

AN APPLICATION OF S-ELEMENTARY WAVELETS IN NUMERICAL SOLUTION OF DIFFERENTIAL AND FRACTIONAL INTEGRAL EQUATIONS

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Dedicated to sincere professor Mashaallah Mashinchi

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ABSTRACT. In this article, we introduce wavelet sets and consider a special wavelet set in \mathbb{R} . We build a basis associated with this type of wavelet sets and use an operational matrix of this basis to solve nonlinear Riccati differential equations and Riemann-Liouville fractional integral equations of order $\alpha > 0$, numerically. Convergence analysis of this method is investigated. Also, we give examples that show the accuracy of the new method by comparing it with previous methods.

Keywords: Fractional integral equation, Differential equation, Wavelet sets, s-elementary wavelets.

2020 MSC: 65T60, 65R20, 42C40, 34A08, 45E16.

1. Introduction

1.1. History. After the development of wavelets by Chui [5], Daubechies [9], Dai and Lu [8], Fang and Wang [12] and Hernandez and Weiss [16] and foundation of Multiresolution Analysis (MRA) method by Meyer [24] and Mallat [23], the word wavelet set and how to make it, was first introduced by Dai and Larson [6], Gabardo and Yu [13] and Benedetto and Sumetkijakan [4]. In the sequel, we introduce a special wavelet set in \mathbb{R} . Then we build a basis for $L^2(\mathbb{R})$ and approximate functions via this basis, and use this it for solving nonlinear Riccati differential equations and Riemann-Liouville fractional integral equations of order $\alpha > 0$, numerically.

The nonlinear Riccati differential equations are of much importance and play a significant role in many fields of applied sciences [14, 29]. There are many ways proposed to solve Riccati differential equations, for example, Adomian's decomposition method [2, 11], Variational Iteration method (VIM), Homotopy Perturbation method (HPM) [1–3], the Legendre wavelets method [25] and Homotopy Analysis method (HAM), a piecewise variational iteration method [29]

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and another methods [17, 30].

Development of the theory of fractional integrals and derivatives has begun by Euler, Liouville, and Abel (1823). There are many real problems in physics, mechanics, chemistry and biology that have been formulated via fractional integral equations. Also, there are several methods for solving fractional integral equations such as, He's homotopy [26], Adomian decomposition [22], collocation [21] and power spectral density methods [31] and another methods [10, 18–20, 28].

The aim of this paper is to introduce bases made of wavelet sets to obtain a method for approximating the solution of the Riemann-Liouville fractional integral equations of order $\alpha > 0$ and obtain the approximate solutions of nonlinear Riccati differential equations. The speed of computer calculations and high accuracy are the advantages of this method compared to other methods. Also, the efficiency and accuracy of the presented method are shown by some example and table.

In section 2, we introduce the s-elementary vector wavelets and use them to determine operational matrices.

In section 3, we will analyse the convergence of s-elementary wavelets approximation series.

In the last section, we use s-elementary wavelets for the numerical solution of nonlinear Riccati differential equation and fractional integral equation and compare the results with the previous methods.

1.2. Notations and definitions. We use the standard notations and results from wavelet and wavelet sets as found in [6, 7, 16]. To complete the discussion, we bring the following definitions and theorems.

We know that the function ψ is a 2-dilation wavelet for $L^2(\mathbb{R})$, if the system:

$$\left\{ \psi_m^j(x) = 2^{\frac{j}{2}} \psi(2^j x - 2m\pi) ; m, j \in \mathbb{Z} \right\},$$

is an orthonormal basis for $L^2(\mathbb{R})$.

Also, let $\{V_j\}_{j \in \mathbb{Z}}$ be a sequence of closed subspaces of functions in $L^2(\mathbb{R})$. The collection of spaces $\{V_j\}_{j \in \mathbb{Z}}$ is called a Multiresolution Analysis (MRA) $L^2(\mathbb{R})$, if the following conditions hold:

- (1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
- (2) $f(\cdot) \in V_j$ iff $f(2\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$,
- (3) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,
- (4) $f(\cdot) \in V_0$ iff $f(\cdot - n) \in V_0$ for all $n \in \mathbb{Z}$,
- (5) There exists a function $\varphi \in V_0$, called a **scaling function**, such that the family $\{\varphi(\cdot - m); m \in \mathbb{Z}\}$ is an orthonormal basis for V_0 .

In this paper, for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the Fourier transform and the inverse Fourier transform will be defined by:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \text{ and } \check{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{i\xi x} d\xi.$$

Lemma 1.1. [16] *Let φ be the scaling function of MRA and let ψ be the corresponding wavelet function. Then $|\hat{\varphi}(\xi)|^2 = \sum_{j=0}^{+\infty} |\hat{\psi}(2^j \xi)|^2$, a.e $\xi \in \mathbb{R}$.*

In the following, We are going to introduce the wavelet set and its characterization. The measurable set $W \subset \mathbb{R}$, with finite measure, is a **wavelet set** for $L^2(\mathbb{R})$, if the inverse Fourier transform of $\hat{\psi} = \chi_W$ is an orthonormal wavelet for $L^2(\mathbb{R})$.

Similarly Dai and Larson in [6] called this type of wavelets an **s-elementary wavelets** (The prefix “s” is for “set”). Also, by Fang and Wang [12], a measurable set $W \subset \mathbb{R}$ is a wavelet set if and only if $\{2^j W; j \in \mathbb{Z}\}$ and $\{W + 2\pi m; m \in \mathbb{Z}\}$ are both partitions of \mathbb{R} .

