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# SCOTT-TOPOLOGY BASED ON TRANSITIVE BINARY RELATION

O.R. Sayed* and N.H. Sayed

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Abstract. In the study of partially ordered sets, topologies such as Scott-topology have shown to be of paramount importance. In order to have analogous topology-like tools in the more general setting of quantitative domains, we introduce a method to construct Scott-topology on a set equipped with a transitive binary relation which we call t-set. As an application of this result there is a Scott-topology associated to any topology induced by its specialization pre-ordered relation. Some relations between this topology and the original topology are investigated.

*Keywords*: Scott-topology; Directed topology; t-set.
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## 1. Introduction and Preliminaries

Bonsangue et al [4] introduced a so-called generalized Alexandroff (Scott) topology, which provides an interesting topological tool for quantitative domains. The Scott topology and its connection with the convergence given in order theoretic terms by S-limits is discussed in [6]. Windels [17] introduced the Scott distance which is a canonical numerification of the Scott topology and at the same time allows for quantitative considerations. He showed that the Scott distance shares the important features of the Scott toplogy. Zhao and Ho [19] defined and studied a new topology constructed from any given topology on a set, using irreducible sets. The manner in which this derived topology is obtained is inspired by how the Scott topology on a poset is constructed from its Alexandroff topology. Zou et al [20] showed how, to each approximating relation,an associated order-compatible topology can be defined in such a way that for the case of a continuous poset the topology associated to the way-below relation is exactly the Scott topology. A preliminary investigation is carried out on this new topology.

Abramsky and Jung [1], introduced a method to construct a canonical partially ordered set from a pre-ordered set and said: "many notions from the theory of partially ordered sets make sense even if reflexivity fails". Finally,

*Corresponding author, ORCID: 0000-0001-9162-1862
E-mail: o_sayed@aun.edu.eg
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they sum up these considerations with the slogan: "Order theory is the study of transitive relations". In the present paper, we naturally put forth an open question whether one may construct a Scott-topology on a t-set (a set equipped with a transitive binary relation). Naturally, in t-sets, in the absence of any sort of join and the absence of the antisymmetric and the reflectivity of the binary relation "$\preceq$", we need new definitions or additional considerations, and proofs become more complicated. Mainly, we consider directed topologies and Scott topologies on t-sets and their interactions with the continuity property of t-sets. Sometimes we need pre-ordered sets instead of t-sets.

This paper is divided into four sections. After the introduction, in this section, in Section 2 we introduced and study the directed-topology and the Scott-topology based on a t-set. Section 3 is devoted to introduce and study interaction between a continuous t-set and its Scott-topology. Furthermore, an application of some of our results is given on topological spaces. We conclude this paper in Section 4 by pointing the reader to possible future directions and posing some open questions.

Throughout the paper, we assume the reader is familiar with general topology and order theory. For general topology, refer to [5, 9].

**Definition 1.1.** A binary operation "$\preccurlyeq$" on a nonempty set $X$ is said to be:

(1) Reflexive [12] if $\forall x \in X, x \preccurlyeq x$;
(2) Antisymmetric [12] if $\forall x, y \in X, x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$;
(3) Transitive [12] if $\forall x, y, z \in X, x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$;
(4) Symmetric [12] if $\forall x, y \in X, x \preccurlyeq y$, then $y \preccurlyeq x$;
(5) Interpolative [8, 14] if $\forall x, z \in X$ with $x \preccurlyeq z$, $\exists y \in X$ such that $x \preccurlyeq y \preccurlyeq z$.

**Definition 1.2.** Let $X$ be a nonempty set with a binary relation "$\preccurlyeq$" on $X$. The pair $(X, \preccurlyeq)$ is said to be:

(1) A poset [3, 12] if "$\preccurlyeq$"' satisfies (1), (2) and (3) in Definition 1.1;
(2) A pre-ordered set [12] if "$\preccurlyeq$" satisfies (1) and (3) in Definition 1.1;
(3) A continuous information system [11, 15] if "$\preccurlyeq$" satisfies (3) and (5) in Definition 1.1.
(4) An abstract base [16] if "$\preccurlyeq$" satisfies (3) in Definition 1.1 and if $\forall y \in A, y \preccurlyeq x \forall x \in X$, $\exists z \in X$ such that $y \preccurlyeq z \preccurlyeq x$, where $A$ is a finite subset of $X$.

**Definition 1.3.** 1001[14]. A t-set is a pair $(X, \preccurlyeq)$, where "$\preccurlyeq$" is a transitive binary relation on a nonempty set $X$.

