) \leq I_N(y) \leq I_N((x \circ y) \circ (x \circ y)) \leq \max\{I_N(x), I_N(y)\}$ and $F_N(x \circ (y \circ y)) \leq F_N(y) \leq F_N((x \circ y) \circ (x \circ y)) \leq \max\{F_N(x), F_N(y)\}$, for all $x, y \in S$. Thereby, S_N is a neutrosophic $\mathcal{N}-$subalgebra of S. □

The inverse of Theorem 4.21 does not generally hold.

Example 4.22. Consider the SBE-algebra S in Example 4.2. Then a neutrosophic $\mathcal{N}-$structure

$$S_N = \{\frac{0}{(-1, -0.88, 0)}, \frac{1}{(0, -1, -0.73)}\} \cup \{\frac{x}{(-0.55, -0.91, -0.43)} : x \in S - \{0, 1\}\}$$

on S is a neutrosophic $\mathcal{N}-$subalgebra of S but it is not a neutrosophic $\mathcal{N}-$filter of S since $T_N(0) - 1 < -0.55 = \min\{T_N(u \circ (0 \circ 0)), T_N(u)\}$.

Lemma 4.23. *Let S_N be a neutrosophic $\mathcal{N}-$subalgebra of a SBE-algebra S satisfying*

(7)

$$\min\{T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))\} \leq T_N(x \circ (z \circ z))$$
$$I_N(x \circ (z \circ z)) \leq \max\{I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y))\}$$
and
$$F_N(x \circ (z \circ z)) \leq \max\{F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y))\},$$

for all $x, y, z \in S$. Then S_N is a neutrosophic $\mathcal{N}-$filter of S.

Proof. Let S_N be a neutrosophic $\mathcal{N}-$subalgebra of S satisfying the condition (7). By Lemma 3.7, it is obvious that $T_N(x) \leq T_N(1), I_N(1) \leq I_N(x)$ and $F_N(1) \leq F_N(x)$, for all $x \in S$. Then it is obtained from Lemma 2.3 (ii) that

$\min\{T_N(x \circ (y \circ y)), T_N(x)\} = \min\{T_N(1 \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))), T_N(1 \circ (x \circ x))\} \leq T_N(1 \circ (y \circ y)) = T_N(y)$, $I_N(y) = I_N(1 \circ (y \circ y)) \leq \max\{I_N(1 \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))), I_N(1 \circ (x \circ x))\} = \max\{I_N(x \circ (y \circ y)), I_N(x)\}$ and $F_N(y) = F_N(1 \circ (y \circ y)) \leq \max\{F_N(1 \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))), F_N(1 \circ (x \circ x))\} = \max\{F_N(x \circ (y \circ y)), F_N(x)\}$, for all $x, y \in S$. Hence, S_N is a neutrosophic $\mathcal{N}-$filter of S. □

Theorem 4.24. *Let S be a self-distributive SBE-algebra. Then every neutrosophic $\mathcal{N}-$filter of S is an implicative neutrosophic $\mathcal{N}-$filter of S.*

Proof. Let S_N be a neutrosophic $\mathcal{N}-$filter of a self-distributive SBE-algebra S. Since S_N be a neutrosophic $\mathcal{N}-$filter of S, it is clear that $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ *and* $F_N(1) \leq F_N(x)$, for all $x \in S$. Then it follows from Definition 2.4 that $\min\{T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))\} = \min\{T_N((x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))), T_N(x \circ (y \circ y))\} \leq T_N(x \circ (z \circ z))$, $I_N(x \circ (z \circ z)) \leq \max\{I_N((x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))), I_N(x \circ (y \circ y))\} = \max\{I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y))\}$ and $F_N(x \circ (z \circ z)) \leq \max\{F_N((x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))), F_N(x \circ (y \circ y))\} = \max\{F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y))\}$, for all $x, y, z \in S$. Thus, S_N is an implicative neutrosophic $\mathcal{N}-$filter of S. □

