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1-DESIGNS CONSTRUCTED FROM THE GROUPS $PSL_2(81)$ AND $PSL_2(89)$

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Abstract. In this paper, some designs from the primitive permutation representations of the groups $PSL_2(81)$ and $PSL_2(89)$ are constructed using the Key-Moori Method 1. We determine the automorphism groups of all the obtained designs and prove that the groups $PSL_2(81)$, $PSL_2(81){}^{\cdot}2$, $PSL_2(81){:}2$, $PGL_2(81)$, $P\Sigma L_2(81)$, $P\Gamma L_2(81)$, $PSL_2(89)$ and $PSL_2(89){:}2$ appear as the automorphism groups of these constructed designs.

Keywords: Design, Group action, Automorphism group, Projective special linear group.
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1. Introduction

Algebra and combinatorics interact in a significant way when we investigate combinatorial structures using algebraic techniques. Related to our work, the algebra could be group theory and the combinatorics might be block designs. The relationship between Mathieu groups and Witt designs is a well known result which leads us to this important interaction. It is noticeable that the designs with large full automorphism groups are most attractive.

In 2002, Key and Moori [11, 12] introduced a method, which is now known as the Key-Moori Method 1, for constructing designs and graphs from a given group. They used the Janko groups J_1 and J_2 and obtained some designs, binary codes and graphs such that J_1, J_2 and $\bar{J}_2$, the extension of J_2 by its outer automorphism, are the automorphism groups of the designs [11]. Moori and Rodrigues [14, 15] applied this method to the McLaughlin group M^cL and the Conway group Co_2 and obtained some designs and codes invariant under these groups. In the following, Moori and Saeidi [18] obtained some designs and their binary codes from a primitive action of degree 1755 of the Tits group ${}^2F_4(2)'$. They constructed a binary $[1755, 26, 1024]$-code on which ${}^2F_4(2)'$ acts irreducibly. Moreover, they constructed some 1-designs from the maximal subgroups and conjugacy classes of the projective special linear groups $PSL_2(2^n)$

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and the Suzuki groups $Sz(2^{2n+1})$ [19, 20]. Recently, the authors constructed some symmetric 1-designs from the groups $PSL_2(q)$ and the small Ree groups ${}^2G_2(q)$ [13,16,17]. Moreover, some designs and codes are constructed using the Mathieu group M_{11} [1].

Motivated by [11], all the primitive representations of $PSL_2(q)$ (for all possible $q \leq 50$) are considered by Darafsheh et al. [6–8]. They constructed designs and the automorphism groups. Darafsheh [5] investigated the group $PSL_2(q)$ and obtained three designs with parameters $1-\left(\binom{q}{2}, q+1, q+1\right)$, $1-\left(\binom{q+1}{2}, 2(q-1), 2(q-1)\right)$ and $1-\left(\binom{q+1}{2}, q-1, q-1\right)$, where q is a power of 2. Moreover, It is shown that the second one is invariant under the full automorphism group S_{q+1}. In [9, 10], the present author examined designs and the automorphism groups from the groups $PSL_2(53)$, $PSL_2(59)$, $PSL_2(61)$ and $PSL_2(64)$. Here, the Key-Moori Method 1 is employed and 1-designs from the groups $PSL_2(81)$ and $PSL_2(89)$ are obtained. For each constructed design, the parameters and the full automorphism group are determined. It is shown that $PSL_2(81)$, $PSL_2(81){}^{\cdot}2$, $PSL_2(81){:}2$, $PGL_2(81)$, $P\Sigma L_2(81)$, $P\Gamma L_2(81)$, $PSL_2(89)$ and $PSL_2(89){:}2$ appear as the automorphism groups of the constructed designs.

