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ON ENUMERATION OF EL-HYPERSTRUCTURES WITH 2 ELEMENTS

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Abstract. EL-hypergroups were defined by Chvalina 1995. Till now, no exact statistics of EL-hypergroups have been done. Moreover, there is no classification of EL-hypergroups and EL^2-hypergroups even over small sets. In this paper we classify all EL-(semi)hypergroups over sets with two elements obtained from quasi ordered semigroups. Also, we characterize all quasi ordered H_v-group and then we enumerate the number of EL^2-H_v-hypergroups and EL^2-hypergroups of order 2.

Keywords: Ends lemma, EL-hypergroups, H_v-group, quasi order relation, partially order relation.
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1. Introduction

A hypergroup as a generalization of the notion of a group, was introduced by F. Marty in 1934. In 1990, Vougiouklis introduced the concept of H_v-structures in Fourth AHA Congress as a generalization of the well-known algebraic hyperstructures [26]. The book on the subject of the hypergroups was written by Corsini [4]. Also, Corsini and Leoreanu in [5] pointed out the application of hyperstructure theory in lattices, binary relations, graphs and hypergraphs. Some other valuable books in hyperstructures have published in [6–8].

The connection between hyperstructures and ordering has been started in 1960s. This connection has been investigated by a number of mathematicians such as Nieminen, Vougiouklis, Corsini, Rosenberg, Davvaz, Chvalina, Krasner, Mittas and Leoreanu. One special aspect of this issue, which falls within the investigation of hyperstructures determined by binary relations, is known as EL-hyperstructures. This concept was first introduced by Chvalina [3] when he was investigated quasi ordered sets and hypergroups. Also, Rosenberg in [23], Hoskova in [13], Rackova in [22] and Novak in [16–21] studied some results on the ordered semigroups and ordered groups connected with EL-hyperstructures. This class of hyperstructures have been generalized and

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extended in some completely different ways. At first, Ghazavi et al. introduced a new class of EL-hyperstructures called EL^2-hyperstrucutres in [10]. EL^2-hyperstructures are hyperstructures based on (partially) quasi ordered (semi)hypergroups instead of a (partially) quasi ordered (semi)groups. Then, Ghazavi et al. generalized EL-hyperstructures to EL_n-hyperstructures and EL-Γ-hyperstructures in [11] and [12] respectively.

As mentioned above, hyperstructure theory has been studied, investigated and extended for more than 5 decades and in different points of view, but fewer works have been done on enumerating and classifying them. For example, Migliorato [14] found, the total number of 23192 of hypergroups of order 3 while Nordo [15] computed the number of 3999 of non-isomorphic hypergroups of the same order [24]. Vougiouklis in [26] showed that there are 8 hypergroups of order 2 and Bayon and Lygeros in [2] showed that there are, up to isomorphism, 20 H_v-groups of order 2. In the other hand, classification of group theory and the closely related representation theory have many important applications in physics, chemistry, materials science and cryptography. Also, the authors in [5,8] present some of the numerous applications of hyperstructures in physics, chemistry, biology, graphs and etc. The classification of hyperstructures will help us better understand its application in other sciences.

All of the above papers, focused on counting and enumerating hyperstructures with two or three elements and no attempt has been done on classifying them. In this paper, first we characterize all quasi ordered semigroups, quasi ordered hypergroup and quasi ordered H_v-groups of order 2. Then, we concentrate on quasi ordered semigroups, hypergroups and H_v-groups in order to find and classify all EL-hypergroups, EL-semihypergroups, EL^2-H_v-groups and EL^2-hypergroups of order 2.

2. Preliminaries

Let us briefly recall some basic notions and results about hypergroups; for a comprehensive overview of this subject, the reader is refereed to [4, 7].

A hypergroupoid is a pair $(H, \circ)$ where H is a nonempty set and $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is a hyperoperation when $\mathcal{P}^*(H)$ is the family of non-empty subsets of H. A semihypergroup is an associative hypergroupoid, i.e. hypergroupoid satisfying the equality $a \circ (b \circ c) = (a \circ b) \circ c$ for every $a, b, c \in H$.

If moreover the semihypergroup H satisfies $a \circ H = H = H \circ a$, for all $a \in H$, it is called a *hypergroup*. This condition is known as *reproduction axiom*. Also, if for all $(a, b, c) \in H^3$ it holds $a \circ (b \circ c) \bigcap (a \circ b) \circ c \neq \emptyset$, then the hyperoperation $\circ$ is called *weak associative* and the hyperstructure $(H, \circ)$ is called an H_v-*semigroup*. Moreover, an H_v-semigroup is called an H_v-*group* if the reproduction axiom holds.

In the above definition, if A an B are two non-empty subsets of H and $x \in H$, then $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$ and $A \circ B = \bigcup \{a \circ b | a \in A, b \in B\}$. For a deeper insight into the basic hyperstructure theory see [4] and [7].