As a specific example, Hernandez et al. in [15] proved that $W_c = [4\pi(c-1), 2\pi(c-1)) \cup (2\pi c, 4\pi c]$, where $c \in (0, 1)$ is a 2-dilation wavelet set. Such that, if $c = 0.5$ then, $W_{0.5} = [-2\pi, -\pi) \cup (\pi, 2\pi]$ is the **Shannon** wavelet set.

2. Operational matrices corresponding to the basis created by wavelet sets

From now on, consider k as a fixed number unless it is mentioned as something else.

We put ${}_c \hat{\psi} = \frac{1}{\sqrt{2\pi}} \chi_{W_c}$. Then it is a s-elementary wavelet and we define ${}_c \hat{\psi}_{mk}(x) := \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}} {}_c \hat{\psi}(2^k x - 2m\pi)$ for $k, m \in \mathbb{Z}$ and $c \in (0, 1)$.

Lemma 2.1. *The system $\{{}_c \hat{\psi}_{mk}; m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Proof. By definition, obviously

$$(1) \quad {}_c \hat{\psi}_{mk}(x) = \begin{cases} 0, & x < \frac{4\pi(c-1)+2m\pi}{2^k}, \\ \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}}, & \frac{4\pi(c-1)+2m\pi}{2^k} \leq x < \frac{2\pi(c-1)+2m\pi}{2^k}, \\ 0, & \frac{2\pi(c-1)+2m\pi}{2^k} \leq x \leq \frac{2\pi c+2m\pi}{2^k}, \\ \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}}, & \frac{2\pi c+2m\pi}{2^k} < x \leq \frac{4\pi c+2m\pi}{2^k}, \\ 0, & \frac{4\pi c+2m\pi}{2^k} < x, \end{cases}$$

and by (1) we have

$$\text{supp} \left({}_c \hat{\psi}_{mk} \right) = \left[\frac{4\pi(c-1)+2m\pi}{2^k}, \frac{2\pi(c-1)+2m\pi}{2^k} \right) \cup \left(\frac{2\pi c+2m\pi}{2^k}, \frac{4\pi c+2m\pi}{2^k} \right],$$

and a simple calculation show that, for $m \neq n$ and fix k ,

$$\lambda \left(\text{supp} \left({}_c \hat{\psi}_{mk} \right) \cap \text{supp} \left({}_c \hat{\psi}_{nk} \right) \right) = 0,$$

where λ is the Lebesgue measure. So we get:

$$\left\langle {}_c \hat{\psi}_{mk}, {}_c \hat{\psi}_{nk} \right\rangle = \int_{-\infty}^{+\infty} {}_c \hat{\psi}_{mk}(x) {}_c \hat{\psi}_{nk}(x) dx = \begin{cases} 0, & \text{if } m \neq n, \\ 1, & \text{if } m = n. \end{cases}$$

Where $\langle \cdot, \cdot \rangle$ is an inner product on $L^2(\mathbb{R})$. \square

Also, if ${}_c \hat{\varphi}$ is the scaling function corresponding to ${}_c \hat{\psi}$, then by Lemma 1.1, ${}_c \hat{\varphi} = \frac{1}{\sqrt{2\pi}} \chi_{Q_c}$, with $Q_c = [2\pi(c-1), 2\pi c]$ is a basis for $L^2(\mathbb{R})$. So, if for $k, m \in \mathbb{Z}$ and $c \in (0, 1)$ we define ${}_c \hat{\varphi}_{mk}(x) := \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}} {}_c \hat{\varphi}(2^k x - 2m\pi)$.

Lemma 2.2. *The system $\{{}_c \hat{\varphi}_{mk} ; m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Proof. It is similarly to proof Lemma 2.1. \square

Let ${}_{(c)}\hat{\Psi}_{mk}$ be the vector of s-elementary wavelets, so we put:

$$(2) \quad {}_{(c)}\hat{\Psi}_{mk} = \left[{}_c \hat{\psi}_{0k}, {}_c \hat{\psi}_{1k}, {}_c \hat{\psi}_{2k}, \dots, {}_c \hat{\psi}_{mk} \right]^T,$$

As a consequence of definition (2), we can obtain appropriate approximation of functions in $L^2[0, 1]$, if we choose k large enough. Also using large k 's one can cover the interval $[0, 1]$ with the union of supports of ${}_c \hat{\psi}_{mk}$'s i. e. $V = \cup_{i=0}^m \text{supp}({}_c \hat{\psi}_{ik})$ so that the measure of $V \setminus [0, 1]$ be smaller than every $\epsilon > 0$. Then for $f \in L^2[0, 1]$, we have:

$$(3) \quad f \simeq \sum_{i=0}^m d_{ik} {}_c \hat{\psi}_{ik} = D^T {}_{(c)}\hat{\Psi}_{mk},$$

where $d_{ik} = \langle f, {}_c \hat{\psi}_{ik} \rangle$, $D^T = [d_{0k}, d_{1k}, \dots, d_{mk}]$.