**Definition 1.4.** [7]. Let $(X, \preccurlyeq)$ be a t-set and $A \subseteq X$. Then

(1) the upper bound subset in $X$ of $A$, denoted by $ub(A)$, is defined as
$$ub(A) = \{x \in X : \forall y \in A, y \preccurlyeq x\};$$
(2) the lower bound subset in $X$ of $A$, denoted by $lb(A)$, is defined as
$$lb(A) = \{x \in A : \forall y \in A, x \preccurlyeq y\};$$

(3) the supremum of a subset $A$ of $X$, denoted by $\sup(A)$, is defined as
$\sup(A) = lb(ub(A)$;

(4) the lower (resp. upper) closure of $A$, denoted by $\downarrow (A)$ (resp. $\uparrow (A)$), is defined as
$\downarrow (A) = \{x \in X : \exists y \in A, x \preccurlyeq y\}$ (resp.$\uparrow (A) = \{x \in X : \exists y \in A, y \preccurlyeq x\}$);

(5) $A$ is an upper (resp. a lower) subset of $X$ if and only if $\uparrow (A) \subseteq A$ (resp. $\downarrow (A) \subseteq A$).

**Definition 1.5.** [14]. Let $(X, \preccurlyeq)$ be a t-set and $A, B \subseteq X$. Then $A$ is said to be:

(1) Directed if $A \neq \phi$ and for every distinct points $\forall x, y \in A, x \neq y$, $\exists z \in (A \cap ub\{x, y\})$;
(2) Cofinal in $B$ if $A \subseteq B \subseteq \downarrow (A)$;
(3) d-closed if for every directed subset $D$ of $X$ such that $D \subseteq A$, $\sup(D) \subseteq A$;
(4) d-open if $A^c$ is d-closed, where $A^c$ is the complement of $A$;
(5) Scott-closed if $A$ is a d-closed lower subset of $X$;
(6) Scott-open if $A^c$ is Scott-closed (or, $A$ is a d-open upper subset of $X$).

**Definition 1.6.** [14]. Let $(X, \preccurlyeq)$ be a t-set. $(X, \preccurlyeq)$ is domain if for every directed subset $A$ of $X$, $\sup A \neq \phi$.

**Definition 1.7.** [2,6,7]. Let $(X, \preccurlyeq)$ be a t-set and $x, y \in X$. We say that $x$ is way-below $y$ (or, $y$ is way-above $x$) , denoted by $x \ll y$, if and only if for every directed subset $D$ of $X$ such that $y \preccurlyeq \sup(D)$ there exists an element $d$ of $X$ such that $d \in D$ and $x \preccurlyeq d$. The family of all elements of $X$ that are way-above (resp. way-below) $x$ is denoted and defined as follows: $\Uparrow x = \{y \in X : x \ll y\}$). (resp. $\Downarrow x = \{y \in X : y \ll x\}$)).

**Definition 1.8.** [2,13]. Let $(X, \preccurlyeq)$ be a t-set and $x \in X$. If $x \ll x$, then $x$ is said to be isolated and we write $K(X) = \{x \in X : x \ll x\}$. The family of all isolated points that below $x \in X$ is denoted and defined as
$\downarrow^\circ x = \{y \in K(X) : y \preccurlyeq x\}$.

**Definition 1.9.** [10]. A subset $B$ of $X$ is a base for a poset $(X, \preccurlyeq)$ if for every $d \in X$ the set $B \cap \Downarrow d$ is directed $X$ and $\sup(B \cap \Downarrow d) = d$.

**Definition 1.10.** [14]. A t-set $(X, \preccurlyeq)$ is said to be continuous if $\forall x \in X$ , the following conditions hold:

(1) $\sup(\{x\}) \neq \phi$;
(2) $\Downarrow x$ is a directed subset of $X$;
(3) $x \in \downarrow (\sup(\bigcup\{\sup(\Downarrow a) : a \in \Downarrow x\}))$.

**Definition 1.11.** [14]. A t-set $(X, \preccurlyeq)$ is said to be algebraic if for every $x \in X$ the following conditions hold:

(1) $\downarrow^\circ x$ is directed;

(2) $x \in \downarrow (\sup(\downarrow^{\circ} x))$.