Lemma 4.25. *Let S_N be a (implicative) neutrosophic $\mathcal{N}-$filter of a SBE-algebra S. Then the subsets $S_{T_N} = \{x \in S : T_N(x) = T_N(1)\}$, $S_{I_N} = \{x \in S : I_N(x) = I_N(1)\}$ and $S_{F_N} = \{x \in S : F_N(x) = F_N(1)\}$ of S are (implicative) SBE-filters of S.*

Proof. Let S_N be a neutrosophic $\mathcal{N}-$filter of S. Then it is obvious that $1 \in S_{T_N}, S_{I_N}, S_{F_N}$. Assume that $x, x \circ (y \circ y) \in S_{T_N}, S_{I_N}, S_{F_N}$. Since $T_N(x) = T_N(1) = T_N(x \circ (y \circ y))$, $I_N(x) = I_N(1) = I_N(x \circ (y \circ y))$ and $F_N(x) = F_N(1) = F_N(x \circ (y \circ y))$, it is obtained that $T_N(1) = \min\{T_N(1), T_N(1)\} = \min\{T_N(x \circ (y \circ y)), T_N(x)\} \leq T_N(y)$, $I_N(y) \leq \max\{I_N(x \circ (y \circ y)), I_N(x)\} = \max\{I_N(1), I_N(1)\} = I_N(1)$ and $F_N(y) \leq \max\{F_N(x \circ (y \circ y)), F_N(x)\} = \max\{F_N(1), I_N(1)\} = F_N(1)$, which imply that $T_N(y) = T_N(1)$, $I_N(y) = I_N(1)$ and $F_N(y) = F_N(1)$. Then $y \in S_{T_N}, S_{I_N}, S_{F_N}$. Hence, S_{T_N}, S_{I_N} and S_{F_N} are SBE-filters of S.

Let S_N be an implicative neutrosophic $\mathcal{N}-$filter of S. Suppose that $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))), x \circ (y \circ y) \in S_{T_N}, S_{I_N}, S_{F_N}$. Since $T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) = T_N(1) = T_N(x \circ (y \circ y))$, $I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) = I_N(1) = I_N(x \circ (y \circ y))$ and $F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) = F_N(1) = F_N(x \circ (y \circ y))$, it follows that $T_N(1) = \min\{T_N(1), T_N(1)\} = \min\{T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))\} \leq T_N(x \circ (z \circ z))$, $I_N(x \circ (z \circ z)) \leq \max\{I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y))\} = \max\{I_N(1), I_N(1)\} = I_N(1)$ and $F_N(x \circ (z \circ z)) \leq \max\{F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y))\} = \max\{F_N(1), F_N(1)\} = F_N(1)$, which imply that $T_N(x \circ (z \circ z)) = T_N(1)$, $I_N(x \circ (z \circ z)) = I_N(1)$ and $F_N(x \circ (z \circ z)) = F_N(1)$. Thus, $x \circ (z \circ z) \in S_{T_N}, S_{I_N}, S_{F_N}$. Therefore, S_{T_N}, S_{I_N} and S_{F_N} are implicative SBE-filters of S. □

Definition 4.26. Let S be a SBE-algebra. Define the subsets

$$S_N^{s_t} := \{x \in S : T_N(s_t) \leq T_N(x)\},$$
$$S_N^{s_i} := \{x \in S : I_N(x) \leq I_N(s_i)\}$$

and

$$S_N^{s_f} := \{x \in S : F_N(x) \leq F_N(s_f)\}$$

of S, for all $s_t, s_i, s_f \in S$. Also, it is obvious that $s_t \in S_N^{s_t}, s_i \in S_N^{s_i}$ and $s_f \in S_N^{s_f}$.