2. Preliminaries

Our notation is standard and we follow the notation of $\mathbb{ATLAS}$ [4] for the structure of groups and their maximal subgroups. The general, split and non-split extensions are denoted by $G.H$, $G{:}H$ and $G{}^{\cdot}H$, respectively. The cyclic group $\mathbb{Z}_m$ is denoted by m. For a prime p and a natural number n, p^n is the elementary abelian group of that order.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure, where $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ are the point set, block set and incidence relation, respectively. When $p \in \mathcal{P}$ and $B \in \mathcal{B}$, we write $p\ \mathcal{I}\ B$ if $(p, B) \in \mathcal{I}$. Moreover, we write $p \in B$ if $\mathcal{I}$ is the membership relation $\in$. The triple $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is named a t-(v, k, λ) design if $|\mathcal{P}| = v$, $|B| = k$ for every $B \in \mathcal{B}$ and any t points is incident with exactly λ blocks. The number of blocks is denoted by b. The design $\mathcal{D}$ is symmetric if $v = b$. The number of blocks through any set of s points, where $s \leq t$, is $\lambda_s := \lambda\binom{\nu-s}{t-s}/\binom{k-s}{t-s}$. The t-(v, k, λ) design $\mathcal{D}$ is thus an s-(v, k, λ_s) design. A t-(v, k, λ) design is called trivial if any subset of $\mathcal{P}$ of size k is a block. The dual of the incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is $\mathcal{S}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I}^t)$, where $(\mathcal{B}, \mathcal{P}) \in \mathcal{I}^t$ if and only if $(\mathcal{P}, \mathcal{B}) \in \mathcal{I}$. The one to one correspondence $\theta : \mathcal{P} \to \mathcal{P}'$ between $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ is called an isomorphism if $p\, \mathcal{I}\, B \leftrightarrow \theta(p)\, \mathcal{I}'\, \theta(B)$ for any $p \in \mathcal{P}$ and $B \in \mathcal{B}$. Now, an isomorphism between $\mathcal{S}$ and itself is called an automorphism of $\mathcal{S}$. The automorphism group of $\mathcal{S}$, i.e. $\mathrm{Aut}(\mathcal{S})$, consists of all the automorphisms of $\mathcal{S}$. See [2] for more information.

Let F_q be the finite field of order q, where $q = p^n$ is a prime power. Set $\bar{F}_q = F_q \cup \{\infty\}$, where ∞ denotes a distinguish symbol. The set of all the invertible 2×2 matrices over F_q, denoted by $GL_2(q)$, forms a group and moreover, the set $SL_2(q) = \{M \in GL_2(q) \mid \det(M) = 1\}$ is a subgroup of $GL_2(q)$. Associated to $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(q)$, a permutation f_M on $\bar{F_q}$ is defined by

$$f_M = \begin{cases} \frac{ax+b}{cx+d}, & \text{if } x \in F_q \text{ and } cx + d \neq 0; \\ \infty, & \text{if } x \in F_q \text{ and } cx + d = 0; \\ a/c, & \text{if } x = \infty \text{ and } c \neq 0; \\ \infty, & \text{if } x = c = 0. \end{cases}$$

The set of all such permutations, denoted by $PGL_2(q)$, is a subgroup of S_{q+1}. Now, the projective special linear group of degree two over F_q is

$$PSL_2(q) = \Big\{x \xmapsto{f_M} \frac{ax+b}{cx+d} \mid M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det(M) \text{ is a non-zero square}\Big\}.$$

The set of all the transformations $f : \bar{F_q} \to \bar{F_q}$ defined by the equation $f(x) = \frac{ax^\sigma+b}{cx^\sigma+d}$ forms the group $P\Gamma L_2(q)$, where $a, b, c, d \in F_q$, $ad - bc \neq 0$ and $\sigma \in \mathrm{Aut}(F_q)$. Moreover, $P\Sigma L_2(q)$ is a subgroup of $P\Gamma L_2(q)$ consisting of those elements of $P\Gamma L_2(q)$ for which $ad - bc = 1$. We have $|PGL_2(q)| = q(q^2 - 1)$ and $|PSL_2(q)| = q(q^2 - 1)/(2, q - 1)$.

Lemma 2.1. *[22] If M is a maximal subgroup of $PSL_2(q)$ then M has one of the following shapes:*

- $D_{2(q-\epsilon)/(2,q-1)}$, *where $\epsilon = \pm 1$, except $\epsilon = 1$, $q = 3, 5, 7, 9, 11$ and $\epsilon = -1$, $q = 2, 7, 9$.*
- *A solvable group of order $q(q-1)/(2, q-1)$.*
- A_4 *if $q > 3$ is prime and $q \equiv 3, 13, 27, 37 \pmod{40}$.*
- S_4 *if $q \neq 2$ is prime and $q \equiv \pm 1 \pmod 8$.*
- A_5 *if $q = 5^m, 4^m$ and m is prime, q is prime and $q \equiv \pm 1 \pmod 5$, or $q \neq 2$ is the square of a prime number and $q \equiv -1 \pmod 5$.*
- $PSL(2, r)$ *if $q = r^m$ and $m \neq 2$ is a prime number.*
- $PGL(2, r)$ *if $q = r^2$.*

For more information on this topic, we refer to [22].