Also, the integration of entries ${}_c \hat{\Psi}_{mk}$ can be expanded in terms of

$$(4) \quad \int_0^x {}_{(c)}\hat{\Psi}_{mk}(t) dt = {}_{(c)}\hat{P} {}_{(c)}\hat{\Psi}_{mk}(x),$$

where the $(m+1) \times (m+1)$ matrix ${}_{(c)}\hat{P} = [{}_{(c)}\hat{p}_{ij}]$ is called the operational matrix of s-elementary wavelets and its entries are:

$${}_{(c)}\hat{p}_{ij} = \left\langle \int_0^{(\cdot)} {}_c \hat{\psi}_{i-1,k}(t) dt, {}_c \hat{\psi}_{j-1,k} \right\rangle.$$

Similarly, if we put:

$$(5) \quad {}_{(c)}\hat{\Phi}_{mk} = [{}_c\hat{\varphi}_{0k}, {}_c\hat{\varphi}_{1k}, {}_c\hat{\varphi}_{2k}, \dots, {}_c\hat{\varphi}_{mk}]^T,$$

then for $f \in L^2(\mathbb{R})$,

$$(6) \quad f \simeq \sum_{i=0}^m h_{ik} {}_c\hat{\varphi}_{ik} = H^T {}_{(c)}\hat{\Phi}_{mk},$$

where $h_{ik} = \langle f, {}_c\hat{\varphi}_{ik} \rangle$ and $H^T = [h_{0k}, h_{1k}, \dots, h_{mk}]$.

Also the integration of entries ${}_{(c)}\hat{\Phi}_{mk}(x)$ can be expanded in terms of

$$(7) \quad \int_0^x {}_{(c)}\hat{\Phi}_{mk}(t) dt = {}_{(c)}\hat{Q} {}_{(c)}\hat{\Phi}_{mk}(x),$$

where the $(m+1) \times (m+1)$ matrix ${}_{(c)}\hat{Q} = [{}_{(c)}\hat{q}_{ij}]$ is called the operational matrix of s-elementary wavelets and its entries are:

$${}_{(c)}\hat{q}_{ij} = \left\langle \int_0^{(\cdot)} {}_c\hat{\varphi}_{i-1,k}(t) dt, {}_c\hat{\varphi}_{j-1,k} \right\rangle.$$

For $a = \frac{\pi}{2^k}$, the $(m+1) \times (m+1)$ operational matrix of s-elementary wavelets are as the following,

$${}_{(c)}\hat{P}_{(m+1)}^k = \begin{bmatrix} a & a(2c^2 - 2c + 2) & 2a & \dots & 2a & 2a \\ 2a(c - c^2) & a & a(2c^2 - 2c + 2) & 2a & \dots & 2a \\ 0 & 2a(c - c^2) & a & a(2c^2 - 2c + 2) & \dots & 2a \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 2a(c - c^2) & a & a(2c^2 - 2c + 2) & 2a \\ 0 & 0 & \dots & 0 & 2a(c - c^2) & a \end{bmatrix},$$

$${}_{(c)}\hat{Q}_{(m+1)}^k = \begin{bmatrix} a & 2a & \dots & 2a & 2a \\ 0 & a & 2a & \dots & 2a \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \ddots & a & 2a \\ 0 & 0 & \dots & 0 & a \end{bmatrix}_{(m+1) \times (m+1)}.$$

In addition, let $E = [e_0, e_1, \dots, e_m]^T$ is a vector and $\omega = \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}}$, we have:

$$(8) \quad E^T {}_{(c)}\hat{\Psi}_{mk} {}_{(c)}\hat{\Psi}_{mk}^T = {}_{(c)}\hat{\Psi}_{mk}^T \text{diag}(\omega E).$$

Lemma 2.3 establishes that how to find the operational matrix to solve the Riemann–Liouville fractional integral equations of order $\alpha > 0$.

First, for the function $u(x)$ and $\alpha > 0$ we put:

$$(9) \quad I^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt,$$

where $\Gamma(\cdot)$ is the Gamma function.

Lemma 2.3. *Let ${}_{(c)}\hat{\Psi}_{mk}$ is the vector of s -elementary wavelets. The fractional integration of its entries can be expanded in terms of*

$$(10) \quad I^\alpha {}_{(c)}\hat{\Psi}_{mk} = {}_{(c)}\hat{P}^{(\alpha),k} {}_{(c)}\hat{\Psi}_{mk},$$

where the $(m+1) \times (m+1)$ matrix ${}_{(c)}\hat{P}^{(\alpha),k} = [{}_{(c)}\hat{p}_{ij}^{(\alpha),k}]$ is called the operational matrix of s -elementary wavelets and its entries are given by:

$${}_{(c)}\hat{p}_{ij}^{(\alpha),k} = \left\langle I^\alpha \hat{\psi}_{i-1,k}, \hat{\psi}_{j-1,k} \right\rangle.$$