Example 4.27. Consider the SBE-algebra S in Example 3.2. Let

$$T_N(x) = \begin{cases} -0.99, & \text{if } x = u, w \\ -0.72, & \text{if } x = 1 \\ 0, & \text{otherwise,} \end{cases} \qquad I_N(x) = \begin{cases} 0, & \text{if } x = t, 1 \\ -1, & \text{otherwise,} \end{cases}$$

$$F_N(x) = \begin{cases} -0.011, & \text{if } x = 0, u, v \\ -0.1, & \text{otherwise,} \end{cases} \qquad s_t = v, s_i = u \text{ and } s_f = w.$$

Then

$$S_N^{s_t} = \{x \in S : T_N(v) \leq T_N(x)\} = \{0, v, t\},$$
$$S_N^{s_i} = \{x \in S : I_N(x) \leq I_N(u)\} = \{0, u, v, w\}$$

and

$$S_N^{s_f} = \{x \in S : F_N(x) \leq F_N(w)\} = \{w, t, 1\}.$$

Theorem 4.28. *Let s_t, s_i and s_f be any elements of a SBE-algebra S. If S_N is a (implicative) neutrosophic $\mathcal{N}-$filter of S, then $S_N^{s_t}, S_N^{s_i}$ and $S_N^{s_f}$ are (implicative) SBE-filters of S.*

Proof. Let s_t, s_i and s_f be any elements of S and S_N be a neutrosophic $\mathcal{N}-$filter of S. Since $T_N(s_t) \leq T_N(1)$, $I_N(1) \leq I_N(s_i)$ and $F_N(1) \leq F_N(s_f)$, for any $s_t, s_i, s_f \in S$, it follows that $1 \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$. Assume that $x \circ (y \circ y), x \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$. Since $T_N(s_t) \leq T_N(x \circ (y \circ y)), T_N(x)$, $I_N(x \circ (y \circ y)), I_N(x) \leq I_N(s_i)$ and $F_N(x \circ (y \circ y)), F_N(x) \leq F_N(s_f)$, it is obtained that $T_N(s_t) \leq \min\{T_N(x \circ (y \circ y)), T_N(x)\} \leq T_N(y)$, $I_N(y) \leq \max\{I_N(x \circ (y \circ y)), I_N(x)\} \leq I_N(s_i)$ and $F_N(y) \leq \max\{F_N(x \circ (y \circ y)), F_N(x)\} \leq F_N(s_f)$, which imply that $y \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$. Thus, $S_N^{s_t}$, $S_N^{s_i}$ and $S_N^{s_f}$ are SBE-filters of S.

Let S_N be an implicative neutrosophic $\mathcal{N}-$filter of S. Suppose that $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))), x \circ (y \circ y) \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$. Since $T_N(s_t) \leq T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))$, $I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y)) \leq I_N(s_i)$ and $F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y)) \leq F_N(s_f)$, we get that $T_N(s_t) \leq \min\{T_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), T_N(x \circ (y \circ y))\} \leq T_N(x \circ (z \circ z))$, $I_N(x \circ (z \circ z)) \leq \max\{I_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), I_N(x \circ (y \circ y))\} \leq I_N(s_i)$ and $F_N(x \circ (z \circ z)) \leq \max\{F_N(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))), F_N(x \circ (y \circ y))\} \leq F_N(s_f)$, which means that $x \circ (z \circ z) \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$. Hence, $S_N^{s_t}$, $S_N^{s_i}$ and $S_N^{s_f}$ are implicative SBE-filters of S. □

Example 4.29. Consider the SBE-algebra S in Example 4.2. For a (implicative) neutrosophic $\mathcal{N}-$filter

$$S_N = \{\frac{x}{(-0.86,-0.75,-0.64)} : x = 0, u, v\} \cup \{\frac{x}{(-0.52,-0.8,-0.7)} : x = w, t, 1\}$$

of S and $s_t = 0, s_i = u, s_f = w \in S$, the subsets

$$S_N^{s_t} = \{x \in S : T_N(0) \leq T_N(x)\} = S,$$
$$S_N^{s_i} = \{x \in S : I_N(x) \leq I_N(u)\} = S$$

and

$$S_N^{s_f} = \{x \in S : F_N(x) \leq F_N(w)\} = \{w, t, 1\}$$

of A are (implicative) SBE-filters of S.