3. The method

Let G be a permutation group which acts on a set Ω. It it known that G acts on the set $\Omega \times \Omega$ by the group action $(\alpha, \beta)^g = (\alpha^g, \beta^g)$, where $\alpha, \beta \in \Omega$ and $g \in G$. Each orbit of G on $\Omega \times \Omega$ is said to be an orbital. For any orbital O, set $O^* = \{(\alpha, \beta) \mid (\beta, \alpha) \in O\}$. Clearly, if O is an orbital then O^* is also an orbital. The set O^* is called the paired orbital of O. Moreover, if $O = O^*$ then O is called self-paired. It is easy to deduce that if $\alpha \in \Omega$ and Δ is an orbit of G_α, where $\Delta \neq \{\alpha\}$, then $\bar{\Delta} = \{(\alpha, \delta)^g \mid g \in G, \delta \in \Delta\}$ is an orbital. The orbit Δ is called self-paired if $\bar{\Delta}$ is a self-paired orbital.

Theorem 3.1. *(Key-Moori Method 1) [11, 12] Let Ω be a set of size n and let G be a finite primitive permutation group acting on Ω. If $\alpha \in \Omega$ and Δ is an orbit of G_α, where $\Delta \neq \{\alpha\}$, then $\mathcal{D} = (\Omega, \Delta^G, \in)$ is a symmetric 1-$(n, |\Delta|, |\Delta|)$ design. The design $\mathcal{D}$ is self-dual if Δ is self-paired. Furthermore, G acts primitively on the points and blocks of $\mathcal{D}$.*

Theorem 3.2. *[21] If $\mathcal{D}$ is the design obtained by Theorem 3.1 then $G \leq \mathrm{Aut}(\mathcal{D})$.*

By a computer program in Magma [3], we construct designs according to the construction method outlined in Theorem 3.1. Tables 1 and 2 contain all of the information about the designs from the groups $PSL_2(81)$ and $PSL_2(89)$, respectively. The shapes of the maximal subgroups are shown under the heading 'Maximal Subgroups' and their indices are under the heading 'Degree'. The column '#' indicates the number of orbits of a stabilizer and the column 'Length' indicates the lengths of the non-singleton orbits. Note that '$m(n)$' means that we have n orbits of length m and the related design is denoted by $\mathcal{D}_{m(n)}$. The orders of the automorphism groups are written under the column '$|\mathrm{Aut}(\mathcal{D})|$'.

4. Designs from the groups $PSL_2(81)$ and $PSL_2(89)$

Computations with Magma show that the projective special linear group $PSL_2(81)$ has five maximal subgroups of orders 82, 80, 720, 720 and 3240, up to conjugacy. By Lemma 2.1, these subgroups are D_{41}, D_{40}, $PGL_2(9)$, $PGL_2(9)$ and a solvable group H, respectively. The action of $PSL_2(81)$ on the cosets of H is 2-transitive and the related design is trivial. So, we don't consider it. Moreover, the third and fourth maximal subgroups give us the same results and we write them in one row. See Table 1.

Theorem 4.1. *(i) For $PSL_2(81)$ of degree 3240, the designs $\mathcal{D}_{41(24)}$, $\mathcal{D}_{41(8)}$, $\mathcal{D}_{41(4)}$, $\mathcal{D}_{41(2)}$, $\mathcal{D}_{41(1)}$ and $\mathcal{D}_{82(20)}$ with parameters 1-$(3240, 41, 41)$, 1-$(3240, 41, 41)$, 1-$(3240, 41, 41)$, 1-$(3240, 41, 41)$, 1-$(3240, 41, 41)$ and 1-$(3240, 82, 82)$ are obtained, respectively, such that $\mathrm{Aut}(\mathcal{D}_{41(24)}) = PSL_2(81)$, $\mathrm{Aut}(\mathcal{D}_{41(8)}) = PSL_2(81){\cdot}2$, $\mathrm{Aut}(\mathcal{D}_{41(4)}) = PSL_2(81){:}2$, $\mathrm{Aut}(\mathcal{D}_{41(2)}) \cong P\Sigma L_2(81)$, $\mathrm{Aut}(\mathcal{D}_{41(1)}) \cong P\Gamma L_2(81)$ and $\mathrm{Aut}(\mathcal{D}_{82(20)}) = PGL_2(81)$.*