Proof. By (1) and (9) for ${}_{(c)}\hat{p}_{1,1}^{(\alpha),k}$, we have:

$$\begin{aligned} I^\alpha \hat{\psi}_{0k}(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \hat{\psi}_{0k}(t) dt \\ &= \begin{cases} \frac{2^{\frac{k}{2}}}{\alpha \Gamma(\alpha) \sqrt{2\pi}} \left(x - \frac{4\pi(c-1)}{2^k}\right)^\alpha, & \frac{4\pi(c-1)}{2^k} \leq x < \frac{2\pi(c-1)}{2^k}, \\ \frac{-2^{\frac{k}{2}}}{\alpha \Gamma(\alpha) \sqrt{2\pi}} \left(\left(x - \frac{2\pi(c-1)}{2^k}\right)^\alpha - \left(x - \frac{4\pi(c-1)}{2^k}\right)^\alpha \right), & \frac{2\pi(c-1)}{2^k} \leq x \leq \frac{2\pi c}{2^k}, \\ \frac{-2^{\frac{k}{2}}}{\alpha \Gamma(\alpha) \sqrt{2\pi}} \left(\left(x - \frac{2\pi(c-1)}{2^k}\right)^\alpha - \left(x - \frac{4\pi(c-1)}{2^k}\right)^\alpha - \left(x - \frac{2\pi c}{2^k}\right)^\alpha \right), & \frac{2\pi c}{2^k} < x \leq \frac{4\pi c}{2^k}, \\ \frac{-2^{\frac{k}{2}}}{\alpha \Gamma(\alpha) \sqrt{2\pi}} \left(\left(x - \frac{2\pi(c-1)}{2^k}\right)^\alpha - \left(x - \frac{4\pi(c-1)}{2^k}\right)^\alpha - \left(x - \frac{2\pi c}{2^k}\right)^\alpha + \left(x - \frac{4\pi c}{2^k}\right)^\alpha \right), & \frac{4\pi c}{2^k} < x, \end{cases} \end{aligned}$$

then we have:

$$\begin{aligned} {}_{(c)}\hat{p}_{1,1}^{(\alpha),k} &= \left\langle I^\alpha \hat{\psi}_{0k}, \hat{\psi}_{0k} \right\rangle \\ &= \int_{\frac{4\pi(c-1)}{2^k}}^1 I^\alpha \hat{\psi}_{0k}(x) \hat{\psi}_{0k}(x) dx \\ &= \int_{\frac{4\pi(c-1)}{2^k}}^{\frac{2\pi(c-1)}{2^k}} I^\alpha \hat{\psi}_{0k}(x) dx + \int_{\frac{2\pi c}{2^k}}^{\frac{4\pi c}{2^k}} I^\alpha \hat{\psi}_{0k}(x) dx \\ &= \frac{-2^k}{\alpha(\alpha+1)\Gamma(\alpha)2\pi} \left(\left(\frac{4\pi c}{2^k} - \frac{2\pi(c-1)}{2^k}\right)^{\alpha+1} - \left(\frac{4\pi c}{2^k} - \frac{4\pi(c-1)}{2^k}\right)^{\alpha+1} - \left(\frac{4\pi c}{2^k} - \frac{2\pi c}{2^k}\right)^{\alpha+1} \right) - \\ &\quad \frac{-2^k}{\alpha(\alpha+1)\Gamma(\alpha)2\pi} \left(\left(\frac{2\pi c}{2^k} - \frac{2\pi(c-1)}{2^k}\right)^{\alpha+1} - \left(\frac{2\pi c}{2^k} - \frac{4\pi(c-1)}{2^k}\right)^{\alpha+1} \right) + \end{aligned}$$

$$\frac{2^k}{\alpha(\alpha+1)\Gamma(\alpha)2\pi} \left(\frac{2\pi(c-1)}{2^k} - \frac{4\pi(c-1)}{2^k} \right)^{\alpha+1}.$$

In the same way, the other entries obtained. \square

For fix $k = 5$ and $m = 5$, $\alpha = 0.5$ and $c = 0.5$ the operator matrix is given by:

$${}_{(0.5)}\hat{P}_{(6)}^{(0.5),5} = \begin{bmatrix} 0.7197 & 0.4180 & 0.5510 & 0.3840 & 0.3230 & 0.2855 \\ 0.2447 & 0.7197 & 0.4180 & 0.5510 & 0.3840 & 0.3230 \\ 0 & 0.2447 & 0.7197 & 0.4180 & 0.5510 & 0.3840 \\ 0 & 0 & 0.2447 & 0.7197 & 0.4180 & 0.5510 \\ 0 & 0 & 0 & 0.2447 & 0.7197 & 0.4180 \\ 0 & 0 & 0 & 0 & 0.2447 & 0.7197 \end{bmatrix}_{(6 \times 6)}.$$

Note that the above method can be stated by basis vector (5).

Lemma 2.4. Let ${}_{(c)}\hat{\Phi}_{mk}(x)$ is the vector of s-elementary wavelets. The fractional integration of its entries can be expanded as follows,

$$(11) \quad I_{(c)}^{\alpha} {}_{(c)}\hat{\Phi}_{mk}(x) = {}_{(c)}\hat{Q}^{(\alpha),k} {}_{(c)}\hat{\Phi}_{mk}(x),$$

where the $(m+1) \times (m+1)$ matrix ${}_{(c)}\hat{Q}^{(\alpha),k} = [{}_{(c)}\hat{q}_{ij}^{(\alpha),k}]$ is called the operational matrix of s-elementary wavelets and its entries are:

$${}_{(c)}\hat{q}_{ij}^{(\alpha),k} = \langle I_{(c)}^{\alpha} \hat{\phi}_{i-1,k}, \hat{\phi}_{j-1,k} \rangle.$$