Since the theory of relations and ordered structures is dealt with ordered relations, we need to recall some definitions in this respect. Binary relation R is called *quasi order* if it is reflexive and transitive. Also, if the binary relation R is reflexive, transitive and antisymmetric, then it is known as a partially order relation. By a quasi (partially) ordered (semi)group, we mean a triple $(S,\cdot,R)$, where $(S,\cdot)$ is a (semi)group and R is a quasi (partially) order relation on S such that for all $x,y,z\in S$ with the property xRy it holds $(x\cdot z)R(y\cdot z)$ and $(z\cdot x)R(z\cdot y)$. This property is known as monotone condition. Moreover, the notation $[x)_R$ used below stands for the set $\{s\in S; xRs\}$ and also $[A)_R=\bigcup\limits_{x\in A}[x)_R$. Similarly, $(x]_R=\{s\in S; sRx\}$ and $(A]_R=\bigcup\limits_{x\in A}(x]_R$. The EL-hyperstructures or *Ends lemma* based hyperstructures are hyperstructures constructed from a quasi (partially) ordered (semi)groups using "Ends lemma". This concept was first introduced by Chvalina in 1995 [3]. In particular, Chvalina proved that:

Lemma 2.1. *([3], Theorem 1.3) Let $(S,\cdot,R)$ be a partially ordered semigroup. Binary hyperoperation $\circ: S\times S\longrightarrow \mathcal{P}^*(S)$ defined by $a\circ b=[a\cdot b)_R=\{x\in S, a\cdot bRx\}$ is associative. The semihypergroup $(S,\circ)$ is commutative if and only if the semigroup $(S,\cdot)$ is commutative.*

Theorem 2.2. *([3], Theorem 1.4) Let $(S,\cdot,R)$ be a partially ordered semigroup. The following conditions are equivalent:*

(1) *For any pair $(a,b)\in S^2$ there exists a pair $(c,c_1)\in S^2$ such that $(b\cdot c)Ra$ and $(c_1\cdot b)Ra$.*
(1) *The associated semihypergroup $(S,\circ)$ is a hypergroup.*

Remark 2.3. If $(S,\cdot,R)$ is a partially ordered group, then if we take $c=b^1\cdot a$ and $c=a\cdot b^1$, then condition II is valid. Therefore, if $(S,\cdot,R)$ is a partially ordered group, then its associated hyperstructure is a hypergroup.

Remark 2.4. The text of the above Theorems is the exact translation of theorems from [3]. The respective proofs, however, do not change in any way, if we regard quasi ordered structures instead of partially ordered ones as the anti-symmetry of the relation R is not needed (with the exception of the $\Leftarrow$ implication of the part on commutativity, which does not hold in this case). The commonly quoted version of the "Ends lemma" is therefore the version assuming quasi ordered structures.

3. Enumeration EL-hyperstructures of order 2

In this section, we discus and investigate EL-hypergroups, EL-semihypergroups, EL^2-H_v-groups and EL^2-hypergroups, with two elements, in three different subsections. As defined before, EL-hyperstructures are a category of hyperstructures constructed on quasi ordered base structures on which a special hyperoperation is defined. So, we have to regard all quasi order relation on an arbitrary set with two elements. Therefore, in all the above these subsections, we need the following theorem.

Theorem 3.1. *Let $H = \{1,2\}$. Then there are 4 quasi order relations on H as follows:.*

$$R_1 = \{(1,1),(2,2)\},$$
$$R_2 = \{(1,1),(2,2),(1,2),(2,1))\} = H \times H,$$
$$R_3 = \{(1,1),(2,2),(2,1),\},$$
$$R_4 = \{(1,1),(2,2),(1,2)\}.$$

3.1. **EL-semihypergroup and EL-hypergroups of order 2.** EL-semihypergroups are hyperstructures constructed on a quasi ordered semigroup using Ends Lemma. By Lemma 2.1, if we start with a quasi ordered semigroup $(S,\cdot,R)$ and set the EL-construction on it, the resulted hyperstructure $(S,\circ)$ would be a semihypergroup. So, for the first step, we need all semigroups of order 2. Distler in [9] classified and enumerated finite semigroups with small order. Then

Theorem 3.2. *[9] There are only 5 non-isomorphic semigroups of order 2 in Table 1.*

Table 1. Classification of the semigroups of order 2

$\cdot_1$	1	2
1	1	2
2	2	1

$\cdot_2$	1	2
1	1	1
2	1	1

$\cdot_3$	1	2
1	1	1
2	1	2

$\cdot_4$	1	2
1	1	2
2	1	2

$\cdot_5$	1	2
1	1	1
2	2	2

By Theorem 3.1 there are 4 quasi ordered relations on a set with two elements. Hence, by Theorem 3.2 there are 4*5=20 triple $(S,\cdot_i,R_j)$ for $1 \leq i \leq 5$ and $1 \leq j \leq 4$. Now, we look after the ones which are quasi ordered semigroups. (i.e. those which has the monotone condition).