(ii) For $PSL_2(81)$ of degree 3321, the designs $\mathcal{D}_{20(2)}$, $\mathcal{D}_{40(24)}$, $\mathcal{D}_{40(12)}$, $\mathcal{D}_{40(2)}$, $\mathcal{D}_{80(20)}$ and $\mathcal{D}_{80(2)}$ with parameters 1-$(3321, 20, 20)$, 1-$(3321, 40, 40)$, 1-$(3321, 40, 40)$, 1-$(3321, 40, 40)$, 1-$(3321, 80, 80)$ and 1-$(3321, 80, 80)$ are obtained, respectively. Moreover, $\mathrm{Aut}(\mathcal{D}_{20(2)}) \cong \mathrm{Aut}(\mathcal{D}_{40(2)}) \cong \mathrm{Aut}(\mathcal{D}_{80(2)}) \cong P\Sigma L_2(81)$, $\mathrm{Aut}(\mathcal{D}_{40(24)}) = PSL_2(81)$, $\mathrm{Aut}(\mathcal{D}_{40(12)}) \cong PSL_2(81){:}2$ and $\mathrm{Aut}(\mathcal{D}_{80(20)}) \cong PGL_2(81)$.

(iii) For $PSL_2(81)$ of degree 369, the designs $\mathcal{D}_{36(1)}$, $\mathcal{D}_{72(1)}$, $\mathcal{D}_{80(1)}$ and $\mathcal{D}_{90(2)}$ with parameters 1-$(369, 36, 36)$, 1-$(369, 72, 72)$, 1-$(369, 80, 80)$ and 1-$(369, 90, 90)$ are obtained, respectively, such that $\mathrm{Aut}(\mathcal{D}_{36(1)}) \cong \mathrm{Aut}(\mathcal{D}_{72(1)}) \cong \mathrm{Aut}(\mathcal{D}_{80(1)}) \cong P\Sigma L_2(81)$ and $\mathrm{Aut}(\mathcal{D}_{90(2)}) \cong PSL_2(81){:}2$.

TABLE 1. 1-designs from $PSL_2(81)$

Maximal Subgroups	Degree	#	Length	$\|Aut(\mathcal{D})\|$
D_{41}	3240	60	41(24)	265680
			41(8)	531360
			41(4)	531360
			41(2)	1062720
			41(1)	2125440
			82(20)	531360
D_{40}	3321	63	20(2)	1062720
			40(24)	265680
			40(12)	531360
			40(2)	1062720
			80(20)	531360
			80(2)	1062720
$PGL_2(9)$	369	6	36(1)	1062720
			72(1)	1062720
			80(1)	1062720
			90(2)	531360

TABLE 2. 1-designs from $PSL_2(89)$

Maximal Subgroups	Degree	#	Length	$\|Aut(\mathcal{D})\|$
S_4	14685	632	6(6)	352440
			8(7)	352440
			12(20)	352440
			24(598)	352440
A_5	5874	112	5(1)	352440
			12(4)	352440
			20(6)	352440
			30(10)	352440
			60(90)	352440
D_{44}	4005	69	22(2)	352440
			44(42)	352440
			88(2)	352440
			88(22)	704880
D_{45}	3916	66	45(42)	352440
			45(1)	704880
			90(22)	704880

Proof. Theorem 3.1 and a Magma program imply that we can construct designs with the above parameters.

By Magma computations, the automorphism groups of $\mathcal{D}_{41(24)}$ and $\mathcal{D}_{40(24)}$ are of order 265680. Since $|PSL_2(81)| = 265680$, Theorem 3.2 implies that the automorphism groups of $\mathcal{D}_{41(24)}$ and $\mathcal{D}_{40(24)}$ are the same and equal to $PSL_2(81)$.

Consider the design $\mathcal{D}_{41(8)}$. By our Magma code, $|\operatorname{Aut}(\mathcal{D}_{41(8)})| = 531360 = 2|PSL_2(81)|$. Computations with Magma show that $\operatorname{Aut}(\mathcal{D}_{41(8)})$ has a maximal subgroup M_0 of index 2 such that $M_0 \cong PSL_2(81)$ and moreover, $\operatorname{Aut}(\mathcal{D}_{41(8)})$ is not a split extension of M_0 by 2. Therefore, $\operatorname{Aut}(\mathcal{D}_{41(8)}) \cong PSL_2(81)^{\cdot}2$.