Proof. The proof is clear. \square

For fix $k = 5$, $m = 5$, $\alpha = 0.5$ and $c = 0.5$ the operator matrix is given by:

$${}_{(0.5)}\hat{Q}_{(6)}^{(0.5),5} = \begin{bmatrix} 0.3333 & 0.2761 & 0.1798 & 0.1454 & 0.1255 & 0.1121 \\ 0 & 0.3333 & 0.2761 & 0.1798 & 0.1454 & 0.1255 \\ 0 & 0 & 0.3333 & 0.2761 & 0.1798 & 0.1454 \\ 0 & 0 & 0 & 0.3333 & 0.2761 & 0.1798 \\ 0 & 0 & 0 & 0 & 0.3333 & 0.2761 \\ 0 & 0 & 0 & 0 & 0 & 0.3333 \end{bmatrix}_{(6 \times 6)}.$$

3. Convergence analysis of s-elementary wavelets approximation

By (1), $\left[\frac{2^k - \pi}{2\pi} \right] + 2$ translations of ${}_c \hat{\psi}_{ik}$ cover the interval $[0, 1]$ (The symbol “[] ” is a bracket.). We put $m = \left[\frac{2^k - \pi}{2\pi} \right] + 1$. Then for $f \in L^2([0, 1])$, the s-elementary wavelets series of f is $f_{mk} = \sum_{i=0}^m d_{ik} {}_c \hat{\psi}_{ik}$, and the corresponding error is defined as follows:

$$e_{mk} = f - f_{mk}.$$

It is clear that

$$e_m = \sum_{i=0}^m d_{ik} \psi_{ik}.$$

Suppose that f satisfies a Lipschitz condition on $[0, 1]$, that is,

$$(12) \text{ there is a } M > 0 \text{ such that for all } x, y \in [0, 1], |f(x) - f(y)| \leq M|x - y|.$$

Now we will prove that $e_m(x)$ tends to zero as k goes to infinity.

$$\begin{aligned} \|e_m\|^2 &= \left\langle \sum_{i=0}^m d_{ik} \psi_{ik}, \sum_{j=0}^m d_{jk} \psi_{jk} \right\rangle \\ &= \sum_{i=0}^m \sum_{j=0}^m d_{ik} d_{jk} \left\langle \psi_{ik}, \psi_{jk} \right\rangle \\ &= \sum_{i=0}^m |d_{ik}|^2, \end{aligned}$$

and by using relation (3), we have:

$$\begin{aligned} d_{ik} &= \left\langle f, \psi_{ik} \right\rangle \\ &= \int_0^1 f(x) \psi_{ik}(x) dx \\ &= \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}} \left(\int_{I_1} f(x) dx + \int_{I_2} f(x) dx \right). \end{aligned}$$

Where $I_1 = \left[\frac{4\pi(c-1)+2n\pi}{2^k}, \frac{2\pi(c-1)+2n\pi}{2^k} \right)$ and $I_2 = \left(\frac{2\pi c+2n\pi}{2^k}, \frac{4\pi c+2n\pi}{2^k} \right]$ with $\lambda(I_1) = \frac{2\pi(1-c)}{2^k}$ and $\lambda(I_2) = \frac{2\pi c}{2^k}$. Now, from (1) and by using the mean value theorem, there are $x_1^{ik} \in I_1$ and $x_2^{ik} \in I_2$, such that

$$d_{ik} = \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}} [\lambda(I_1)f(x_1^{ik}) - \lambda(I_2)f(x_2^{ik})].$$

For $c \neq 0.5$, $\lambda(I_1) \neq \lambda(I_2)$. Without loss of generality, suppose $\lambda(I_2) < \lambda(I_1)$,

there is $I_3 \subset I_1$ such that $\lambda(I_2) = \lambda(I_3)$ and $x_1^{ik} \in I_3$. So we have:

$$\begin{aligned} d_{ik} &\leq \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}} [\lambda(I_2)f(x_1^{ik}) - \lambda(I_2)f(x_2^{ik})] \\ &= \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}} \times \frac{2\pi c}{2^k} [f(x_1^{ik}) - f(x_2^{ik})] \\ &\leq \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}} \times \frac{2\pi c}{2^k} \times M (x_1^{ik} - x_2^{ik}) \\ &\leq \frac{2^{\frac{k}{2}}}{\sqrt{2\pi}} \times \frac{2\pi c}{2^k} \times M \times \frac{2\pi}{2^k} \\ &= \frac{2\pi c\sqrt{2\pi}}{2^{\frac{3k}{2}}} \times M, \end{aligned}$$

therefore, $|d_{ik}|^2 \leq \frac{8\pi^3 c^2}{2^{3k}} M^2$

$$\begin{aligned} \|e_m\|^2 &= \sum_{i=0}^m |d_{ik}|^2 \\ &\leq \sum_{i=0}^m \frac{8\pi^3 c^2}{2^{3k}} M^2 \\ &= (m+1) \frac{8\pi^3 c^2}{2^{3k}} M^2 \\ &= \left(\left\lceil \frac{2^k - \pi}{2\pi} \right\rceil + 2 \right) \frac{8\pi^3 c^2}{2^{3k}} M^2 \\ &\leq 2^k \frac{8\pi^3 c^2}{2^{3k}} M^2 \\ &= \frac{8\pi^3 c^2}{2^{2k}} M^2, \end{aligned}$$

By the above proof, we can obtain a bound for $\|e_m\|^2$,

$$\|e_m\| \leq N \left(\frac{1}{2^{2k}} \right),$$

where $N = 2\pi cM\sqrt{2\pi}$.