Theorem 3.3. *Let $S = \{1,2\}$. Then*

1. *The triple $(S,\cdot_i,R_j)$ is quasi ordered semigroups for all $i = 1,2,\cdots,5$ and $j = 1,2$.*
2. *The triple $(S,\cdot_1,R_3)$ is not a quasi ordered semigroup.*
3. *The triple $(S,\cdot_i,R_3)$ is a quasi ordered semigroup for all $i = 2,3,4,5$.*
4. *The triple $(S,\cdot_1,R_4)$ is not a quasi ordered semigroup.*
5. *The triple $(S,\cdot_i,R_4)$ is a quasi ordered semigroup for all $i = 2,3,4,5$.*

Proof. (1) We consider the following two cases:

(i) For $j = 1$, the proof is straightforward, since R_1 consists diagonal pairs $(x,x), x \in S$ and product of any element in these pairs are again diagonal.

(ii) Because $R_2 = S \times S$, all possible pairs $(x \cdot y, x \cdot z)$ and $(y \cdot x, z \cdot x)$ for all $x, y, z, \in S$, are contained in R_2 and therefor the monotone condition holds.

To prove part (2) and (3) we should focus on $(2,1) \in R_3$ and ignore diagonal elements.

(2) Look at $(2,1) \in R_3$. Since $(2 \cdot_1 2, 2 \cdot_1 1) = (1,2) \notin R_3$, the triple $(S, \cdot_1, R_3)$ is not a quasi ordered semigroup.

(3) We consider the following four cases:
   (i) The triple $(S, \cdot_2, R_3)$ has monotone condition since in $(S, \cdot_2)$ we have $x \cdot_2 y = 1$ for all $x, y \in S$ and $(1,1) \in R_3$.
   (ii) Consider $(S, \cdot_3, R_3)$. It is clear that we should focus on $(2,1) \in R_3$ and ignore diagonal elements. Now, it holds $(2 \cdot_3 2, 1 \cdot_3 2) = (1,1) \in R_3$ and $(2 \cdot_3 1, 1 \cdot_3 1) = (1,1) \in R_3$. Notice that $(S, \cdot_3)$ is abelian.
   (iii) Look at $(S, \cdot_4, R_3)$ and $(2,1) \in R_3$. It holds $(2 \cdot_4 2, 1 \cdot_4 2) = (2,2) \in R_3$, $(2 \cdot_4 2, 2 \cdot_4 1) = (2,1) \in R_3$, $(1 \cdot_4 2, 1 \cdot_4 1) = (2,1) \in R_3$ and $(2 \cdot_4 1, 1 \cdot_4 1) = (1,1) \in R_3$. Hence, the monotone condition holds.
   (iv) Look at $(S, \cdot_5, R_3)$ and $(2,1) \in R_3$. It holds $(2 \cdot_5 2, 2 \cdot_5 1) = (2,2) \in R_3$, $(2 \cdot_5 2, 1 \cdot_5 2) = (2,1) \in R_3$, $(2 \cdot_5 1, 1 \cdot_5 1) = (2,1) \in R_3$ and $(1 \cdot_5 2, 1 \cdot_5 1) = (1,1) \in R_3$. Therefore $(S, \cdot_5, R_3)$ is a quasi ordered semigroup.

To prove part (4) and (5) we should focus on $(1,2) \in R_4$ and ignore diagonal elements.

(4) Look at $(1,2) \in R_4$. Since $(1 \cdot_1 2, 2 \cdot_1 2) = (2,1) \notin R_4$, the triple $(S, \cdot_1, R_4)$ is not a quasi ordered semigroup.

(5) We consider the following four cases:
   (i) The triple $(S, \cdot_2, R_4)$ has monotone condition, since in $(S, \cdot_2)$ we have $x \cdot_2 y = 1$ for all $x, y \in S$ and $(1,1) \in R_3$.
   (ii) Consider $(S, \cdot_3, R_4)$. Now, it holds $(1 \cdot_3 1, 1 \cdot_3 2) = (1,1) \in R_4$ and $(1 \cdot_3 2, 2 \cdot_3 2) = (1,2) \in R_4$. Notice that $(S, \cdot_3)$ is abelian.
   (iii) Look at $(S, \cdot_4, R_4)$ and $(1,2) \in R_4$. It holds $(1 \cdot_4 1, 1 \cdot_4 2) = (1,1) \in R_4$, $(1 \cdot_4 1, 2 \cdot_4 1) = (1,2) \in R_4$, $(2 \cdot_4 1, 2 \cdot_4 2) = (1,2) \in R_4$ and $(1 \cdot_4 2, 2 \cdot_4 2) = (2,2) \in R_4$. Hence, the monotone condition holds.
   (iv) Look at $(S, \cdot_5, R_4)$ and $(1,2) \in R_4$. It holds $(1 \cdot_5 1, 1 \cdot_5 2) = (1,1) \in R_4$, $(1 \cdot_5 1, 2 \cdot_5 1) = (1,2) \in R_4$, $(2 \cdot_5 1, 2 \cdot_5 2) = (2,2) \in R_4$ and $(1 \cdot_5 2, 2 \cdot_5 2) = (1,2) \in R_4$. Therefore $(S, \cdot_5, R_4)$ is a quasi ordered semigroup.