By Magma computations, it is implied that the automorphism groups of $\mathcal{D}_{41(4)}$, $\mathcal{D}_{40(12)}$ and $\mathcal{D}_{90(2)}$ are isomorphic to each other and $|\operatorname{Aut}(\mathcal{D}_{41(4)})| = 531360 = 2|PSL_2(81)|$. Furthermore, we obtain a maximal subgroup M_1 of $\operatorname{Aut}(\mathcal{D}_{41(4)})$ and an involution in $\operatorname{Aut}(\mathcal{D}_{41(4)}) \setminus M_1$ such that $M_1 \cong PSL_2(81)$. Therefore, $\operatorname{Aut}(\mathcal{D}_{41(4)}) \cong PSL_2(81){:}2$.

By Magma computations, it is deduced that the automorphism groups of $\mathcal{D}_{82(20)}$ and $\mathcal{D}_{80(20)}$ are isomorphic to each other. Moreover, $\operatorname{Aut}(\mathcal{D}_{82(20)})$ has a maximal subgroup isomorphic to $PSL_2(81)$ of index 2. Computations show that $\operatorname{Aut}(\mathcal{D}_{82(20)}) \cong PGL_2(81)$.

Magma computations imply that the automorphism groups of $\mathcal{D}_{41(2)}$, $\mathcal{D}_{20(2)}$, $\mathcal{D}_{40(2)}$, $\mathcal{D}_{80(2)}$, $\mathcal{D}_{36(1)}$, $\mathcal{D}_{72(1)}$ and $\mathcal{D}_{80(1)}$ are isomorphic to each other. Moreover, $\operatorname{Aut}(\mathcal{D}_{41(2)})$ has a normal subgroup M_2 of index $1062720/265680 = 4$ such that $M_2 \cong PSL_2(81)$ and $\operatorname{Aut}(\mathcal{D}_{41(2)})/M_2 \cong 4$. In fact, $\operatorname{Aut}(\mathcal{D}_{41(2)}) \cong P\Sigma L_2(81)$.

Finally, consider the design $\mathcal{D}_{41(1)}$. Magma shows that $|\operatorname{Aut}(\mathcal{D}_{41(1)})| = 2125440$ and $\operatorname{Aut}(\mathcal{D}_{41(1)})$ has a normal subgroup M_3 of index $2125440/265680 = 8$ such that $M_3 \cong PSL_2(81)$. We can see that $\operatorname{Aut}(\mathcal{D}_{41(1)})/M_3 \cong 2 \times 4$. In fact, Magma shows that $\operatorname{Aut}(\mathcal{D}_{41(1)}) \cong P\Gamma L_2(81)$. □

Magma computations show that the projective special linear group $PSL_2(89)$ has seven maximal subgroups of orders 24, 24, 60, 60, 88, 90 and 3916, up to conjugacy. It follows from Lemma 2.1 that these maximal subgroups are isomorphic to S_4, S_4, A_5, A_5, D_{44}, D_{45} and a solvable group H of order 3916, respectively. The action of $PSL_2(89)$ on the cosets of H is 2-transitive and so, the constructed design from this maximal subgroup is trivial and we don't consider it. Moreover, the maximal subgroups with the same orders give us the same results and so, we set them in one row. See Table 2.

Theorem 4.2. *(i) For* $PSL_2(89)$ *of degree 14685, designs* $\mathcal{D}_{6(6)}$, $\mathcal{D}_{8(7)}$, $\mathcal{D}_{12(20)}$ *and* $\mathcal{D}_{24(598)}$ *with parameters* 1-$(14685, 6, 6)$, 1-$(14685, 8, 8)$, 1-$(14685, 12, 12)$ *and* 1-$(14685, 24, 24)$ *are obtained, respectively. Moreover, we have* $\operatorname{Aut}(\mathcal{D}_{6(6)}) = \operatorname{Aut}(\mathcal{D}_{8(7)}) = \operatorname{Aut}(\mathcal{D}_{12(20)}) = \operatorname{Aut}(\mathcal{D}_{24(598)}) = PSL_2(89)$.