4. Numerical solution of nonlinear differential equations and fractional integral equations

In this section, we present an operational method for solving nonlinear Riccati differential equations and Riemann–Liouville fractional integral equations by using the s-elementary wavelets as an application of them.

4.1. Numerical solution of nonlinear Riccati differential equations.

Consider the following nonlinear Riccati differential equation:

$$(13) \quad \begin{cases} u'(t) + f(t)u^2(t) + g(t)u(t) + h(t) = 0, & f(t) \neq 0, \quad 0 \leq t \leq a, \\ u(0) = \alpha. \end{cases}$$

First, we express the functions u , f , g , h and α in terms of the basis (2) as follows:

$$(14) \quad u \simeq X^T {}_{(c)}\hat{\Psi}_{mk}, \quad f \simeq F^T {}_{(c)}\hat{\Psi}_{mk}, \quad g \simeq G^T {}_{(c)}\hat{\Psi}_{mk}, \quad h \simeq H^T {}_{(c)}\hat{\Psi}_{mk}, \quad \alpha \simeq D^T {}_{(c)}\hat{\Psi}_{mk}.$$

Moreover, we have:

$$(15) \quad \int_0^x u'(t)dt = u(x) - u(0) = u(x) - \alpha \simeq X^T {}_{(c)}\hat{\Psi}_{mk}(x) - D^T {}_{(c)}\hat{\Psi}_{mk}(x),$$

and so by substituting (14) and (15) in (13) and integrating, we get:

$$\begin{aligned} & \int_0^x u'(t)dt + \int_0^x f(t)u^2(t)dt + \int_0^x g(t)u(t)dt + \int_0^x h(t)dt = 0 \\ & X^T {}_{(c)}\hat{\Psi}_{mk}(x) - D^T {}_{(c)}\hat{\Psi}_{mk}(x) + \int_0^x F^T {}_{(c)}\hat{\Psi}_{mk}(t)dt \omega (X^T)^2 {}_{(c)}\hat{\Psi}_{mk}(t)dt + \\ & \int_0^x G^T {}_{(c)}\hat{\Psi}_{mk}(t) X^T {}_{(c)}\hat{\Psi}_{mk}(t)dt + \int_0^x H^T {}_{(c)}\hat{\Psi}_{mk}(t)dt = 0 \\ & {}_{(c)}\hat{\Psi}_{mk}^T(x) X - {}_{(c)}\hat{\Psi}_{mk}^T(x) D + \int_0^x F^T {}_{(c)}\hat{\Psi}_{mk}(t) {}_{(c)}\hat{\Psi}_{mk}^T(t) \omega X^2 dt + \\ & \int_0^x G^T {}_{(c)}\hat{\Psi}_{mk}(t) {}_{(c)}\hat{\Psi}_{mk}^T(t) X dt + \int_0^x H^T {}_{(c)}\hat{\Psi}_{mk}(t) dt = 0, \end{aligned}$$

by using relation (8), we have:

$$\begin{aligned} & {}_{(c)}\hat{\Psi}_{mk}^T(x) X - {}_{(c)}\hat{\Psi}_{mk}^T(x) D + \int_0^x {}_{(c)}\hat{\Psi}_{mk}^T(t) \text{diag}(\omega F) \omega X^2 dt + \\ & \int_0^x {}_{(c)}\hat{\Psi}_{mk}^T(t) \text{diag}(\omega G) X dt + \int_0^x {}_{(c)}\hat{\Psi}_{mk}^T(t) H dt = 0, \end{aligned}$$

and by substitute relation (4), we get:

$$\begin{aligned} & {}_{(c)}\hat{\Psi}_{mk}^T(x) X - {}_{(c)}\hat{\Psi}_{mk}^T(x) D + {}_{(c)}\hat{\Psi}_{mk}^T(x) \left({}_{(c)}\hat{P}_{(m+1)}^k \right)^T \text{diag}(\omega F) \omega X^2 + \\ & {}_{(c)}\hat{\Psi}_{mk}^T(x) \left({}_{(c)}\hat{P}_{(m+1)}^k \right)^T \text{diag}(\omega G) X + {}_{(c)}\hat{\Psi}_{mk}^T(x) \left({}_{(c)}\hat{P}_{(m+1)}^k \right)^T H = 0, \end{aligned}$$

then

$$X - D + \left({}_{(c)}\hat{P}_{(m+1)}^k \right)^T \text{diag}(\omega F) \omega X^2 +$$

$$(16) \quad \left({}_{(c)}\hat{P}_{(m+1)}^k \right)^T \text{diag}(\omega G) X + \left({}_{(c)}\hat{P}_{(m+1)}^k \right)^T H = 0.$$

We solve this nonlinear system by the Newton method and obtain $u(t)$. Note that the above method can be stated by basis vector (5).

Example 4.1. *The nonlinear Riccati differential*

$$(17) \quad \begin{cases} u'(t) = 1 + u(t) - u^2(t), \\ u(0) = \alpha, \end{cases}$$

with the exact solution $u(t) = 1 + \sqrt{2} \tanh \left(\sqrt{2} t + \frac{\log \left(\frac{-1+\sqrt{2}}{1+\sqrt{2}} \right)}{2} \right)$ has been solved in [1] by the methods ADM and VIM and also in [3] by the method HPM. According to Fig.1, the approximate solutions of all methods diverge outside the interval $[0, 1]$, while the approximate solution obtained with the latter method converge to the exact solution at any desired interval with high accuracy and is clearly seen in Fig.2, (a) and (b) when we used ${}_{(c)}\hat{\Phi}_{mk}$ and ${}_{(c)}\hat{\Psi}_{mk}$ in our method, respectively.