□

Therefore, by Theorem 3.3 we obtain exactly 18 quasi ordered semigroups with two elements. Setting EL-construction on them and we can achieve 12 different semihypergroups as below:

Theorem 3.4. *The all EL-semihypergroups obtained from quasi ordered semigroups of order two are given in Table 2.*

Table 2. Classification of the EL-semihypergroups of order 2

$\circ_1$	1	2
1	1	2
2	2	1

$\circ_2$	1	2
1	1	1
2	1	1

$\circ_3$	1	2
1	1	1
2	1	2

$\circ_4$	1	2
1	1	2
2	1	2

$\circ_5$	1	2
1	1	1
2	2	2

$\circ_6$	1	2
1	1	1
2	1	H

$\circ_7$	1	2
1	H	H
2	H	2

$\circ_8$	1	2
1	1	H
2	1	H

$\circ_9$	1	2
1	H	2
2	H	2

$\circ_{10}$	1	2
1	1	1
2	H	H

$\circ_{11}$	1	2
1	H	H
2	2	2

$\circ_{12}$	1	2
1	H	H
2	H	H

Proof. In the following, we discover the ways how the above 12 semihypergroups can be gotten:

(1) Clearly, setting EL-construction on $(S, \cdot_i, R_1)$, we can achieve semihypergroup $(S, \circ_i)$.
(2) Setting EL-construction on $(S, \cdot_2, R_3)$, we can achieve semihypergroup $(S, \circ_2)$.
(3) Semihypergroup $(S, \circ_6)$ can be achieved by setting EL-construction on $(S, \cdot_3, R_3)$.
(4) Semihypergroup $(S, \circ_7)$ can be achieved by setting EL-construction on $(S, \cdot_3, R_4)$.
(5) Semihypergroup $(S, \circ_8)$ can be achieved by setting EL-construction on $(S, \cdot_4, R_3)$.
(6) Semihypergroup $(S, \circ_9)$ can be achieved by setting EL-construction on $(S, \cdot_4, R_4)$.
(7) Semihypergroup $(S, \circ_{10})$ can be achieved by setting EL-construction on $(S, \cdot_5, R_3)$.
(8) Semihypergroup $(S, \circ_{11})$ can be achieved by setting EL-construction on $(S, \cdot_5, R_4)$.
(9) Semihypergroup $(S, \circ_{12})$ can be achieved by setting EL-construction on $(S, \cdot_2, R_4)$ and $(S, \cdot_i, R_2), i = 1, 2, \cdots, 5$.

□

Theorem 3.5. *Among 8 hypergroups of order 2, there are 3 ones which have EL-construction. (i.e. there are 3 EL-hypergroups of order 2)*

Proof. Vougiouklis in [26] showed that there are 8 hypergroups of order 2. Now, considering the above 12 EL-semihypergroups, we can see that $(H, \circ_1)$, $(H, \circ_7)$ and $(H, \circ_{12})$ have reproduction property and therefore they are the only

EL-hypergroups with 2 elements. Moreover, as it can be seen, $(H, \circ_7)$ is the only non-trivial EL-hypergroup (hypergroup which is not total hypergroup nor group $\mathbb{Z}_2$) of rank 2. □

Corollary 3.6. *The group $\mathbb{Z}_2$ is an EL-hypergroup of order 2 which can be constructed by only one way in EL-construction.*

Proof. Look at $(S, \cdot_1)$ (which is $\mathbb{Z}_2$). By part (1) of Theorem 3.3, the triple $(S, \cdot_1, R_i)$ is quasi ordered semigroup for $i = 1, 2$. Now, creating the EL-hyperstructure associated to $(S, \cdot_1, R_1)$, we get $\mathbb{Z}_2$ itself. □

3.2. **EL^2-H_v-hypergroups of order 2.** In this section we try to count and classify all EL^2-H_v-hypergroups with two elements. Before this and for more convenience we recall some preliminaries of EL^2-hyperstructures from [10].

Definition 3.7. An algebraic hyperstructure $(H, \circ, R)$ is called a *(partially) quasi ordered* hypergroupoid if $(H, \circ)$ is a hypergroupoid and "R" is a (partially) quasi order relation on H such that for all $a, b, c \in H$ with the property $a\overline{R}b$ there holds $a \circ c\overline{R}b \circ c$ and $c \circ a\overline{R}c \circ b$ (monotone condition), where if A and B are non-empty subsets of H, then we say ARB whenever for all $a \in A$, there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that aRb.