(ii) For $PSL_2(89)$ *of degree 5874, designs* $\mathcal{D}_{5(1)}$, $\mathcal{D}_{12(4)}$, $\mathcal{D}_{20(6)}$, $\mathcal{D}_{30(10)}$ *and* $\mathcal{D}_{60(90)}$ *with parameters* 1-$(5874, 5, 5)$, 1-$(5874, 12, 12)$, 1-$(5874, 20, 20)$, 1-$(5874, 30, 30)$ *and* 1-$(5874, 60, 60)$ *are obtained, respectively. Furthermore, we have* $\operatorname{Aut}(\mathcal{D}_{5(1)}) = \operatorname{Aut}(\mathcal{D}_{12(4)}) = \operatorname{Aut}(\mathcal{D}_{20(6)}) = \operatorname{Aut}(\mathcal{D}_{30(10)}) = \operatorname{Aut}(\mathcal{D}_{60(90)}) = PSL_2(89)$.

(iii) For $PSL_2(89)$ *of degree 4005, designs* $\mathcal{D}_{22(2)}$, $\mathcal{D}_{44(42)}$, $\mathcal{D}_{88(2)}$ *and* $\mathcal{D}_{88(22)}$ *with parameters* 1-$(4005, 22, 22)$, 1-$(4005, 44, 44)$, 1-$(4005, 88, 88)$ *and* 1-$(4005, 88, 88)$ *are obtained, respectively, such that* $\mathrm{Aut}(\mathcal{D}_{22(2)}) = \mathrm{Aut}(\mathcal{D}_{44(42)}) = \mathrm{Aut}(\mathcal{D}_{88(2)}) = PSL_2(89)$ *and* $\mathrm{Aut}(\mathcal{D}_{88(22)}) \cong PSL_2(89){:}2$.

(iv) For the group $PSL_2(89)$ *of degree 3916, designs* $\mathcal{D}_{45(42)}$, $\mathcal{D}_{45(1)}$ *and* $\mathcal{D}_{90(22)}$ *with parameters* 1-$(3916, 45, 45)$, 1-$(3916, 45, 45)$ *and* 1-$(3916, 90, 90)$ *are constructed, respectively. We have* $\mathrm{Aut}(\mathcal{D}_{45(1)}) \cong \mathrm{Aut}(\mathcal{D}_{90(22)}) \cong PSL_2(89)$ *:2 and* $\mathrm{Aut}(\mathcal{D}_{45(42)}) = PSL_2(89)$.

Proof. Firstly, notice that the designs with the mentioned parameters are obtained by Magma computations and Theorem 3.1.

Magma computations imply that the order of the automorphism groups of $\mathcal{D}_{6(6)}$, $\mathcal{D}_{8(7)}$, $\mathcal{D}_{12(20)}$, $\mathcal{D}_{24(598)}$, $\mathcal{D}_{5(1)}$, $\mathcal{D}_{12(4)}$, $\mathcal{D}_{20(6)}$, $\mathcal{D}_{30(10)}$, $\mathcal{D}_{60(90)}$, $\mathcal{D}_{22(2)}$, $\mathcal{D}_{44(42)}$, $\mathcal{D}_{88(2)}$ and $\mathcal{D}_{45(42)}$ is equal to 352440. Since $|PSL_2(89)| = 352440$, it is deduced from Theorem 3.2 that the full automorphism groups of the above designs are equal to $PSL_2(89)$.

By Magma, it is deduced that the automorphism groups of $\mathcal{D}_{88(22)}$, $\mathcal{D}_{45(1)}$ and $\mathcal{D}_{90(22)}$ are isomorphic to each other. Theorem 3.2 implies that $PSL_2(89) \leq \mathrm{Aut}(\mathcal{D}_{88(22)})$. Computations with Magma show that $|\mathrm{Aut}(\mathcal{D}_{88(22)})| = 704880 = 2|PSL_2(89)|$ and there exists a maximal subgroup $N \leq \mathrm{Aut}(\mathcal{D}_{88(22)})$ of order 352440 such that $N \cong PSL_2(89)$. Moreover, we find an involution with the cycle type $2^{1980}1^{45}$ in $\mathrm{Aut}(\mathcal{D}_{88(22)}) \setminus N$. So, $\mathrm{Aut}(\mathcal{D}_{88(22)}) \cong PSL_2(89){:}2$. □

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