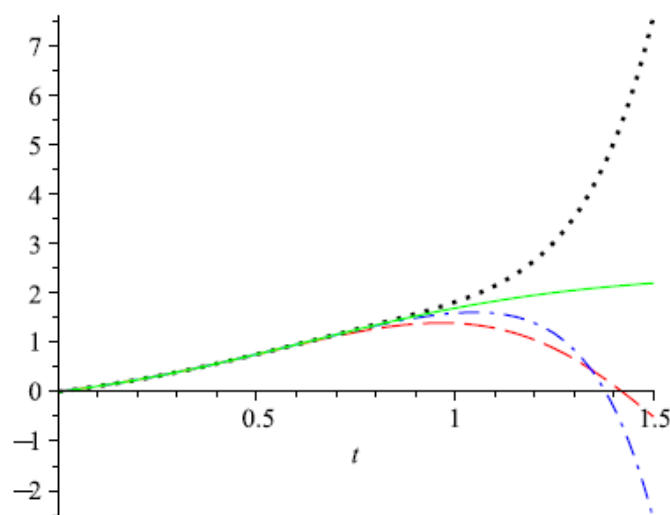


FIGURE 1. The exact solution (solid) versus ADM (dot), VIM (dot dash) and HPM (dash)

4.2. Numerical solution of the Riemann–Liouville fractional integral solution of order $\alpha > 0$. Consider the following Riemann–Liouville fractional

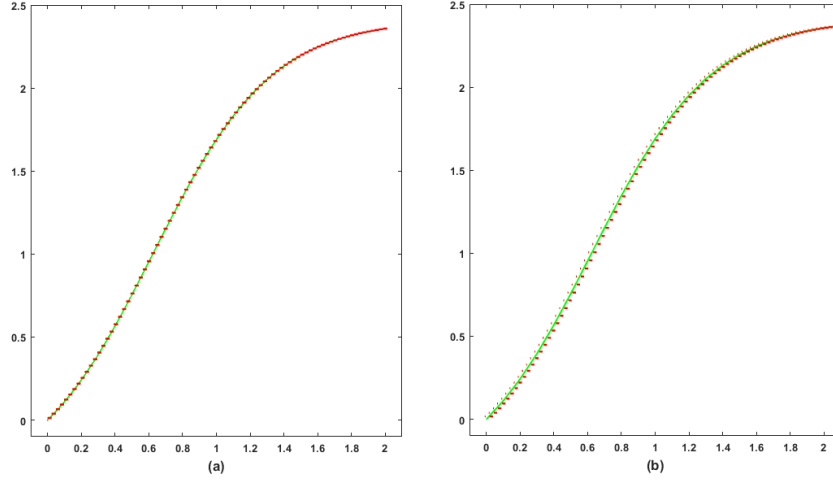


FIGURE 2. Comparison of the approximate solution (dashed red line) by using $(0.99)\hat{\Psi}_{m,8}$ and $(0.8)\hat{\Psi}_{m,8}$ with the exact solution (green line) (Example: 4.1).

integral solution of order $\alpha > 0$ with the unknown function u ,

$$(18) \quad u(x) = f(x) + \int_0^x (x-t)^{\alpha-1} u(t) dt.$$

For solution of the equation (18), we put $f \simeq F^T_{(c)} \hat{\Psi}_{mk}$ and $u \simeq X^T_{(c)} \hat{\Psi}_{mk}$. Then from relations (9) and (10) we have:

$$\begin{aligned} X^T_{(c)} \hat{\Psi}_{mk}(x) &= F^T_{(c)} \hat{\Psi}_{mk}(x) + \int_0^x (x-t)^{\alpha-1} X^T_{(c)} \hat{\Psi}_{mk}(t) dt, \\ X^T_{(c)} \hat{\Psi}_{mk}(x) &= F^T_{(c)} \hat{\Psi}_{mk}(x) + \Gamma(\alpha) X^T_{(c)} \hat{P}^{(\alpha),k}_{(c)} \hat{\Psi}_{mk}(x). \end{aligned}$$

Then we have:

$$\begin{aligned} X^T &= F^T + \Gamma(\alpha) X^T_{(c)} \hat{P}^{(\alpha),k}_{(c)} \\ X^T - \Gamma(\alpha) X^T_{(c)} \hat{P}^{(\alpha),k}_{(c)} &= F^T \\ X^T \left(I - \Gamma(\alpha)_{(c)} \hat{P}^{(\alpha),k}_{(c)} \right) &= F^T \\ (19) \quad X^T &= F^T \left(I - \Gamma(\alpha)_{(c)} \hat{P}^{(\alpha),k}_{(c)} \right)^{-1}. \end{aligned}$$

After finding the vector X , the approximate value of the function $u(t)$ can be obtained. Note that the above method can be stated by basis vector (5).

Example 4.2. Consider the following Riemann-Liouville fractional integral equation (Abel's integral equation) [20, 27]:

$$\int_0^x \frac{f(t)}{\sqrt{x-t}} dt = x,$$

the exact solution of which is $f(x) = \frac{2}{\pi}\sqrt{x}$. In Table 1, we compare the error of our approximations with the method in [27]. Fig. 3, (a) and (b) shows the comparisons between the approximate solutions and the exact solutions, when we have used ${}_{(c)}\hat{\Phi}_{mk}$ and ${}_{(c)}\hat{\Psi}_{mk}$ in our method, respectively.