Example 3.8. *Let $(S, \cdot, R)$ be a (partially) quasi ordered semigroup. If for every $x, y \in S$, set $x \circ y = \{x^i : i \in \mathbb{N}\}$, then $(S, \circ, R)$ is a (partially) quasi ordered semihypergroup.*

Example 3.9. *Let (X, R) be a (partially) quasi ordered set. If for every $x, y \in X$, set $x \circ y = \{x, y\}$, then $(X, \circ, R)$ is a (partially) quasi ordered hypergroup.*

Example 3.10. *Consider $(H = \{1, 2, 3\}, \circ, \leq)$ where "$\leq$" is ordinary "$\leq$" relation and hyperoperation "$\circ$" is given by the Table 3.*

TABLE 3. Ordered hypergroup of order 3

$\circ$	1	2	3
1	1	1, 2	1, 3
2	1, 2	2	2, 3
3	1, 3	2, 3	3

It easy to check that $(H, \circ, \leq)$ is an ordered hypergroup.

Definition 3.11. Suppose $(H, \circ, R)$ is a (partially) quasi ordered hypergroupoid. For $a, b \in H$, we define the new hyperoperation $* : H \times H \longrightarrow \mathcal{P}^*(H)$ as follows:

$$a * b = [a \circ b)_R = \bigcup_{m \in a \circ b} [m)_R.$$

Remark 3.12. From now on, we name $(H, *)$ as the EL^2-hypergroupoid associated to (partially) quasi ordered hypergroupoid $(H, \circ, R)$.

Theorem 3.13. *Let $(S, \cdot, R)$ be a (partially) quasi ordered H_v-semigroup i.e. the hyperoperation $\cdot$ is weak associative. Then, the hyperoperation $*$ on H, defined in Definition 3.11, is weak associative and therefore $(H, *)$ is an H_v-semigroup.*

Corollary 3.14. *If $(H, \circ, R)$ is a (partially) quasi ordered H_v-group, then $(H, *)$ is an H_v-group.*

The converse of the above corollary does not hold. Look at the following example.

Example 3.15. *Consider the hypergroupoid $(H = \{1, 2\}, \circ)$ in Table 4. $(H =$*

TABLE 4. hypergroupoid

$\circ$	1	2
1	1	1
2	2	1

$\{1, 2\}, \circ, R)$ which $R = H \times H$ is a quasi ordered hypergroupoid.

Since $(2\circ 1)\circ 2 \bigcap 2\circ(1\circ 2) = \emptyset$, then $(H, \circ)$ is not an H_v-group. Clearly it has the monotone condition, so we can set EL^2-construction on it. Its associated EL^2-hyperstructure has the Table 5. the hypergroupoid $(H, *)$ is an H_v-group.

TABLE 5. EL^2-hypergroup

$*$	1	2
1	H	H
2	H	H

Now, in order to find and study EL^2-H_v-groups of order 2, we need all H_v-groups. Bayon and Lygeros in [2] showed that the next Theorem:

Theorem 3.16. *[2] There are, up to isomorphism, 20 H_v-groups of order 2 are give in Table 6.*

Remark 3.17. Among these 20 H_v-groups there are 8 ones which are hypergroups. We mention them by a " $*$ " sign in the related Cayley tables of Table 6.

In EL^2-construction, as in EL-construction, we need the triple $(H, \star_i, R_j)$ which has the monotone condition. (i.e. those who are quasi ordered H_v-groups). So, we can see that:

Proposition 3.18. *For all $i \in \{1, 2, \ldots, 20\}$ and $j \in \{1, 2\}$, the triple $(H, \star_i, R_j)$ is a quasi ordered H_v-group.*

Proof. It is straightforward. □

TABLE 6. All H_v-groups of order 2

$\star_1^*$	1	2
1	1	2
2	2	1

$\star_2$	1	2
1	H	2
2	2	1

$\star_3$	1	2
1	1	H
2	2	1

$\star_4$	1	2
1	1	2
2	H	1

$\star_5^*$	1	2
1	H	1
2	1	2

$\star_6$	1	2
1	H	H
2	2	1

$\star_7$	1	2
1	H	2
2	H	1

$\star_8$	1	2
1	1	H
2	H	1

$\star_9$	1	2
1	2	H
2	H	1

$\star_{10}^*$	1	2
1	H	H
2	1	2

$\star_{11}^*$	1	2
1	H	1
2	H	2

$\star_{12}^*$	1	2
1	1	H
2	H	2

$\star_{13}$	1	2
1	H	1
2	1	H

$\star_{14}$	1	2
1	H	2
2	1	H

$\star_{15}$	1	2
1	H	1
2	2	H

$\star_{16}^*$	1	2
1	H	H
2	H	1

$\star_{17}^*$	1	2
1	H	H
2	H	2

$\star_{18}$	1	2
1	H	H
2	1	H

$\star_{19}$	1	2
1	H	H
2	2	H

$\star_{20}^*$	1	2
1	H	H
2	H	H

Proposition 3.19. *For $j \in \{3,4\}$, triples $(H, \star_{12}, R_j)$ and $(H, \star_{17}, R_j)$ are quasi ordered H_v-groups.*