Theorem 4.30. *Let s_t, s_i and s_f be any elements of a SBE-algebra S and S_N be a neutrosophic $\mathcal{N}-$structure on S.*

(i) *If $S_N^{s_t}, S_N^{S_i}$ and $S_N^{s_f}$ are SBE-filters of S, so if*

$T_N(x) \leq \min\{T_N(y \circ (z \circ z)), T_N(y)\}$, *then* $T_N(x) \leq T_N(z)$,

(8) $\max\{I_N(y \circ (z \circ z)), I_N(y)\} \leq I_N(x)$, *then* $I_N(z) \leq I_N(x)$ *and*

$\max\{F_N(y \circ (z \circ z)), F_N(y)\} \leq F_N(x)$, *then* $F_N(z) \leq F_N(x)$,

for all $x, y, z \in S$.

(ii) *If S_N satisfies the condition (8) and*

(9) $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ *and* $F_N(1) \leq F_N(x)$,

for all $x \in S$, then $S_N^{s_t}, S_N^{s_i}$ and $S_N^{s_f}$ are SBE-filters of S, for all $s_t \in T_N^{-1}$, $s_i \in I_N^{-1}$ and $s_f \in F_N^{-1}$.

Proof. Let s_t, s_i and s_f be any elements of S and S_N be a neutrosophic $\mathcal{N}-$structure on S.

(i) Assume that $S_N^{s_t}, S_N^{s_i}$ and $S_N^{s_f}$ be SBE-filters of S and

$$T_N(x) \leq \min\{T_N(y \circ (z \circ z)), T_N(y)\},$$
$$\max\{I_N(y \circ (z \circ z)), I_N(y)\} \leq I_N(x)$$

and

$$\max\{F_N(y \circ (z \circ z)), F_N(y)\} \leq F_N(x).$$

Since $y \circ (z \circ z), y \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$ where $s_t = s_i = s_f = x$, it follows that $z \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$ where $s_t = s_i = s_f = x$. Therefore, $T_N(x) \leq T_N(z)$, $I_N(z) \leq I_N(x)$ and $F_N(z) \leq F_N(x)$, for all $x, y, z \in S$.

(ii) Let S_N be a neutrosophic $\mathcal{N}-$structure on S satisfying the conditions (8) and (9), for $s_t \in T_N^{-1}$, $s_i \in I_N^{-1}$ and $s_f \in F_N^{-1}$. It is obvious from from the condition (9) that $1 \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$. Suppose that $x \circ (y \circ y), x \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$. Then $T_N(s_t) \leq T_N(x \circ (y \circ y)), T_N(x)$, $I_N(x \circ (y \circ y)), I_N(x) \leq I_N(s_i)$ and $F_N(x \circ (y \circ y)), F_N(x) \leq F_N(s_f)$. Since

$T_N(s_t) \leq \min\{T_N(x \circ (y \circ y)), T_N(x)\}$, $\max\{I_N(x \circ (y \circ y)), I_N(x)\} \leq I_N(s_i)$ and $\max\{F_N(x \circ (y \circ y)), F_N(x)\} \leq F_N(s_f)$, it is obtained from the condition (8) that $T_N(s_t) \leq T_N(y)$ $I_N(y) \leq I_N(s_i)$ and $F_N(y) \leq F_N(s_f)$, which imply that $y \in S_N^{s_t}, S_N^{s_i}, S_N^{s_f}$. Thereby, $S_N^{s_t}, S_N^{s_i}$ and $S_N^{s_f}$ are SBE-filters of S.