TABLE 1. Comparison the error of the method presented in [27] with our method (Example: 4.2).

Example	[27]	New method by ${}_{(0.5)}\hat{\Psi}_{m,17}$, $\ e_m\ _2$	New method by ${}_{(0.5)}\hat{\Phi}_{m,17}$, $\ e_m\ _2$
Example 4.2	4.68×10^{-6}	5.61×10^{-7}	2.08×10^{-7}

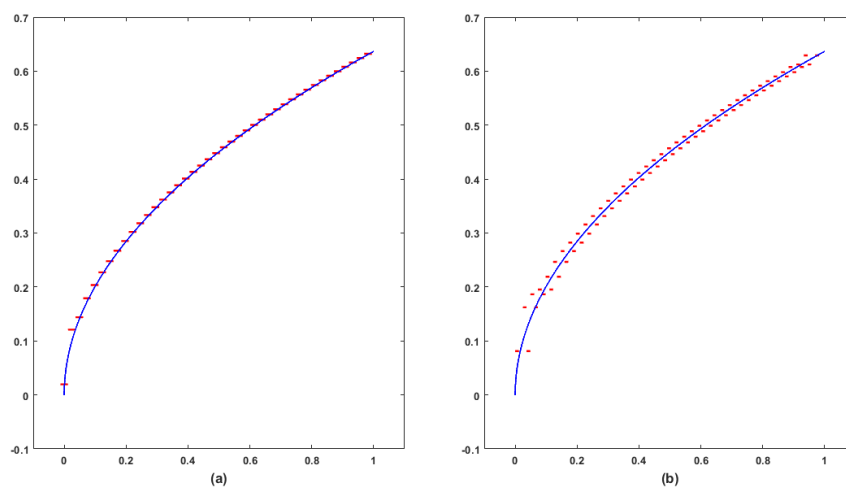


FIGURE 3. Comparison of the approximate solution (dashed red line) using ${}_{(0.5)}\hat{\Phi}_{m,8}$ and ${}_{(0.5)}\hat{\Psi}_{m,8}$ with the exact solution (blue line) (Example: 4.2).

Example 4.3. Consider the following Riemann-Liouville fractional integral equation [27]:

$$f(x) + \int_0^x \frac{f(t)}{\sqrt{x-t}} dt = \frac{1}{2}\pi x + \sqrt{x}, \quad 0 \leq x \leq 1,$$

the exact solution of which is $f(x) = \sqrt{x}$. In Table 2, we compare our error approximations with the method in [27]. Fig. 4, (a) and (b) shows the comparisons between the approximate solutions and the exact solutions, when we have used ${}_{(c)}\hat{\Phi}_{mk}$ and ${}_{(c)}\hat{\Psi}_{mk}$ in our method, respectively.

TABLE 2. Comparison the error of the method presented in [27] with our method (Example: 4.3).

Example	[27]	New method by ${}_{(0.5)}\hat{\Psi}_{m,17}$, $\ e_m\ _2$	New method by ${}_{(0.5)}\hat{\Phi}_{m,17}$, $\ e_m\ _2$
Example 4.3	1.13×10^{-5}	8.81×10^{-7}	3.27×10^{-7}

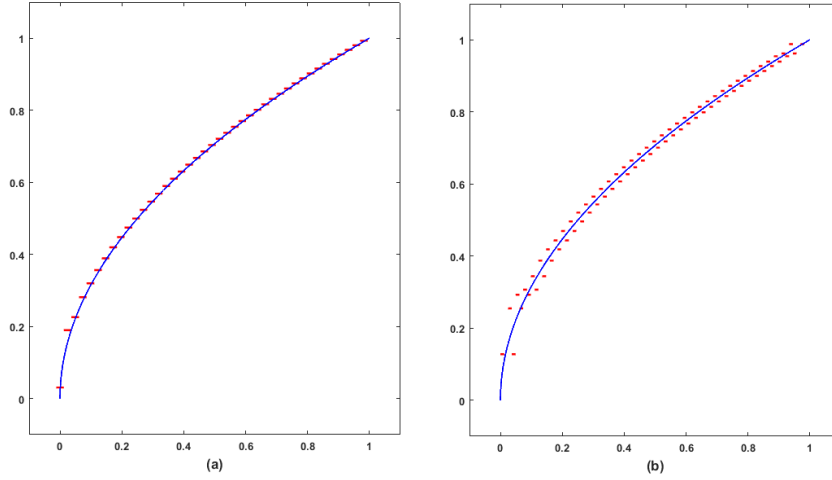


FIGURE 4. Comparison of the approximate solution (dashed red line) ${}_{(0.5)}\hat{\Phi}_{m,8}$ and ${}_{(0.5)}\hat{\Psi}_{m,8}$ with the exact solution (blue line) (Example: 4.3).

5. Conclusion

Using s-elementary wavelets method is very simple and easy to implement and is able to approximate the solution of equations more accurate in a larger interval compared with other discussed methods. Also, we use them to solve the fractional Volterra integral equations which have a weakly singular kernel. Numerical examples and their error analysis show that more accurate results are obtained when finer resolutions are used. We hope the method to be generalized to the case of fractional Fredholm integral equations and other differential equations.

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