Proof. Consider $(H, \star_{12}, R_3)$. As mentioned before, it is enough to focus on $(2,1) \in R_3$. We have $(2 \star_{12} 2, 2 \star_{12} 1) = (2, H)$ and $(2 \star_{12} 1, 1 \star_{12} 1) = (H, H)$ are both in $\overline{R}_3$ and since $(H, \star_{12})$ is abelian, it can be concluded that $(H, \star_{12}, R_3)$ has the monotone condition. (i.e. it is a quasi ordered H_v-group). Notice that, by the relation $(A, B) \in \overline{R}$ we mean $A\overline{R}B$ (See Definition 3.7). The proof of the rest are similar. □

Proposition 3.20. *For all $i \in \{1, 2, \ldots, 19\} - \{12, 17\}$ and $j \in \{3,4\}$, the triple $(H, \star_i, R_j)$ is not a quasi ordered H_v-group.*

Proof. We consider the following 7 cases to prove:

(1) $(H, \star_1, R_3)$ and $(H, \star_1, R_4)$ do not have monotone condition by parts (2) and (4) of Theorem 3.3.
(2) $(H, \star_i, R_3)$ is not a quasi ordered H_v-group because $(2,1) \in R_3$ and $(2 \star_i 2, 2 \star_i 1) \notin \overline{R}_3$ for all $i = 2, 3, 4, 6, 7, 8, 9, 15, 16, 19$.
(3) $(H, \star_i, R_3)$ is not a quasi ordered H_v-group because $(2,1) \in R_3$ and $(2 \star_i 1, 1 \star_i 1) \notin \overline{R}_3$ for all $i = 5, 10, 11, 13, 14, 18$.
(4) $(H, \star_i, R_4)$ is not a quasi ordered H_v-groups because $(1,2) \in R_4$ and $(1 \star_i 2, 2 \star_i 2) \notin \overline{R}_4$ for all $i = 2, 3, 4, 6, 7, 8, 9, 16$.
(5) $(H, \star_i, R_4)$ is not a quasi ordered H_v-groups because $(1,2) \in R_4$ and $(1 \star_4 1, 2 \star_i 1) \notin \overline{R}_4$ for all $i = 5, 10, 13, 14, 18$.
(6) $(H, \star_i, R_4)$ is not a quasi ordered H_v-groups because $(1,2) \in R_4$ and $(1 \star_4 1, 1 \star_i 2) \notin \overline{R}_4$ for $i = 11, 15$.

(7) $(H, \star_{19}, R_4)$ is not a quasi ordered H_v-groups because $(1,2) \in R_4$ and $(2 \star_4 1, 2 \star_i 2) \notin \overline{R}_4$.

Therefore the proof is complete. □

Now, by Proposition 3.18 and 3.19, we have:

Corollary 3.21. *There are 44 quasi ordered H_v-groups of order 2.*

Definition 3.22. Suppose $(H, *)$ is an H_v-group. Then, $(H, *)$ is said to be a nontrivial H_v-group if it is not total H_v-group (i.e. $a * b = H$ for all $(a, b) \in H$) nor it is not associated to $(H, \star_i, R_1)$, $i \in \{1, 2, \cdots, 20\}$ in EL^2-construction.

Theorem 3.23. *There is only one non-trivial EL^2-H_v-group of order 2.*

Proof. First of all, consider the quasi ordered H_v-groups founded in 3.18. It is clear that $(H, \star_i, R_1)$ leads to $(H, \star_i)$, for all $i = 1, 2, \cdots, 20$, via EL^2-construction, which are trivial by Definition 3.22. Also, for all $i \in \{1, 2, \cdots, 20\}$, the triple $(H, \star_i, R_2)$ leads to total H_v-group. Now, consider 4 non-trivial quasi ordered H_v-groups founded in 3.19. (i.e. $(H, \star_{12}, R_i)$ and $(H, \star_{17}, R_i)$ for $i = 3, 4$).

Setting EL^2-construction on $(H, \star_{12}, R_4)$ and $(H, \star_{17}, R_3)$, we can get $(H, \star_{17})$. Also, if we do the same for $(H, \star_{12}, R_3)$, we achieve an H_v-group with the following Cayley table: The H_v-group in Table 7, is isomorphic to $(H, \star_{17})$. Also,

Table 7. H_v-group

$*$	1	2
1	1	H
2	H	H

$(H, \star_{17}, R_4)$ leads to total H_v-group. Hence, $(H, \star_{17})$ is the only proper H_v-group which has EL^2-construction. □

Definition 3.24. The H_v-group $(H, *)$ is said to be a proper H_v-group if it is not a hypergroup. (i.e. the hyperoperation $*$ is weak associative but not associative.)