□

Example 4.31. Consider the SBE-algebra S in Example 4.2. Let

$$T_N(x) = \begin{cases} -0.83, & \text{if } x = 0, w, t \\ 0, & \text{otherwise,} \end{cases} \qquad I_N(x) = \begin{cases} -0.79, & \text{if } x = u, v, 1 \\ -0.3, & \text{otherwise,} \end{cases}$$

$$F_N(x) = \begin{cases} -0.67, & \text{if } x = 1 \\ -0.17, & \text{otherwise,} \end{cases} \qquad \text{and } s_t = t,\ s_i = u,\ s_f = v \in S.$$

Then the SBE-filters

$$S_N^{s_t} = S, S_N^{s_i} = \{u, v, 1\} \text{ and } S_N^{s_f} = S$$

of S satisfy the condition (8).

Also, let

$$S_N = \{\frac{x}{(0, -1, -1)} : x = w, t, 1\} \cup \{\frac{x}{(-0.09, 0, -0.9)} : x = 0, u, v\}$$

be a neutrosophic $\mathcal{N}-$structure on S satisfying the conditions (8) and (9). Then the subsets $S_N^{s_t} = S$, $S_N^{s_i} = \{w, t, 1\}$ and $S_N^{s_f} = \{w, t, 1\}$ of S are SBE-filters of S, where $s_t = u, s_i = w$ and $s_f = 1$.

5. Conclusion

In this study, an implicative SBE-filter, a neutrosophic $\mathcal{N}-$subalgebra, a (implicative) neutrosophic $\mathcal{N}-$filter and a level set on neutrosophic $\mathcal{N}-$ structures are introduced on SBE-algebras. Then we prove that the level set of a neutrosophic $\mathcal{N}-$subalgebra (a (implicative) neutrosophic $\mathcal{N}-$filter) of a SBE-algebra is its SBE-subalgebra (a (implicative) SBE-filter) and vice versa, and that the family of all neutrosophic $\mathcal{N}-$subalgebras of the algebraic structure forms a complete distributive modular lattice. We present the situations which $\mathcal{N}-$functions are constant. Additionally, the new statement equivalent to the definition of a neutrosophic $\mathcal{N}-$filter of a SBE-algebra is given. We restate a neutrosophic $\mathcal{N}-$filter of a SBE-algebra by means of upper sets on this algebra. It is illustrated that every implicative neutrosophic $\mathcal{N}-$filter of a SBE-algebra is the neutrosophic $\mathcal{N}-$filter but the inverse does not mostly hold, and that level set of a (implicative) neutrosophic $\mathcal{N}-$filter of a SBE-algebra is its (implicative) SBE-filter and vice versa. Infact, we reveal relationships between (implicative) neutrosophic $\mathcal{N}-$filters of two SBE-algebras by the hepl of an onto SBE-homomorphism. It is demontrated that every neutrosophic $\mathcal{N}-$filter of a SBE-algebra is the neutrosophic $\mathcal{N}-$subalgebra but the inverse is not valid in general. Also, it is shown that a neutrosophic $\mathcal{N}-$filter of a self-distributive SBE-algebra is its implicative neutrosophic $\mathcal{N}-$filter. Besides, the subsets S_{T_N},

S_{I_N} and S_{F_N} of a SBE-algebra are its (implicative) SBE-filters for the (implicative) $\mathcal{N}-$filter. At the end, it is proved that the subsets $S_N^{s_t}$, $S_N^{s_i}$ and $S_N^{s_f}$ of a SBE-algebra defined by any elements s_t, s_i, s_f of the algebraic structure and $\mathcal{N}-$functions are its (implicative) SBE-filters, if a neutrosophic $\mathcal{N}-$structure on this algebraic structure is the (implicative) neutrosophic $\mathcal{N}-$filter.

In future works, we plan to study on plithogenic structures and relationships between neutrosophic $\mathcal{N}-$structures on some algebraic structures.

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