Proposition 3.25. *There is no proper EL^2-H_v-group created by semigroups.*

Proof. Let $(S, \cdot, \leq)$ be a quasi ordered semigroup and $(S, *)$ be its associated EL-hyperstructure. By Lemma 2.1, the hyperoperation $*$ is associative and therefore $(S, *)$ can not be a proper H_v-group. □

Corollary 3.26. *There is no non-trivial proper H_v-group with EL^2-construction.*

Proof. By Theorem 3.23, there is only one EL^2-H_v-group $(H, \star_1 7)$ which is not proper. □

3.3. **EL^2-hypergroups of order 2.** Chvalina obtained semihypergroup from a quasi (partially) ordered semigroup $(S, \cdot, R)$, defined a special hyperoperation on it and in this way achieved a semihypergroup called EL-semihypergroup [3]. It is known as "Ends lemma". Ghazavi et. al., extended the hyperoperation used by Chvalina and started from a quasi (partially) ordered semihypergroup. They called the resulted hyperstructure as EL^2-semihypergroup. In other words, EL^2-hypergroups are hyperstructures based on (partially) quasi ordered (semi)hypergroup instead of a (partially) quasi ordered (semi)group. More details of this family of hyperstructures can be found in [10]. In this section, we try to classify and enumerate all EL^2-hypergroups. Before this, we recall two important Theorems from [10].

Theorem 3.27. *[10] Let $(H, \circ, R)$ be a (partially) quasi ordered semihypergroup. The hyperoperation $*$ on H, defined in Definition 3.11, is associative and therefore $(H, *)$ is a semihypergroup.*

Corollary 3.28. *[10] If $(H, \circ, R)$ is a (partially) quasi ordered hypergroup, then $(H, *)$ is a hypergroup.*

Theorem 3.29. *There is only one non-trivial EL^2-hypergroup of order 2.*

Proof. As mentioned in Remark 3.17, there are eight hypergroup of order 2. ($(H, \star_i)$ for $i \in \{1, 5, 10, 11, 12, 16, 17, 20\}$). Now, if we repeat the proof of Theorem 3.23, we see that $(H, \star_{17})$ is the only non-trivial EL^2-hypergroup of order 2. □

4. Conclusion

In this paper, we enumerated all possible EL-hypergroups, EL-semihypergroups obtained from quasi ordered semigroups of order 2 and they are given in Table 8 (See Theorem 3.4 and Table 2). In Table 8 the Cayley table $(abcd)$ means that $a = 1 \circ 1$, $b = 1 \circ 2$, $c = 2 \circ 1$ and $d = 2 \circ 2$. Also $EL-s.$ means that EL-semihypergroup and *Q. o. semigroup* means that quasi order semigroup.

Table 8. EL-semihypergroups obtain from quasi ordered semigroups

	$EL-s.$	$Q.\ o.\ semigroup$		$EL-s.$	$Q.\ o.\ semigroup$
1	$(S, \cdot_i)$	$(S, \cdot_i, R_1)$ $where\ i = 1, 3, 4, 5.$	6	$(HHHH)$	$(S, \cdot_2, R_4)\ and\ (S, \cdot_i, R_2)$ $where\ i = 1, 2, \ldots, 5.$
2	(1111)	$(S, \cdot_2, R_3)\ and\ (S, \cdot_2, R_1)$	7	$(111H)$	$(S, \cdot_3, R_3)$
3	$(HHH1)$	$(S, \cdot_3, R_4)$	8	$(1H1H)$	$(S, \cdot_4, R_3)$
4	$(H2H2)$	$(S, \cdot_4, R_4)$	9	$(11HH)$	$(S, \cdot_5, R_3)$
5	$(HH22)$	$(S, \cdot_5, R_4)$			

By Table 8, there are, up to isomorphism, 12 EL-semihypergroups and 3 EL-hypergroups obtained from quasi ordered semigroups.

Also, by Theorems 3.23 and 3.29, we computed all EL^2-hyperstructures with 2 elements obtained from quasi ordered H_v-groups(see Table 9).

TABLE 9. EL^2-hyperstructures obtain from quasi ordered H_v-groups

$EL^2 - H_v - group$	$Quasi\ ordered\ H_v - group$
$(H, \star_i)$	$(H, \star_i, R_1)$ $where\ i = 1, 2, \ldots, 16.$
$(H, \star_{17})$	$(H, \star_{12}, R_4)\ and\ (H, \star_{17}, R_3)$ $(H, \star_{12}, R_3) and\ (H, \star_{17}, R_1)$
$(H, \star_i)$	$(H, \star_i, R_1)$ $where\ i = 18, 19.$
$(H, \star_{20})$	$(H, \star_{20}, R_1)\ and\ (H, \star_{17}, R_4)$ $(H, \star_i, R_2)$ $where\ i = 1, 2, \ldots, 20.$

By the Table 9, there are, up to isomorphism, 20 EL^2-H_v-groups and 8 EL-hypergroups obtained from quasi ordered H_v-groups.

For future work, it will be interesting to introduce computation EL-hypergroups of order 3. Also, In [7] a classification of polygroups of order 3 is performed. It will be interesting to calculate all quasi ordered polygroups of order 3 and enumerate the EL-hypergroups of order 3 obtained from polygroups. Moreover, here are other types of hyperstructures, for example e-hyperstructure [25], strongly H_v-groups [1] and etc. Using these hyperstructures and "Ends Lemma" method can achieve a new results about EL-hyperstructures.

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References

[1] F. Arabpur, M. Jafarpour, On strongly H_v-groups, *J. Mahani Math. Res. Cent.*, 8(1)(2019) 13-21.

[2] R. Bayon, N. Lygeros, Advanced results in enumeration of hyperstructures, *Journal of Algebra*, 320(2)(2008), 821-835,

[3] J. Chvalina, Functional Graphs, Quasi-Ordered Sets and Commutative Hypergroups, *Masaryk University*, Brno, (1995), (in Czech).

[4] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani, Udine, 2003.

[5] P. Corsini, V. Leoreanu-Fotea, *Applications of Hyperstruture Theory*, Kluwer Academic Publishers, 2003.

[6] B. Davvaz, *Semihypergroup Theory*, Elsevier, 2016.
[7] B. Davvaz, *Polygroup Theory and Related System*, World Scientific Publishing Co. Pte. Ltd., 2012.
[8] B. Davvaz and T. Vougiouklis, *A Walk Through Weak Hyperstructures; H_v-Structure*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019.
[9] A. Distler, *Classification and Enumeration of Finite Semigroups*, Ph.D. Thesis, University of St Andrews, (2010).
[10] S. H. Ghazavi, S. M. Anvarieh and S. Mirvakili, *EL^2–hyperstructures derived from (partially) quasi ordered hyperstructures*, *Iranain Journal of Mathematical Sciences and Informatics*, 10(2) (2015), 99-114.
[11] S. H. Ghazavi, S. M. Anvarieh and S. Mirvakili, Ideals in EL-semihypergroups associated to ordered semigroups, *Journal of Algebraic Systems*, 3(2) (2016), 109-125.
[12] S. H. Ghazavi and S. M. Anvarieh, EL-hyperstructures associated to n-ary relations, *Soft Computing*, 21(2013), 5841-5850.
[13] S. Hoskova, Binary Hyperstructures Determined by Relational and Transformation Systems, Habilitation Thesis, Faculty of Science, Universityof Ostrava, (2008).
[14] R. Migliorato, Ipergruppi di cardinalit 3 e isomorfismi di ipergruppoidi commutativi totalmente regolari, *Atti Convegno su Ipergruppi*, Udine, (1985).
[15] G. Nordo, An algorithm on number of isomorphism classes of hypergroups of order 3, *Italian J. Pure Appl. Math.* 2(1997), 37-42.
[16] M. Novak, Potential of the Ends lemma to create ring-like hyperstructures from quasi-ordered (semi)groups, *South Bohemia Mathem. Letters*, 17(1)(2009), 39-50.
[17] M. Novak, The notion of subhyperstructure of"Ends lemma"-based hyperstructures, *Journal of Applied Mathematics*, 3(2) (2010), 237-247.
[18] M. Novak, Important elements of EL-hyperstructures, in *APLIMAT:10th International Conference*, STU in Bratislava, Bratislava, (2011), 151-158.
[19] M. Novak, *El*-hyperstructures: an overview, *Ratio Mathematica*, 23(2012), 65-80.
[20] M. Novak, Some basic properties of EL-hyperstructure, *European Journal of Combinatorics*, 34(2013), 446-459.
[21] M. Novak, On EL-semihypergroup, *European Journal of Combinatorics*, 44(2015), 274-286.
[22] P. Rackova, Hypergroups of symmetric matrices, *10th International Congress of Algebraic Hyperstructures and Applications*, Proceeding of AHA, (2008).
[23] I. G. Rosenberg, Hypergroups and join spaces determined by relations, *Italian Journal of Pure and Applied Mathematics*, 4(1998), 93-101.
[24] Ch. Tsitouras and Ch. G. Massouros, On enumeration of hypergroups of order 3, *Comput. Math. Appl.* 59(2010) 519-523.
[25] T. Vougiouklis, The e-hyperstructures, *J. Mahani Math. Res. Cent.*, 1(1)(2012) 13-28.
[26] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Florida, 1994.

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