**Proposition 1.12.** *[14]. Let $(X, \preccurlyeq)$ be a t-set and $x, y, z \in X$.*

(1) *If $x \preccurlyeq y$ and $y \ll z$, then $x \ll z$;*
(2) *If $x \ll y$ and $y \preccurlyeq z$ , then $x \ll z$;*
(3) *If $\sup(\{y\}) \neq \emptyset$ and $x \ll y$, then $x \preccurlyeq y$;*
(4) *If $\sup(\{y\}) \neq \emptyset$ or $\sup(\{z\}) \neq \emptyset$, $x \ll y$ and $y \ll z$, then $x \ll z$.*

**Proposition 1.13.** *[14]. Let $\{A_j : j \in J\}$ be a family of subsets of a t-set $(X, \preccurlyeq)$. Then:*

(1) $\downarrow (\bigcup_{j\in J} A_j) = \bigcup_{j\in J}\{\downarrow (A_j)\}$;
(2) $\uparrow (\bigcup_{j\in J} A_j) = \bigcup_{j\in J}\{\uparrow (A_j)\}$;
(3) $\downarrow (\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J}\{\downarrow (A_j)\}$;
(4) $\uparrow (\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J}\{\uparrow (A_j)\}$.

**Theorem 1.14.** *[14]. $(X, \preccurlyeq)$ is a continuous t-set if and only if the following conditions hold:*

(1) *for all $x \in X$, $\sup(\{x\}) \neq \phi$;*
(2) *for all $x \in X$, $\Downarrow x$ is directed;*
(3) *"$\ll$" is interpolative;*
(4) *for all $x \in X$, $x \in \downarrow (\sup(\Downarrow x))$.*

## 2. Directed-topology and Scott-topology on a t-set

To solve the problem of constructing Scott-topology on a t-set, we first investigate those sets which are closed with respect to suprema of directed subsets. Also, we restrict them to more important Scott closed sets.

**Definition 2.1.** Let $(X, \preccurlyeq)$ be a t-set. A subset $D$ of $X$ is S-directed if and only if $D$ is directed and for all $x \in D$, $\sup(\{x\}) \cap D \neq \phi$.

It is clear that any S-directed subset is directed but the converse may not be true as illustrated by the following example.

**Example 2.2.** *Let $X = \{a, b, c, d, e\}$ and consider the transitive binary relation "$\preccurlyeq$" on $X$ as follows: $\preccurlyeq = \{(a, b)\}$. Then $\{a\}$ is directed but it is not S-directed.*

**Definition 2.3.** Let $(X, \preccurlyeq)$ be a t-set and $A \subseteq X$.

(1) $A$ is d*-closed if and only if for every S-directed subset $D$ of $A$, $\sup(D) \subseteq A$; and for all $x \in A, \sup(\{x\}) \subseteq A$;
(2) $A$ is Scott*-closed if and only if $A$ is d*-closed lower subset;
(3) $A$ is d*-open (resp. Scott*-open) if and only if $A^c$ is d*-closed (resp. Scott*-closed).

**Example 2.4.** *Let $(X, \preccurlyeq)$ be a t-set, where $X = \{a, b, c, d\}$ and $\preccurlyeq = \{(a, b), (b, d), (a, d), (d, d)\}$. The the set of all directed subsets of $X$ is $\{\{a\}, \{b\}, \{c\}, \{d\}, \{b, d\}\}$ and the set of all S-directed subsets of $X$ is $\{\{d\}, \{b, d\}\}$.*

*Furthermore, the set of all d*-closed subsets of* $X$ *is* $\{\phi, \{b, d\}, \{a, b, d\}, \{c, b, d\}, X\}$ *and the set of all d*-open subsets of* $X$ *is* $\{\phi, \{a\}, \{c\}, \{a, c\}, X\}$.

Now, we consider the notion of Scott*-open sets.

**Proposition 2.5.** *Let* $(X, \preccurlyeq)$ *be a t-set* $aB \subseteq X$.

(1) $B$ *is d*-open if and only if* $\forall x \in X$ *and for every S-directed subset* $D$ *of* $X$ *with* $\sup(D) \cap B \neq \phi, D \cap B \neq \phi$ *and* $\sup(\{x\}) \cap B \neq \phi$*, we have* $x \in B$*;*

(2) $B$ *is Scott*-open if and only if* B *is a d*-open upper subset.*

*Proof.* (1) The proof is obtained logically as follows:

Suppose that $P \equiv$ "$D$ is S-directed subset ($D \in E_S$)", where $E_S$ is the set of all S-directed subset of $X$, $Q \equiv$ "$D \subseteq A$", $R \equiv$ "$\sup(D) \subseteq A$", $L \equiv$ "$x \in X$", $S \equiv$ "$x \in A$", and $H \equiv$ "$\sup(\{x\}) \subseteq A$". Now, $A$ is d*-closed $\equiv P \wedge (Q \Rightarrow R) \wedge L \wedge (S \Rightarrow H) \equiv P \wedge (\neg R \Rightarrow \neg Q) \wedge L \wedge (\neg H \Rightarrow \neg S) \equiv A^c$ is d*-open. Then, $A^c$ is d*-open $\equiv (D \in E_S) \wedge ((\sup(D) \cap A^c) \neq \phi \Rightarrow (D \cap A^c \neq \phi) \wedge (x \in X) \wedge ((\sup(\{x\}) \cap A^c) \neq \phi \Rightarrow x \in A^c)$. Now, we can say that, $B$ is a d*-open if and only if for every $D \in E_S$ and for all $x \in X$ with $\sup(D) \cap B \neq \phi, D \cap B \neq \phi$ and $\sup(\{x\}) \cap B \neq \phi$, we have $x \in B$.

(2) Immediate.

□

**Proposition 2.6.** *Let* $(X, \preccurlyeq)$ *be a t-set. If* $A$ *is a d-closed (resp. d-open) subset of* $X$*, then* $A$ *is d*-closed (resp. d*-open).*

*Proof.* We prove only one statement since the other follows easily in this case. So, suppose $D$ is an S-directed subset of $A$. Since $A$ is d-closed, $\sup(D) \subseteq A$. Furthermore, for all $x \in A, \{x\}$ is a directed subset of $A$. Thus $\sup(\{x\}) \subseteq A$. Therefore, $A$ is d*-closed. □

**Corollary 2.7.** *Let* $(X, \preccurlyeq)$ *be a t-set. If* $A$ *is a Scott-closed (resp. Scott-open) subset of* $X$*, then* $A$ *is Scott*-closed (resp. Scott*-open).*

*Proof.* Follows directly from Proposition 2.2. □

**Proposition 2.8.** *Let* $(X, \preccurlyeq)$ *be a pre-ordered set and* $A \subseteq X$*. Then*

(1) $A$ *is directed if and only if* $A$ *is an S-directed;*

(2) $A$ *is d-open (resp. d-closed, Scott-open, Scott-closed) if and only if* $A$ *is d*-open (resp. d*-closed, Scott*-open, Scott*-closed).*

*Proof.* The proof, being easy, is omitted. □

**Lemma 2.9.** *Let* $A, B \subseteq X$*. If* B *is directed and cofinal in* $A$*, then* $A$ *is directed and* $\sup(A) = \sup(B)$.

**Theorem 2.10.** *Let* $(X, \preccurlyeq)$ *be a t-set. If* $\forall x \in X, \sup(\{x\}) \neq \phi$, $\tau_{d^*} = \{A \subseteq X : A$ *is d*-open*$\}$ *is a topology on* $X$ *(called the directed*-topology on* $X$*).*

*Proof.* (1) One can easily show that $X$ and $\phi$ are d*-closed sets. So, $\phi$ and $X$ are d*-open sets.

(2) Let $A$ and $B$ be d*-open sets. Then $A^c$ and $B^c$ are d*-closed sets. Suppose $D$ is an S-directed subset of $A^c \cup B^c$. Thus $D = (D \cap A^c) \cup (D \cap B^c)$. We need to prove that either $D \cap A^c$ or $D \cap B^c$ is cofinal in $D$. Suppose $D \cap A^c$ is not cofinal in $D$. Then there exists $d_\circ \in D$ such that for all $a \in D \cap A^c, d_\circ \not\preccurlyeq a$. If $d_\circ \in A^c$, then $\sup(\{d_\circ\}) \subseteq A^c$. Thus, there exists $m \in D \cap A^c$ such that $d_\circ \preccurlyeq m$, a contradiction. Hence, $d_\circ \in B^c$. Suppose that $d \in D$ such that $d \neq d_\circ$. Then there exists $d' \in D \cap ub\{d, d_\circ\}$. If $d' \in A^c$, then $d_\circ \preccurlyeq d'$ which leads to a contradiction. So, $d' \in B^c$. Thus $D \subseteq \downarrow (D \cap B^c)$ and $D \cap B^c$ is cofinal in $D$. Now, we prove that $D \cap B^c$ is directed. Suppose that $b_1, b_2 \in D \cap B^c$ such that $b_1 \neq b_2$. Then there exists $d \in D \cap ub\{b_1, b_2\}$. So, there exists $k \in D \cap B^c$ such that $d \preccurlyeq k$. Thus, $k \in (D \cap B^c) \cap ub\{b_1, b_2\}$ and $D \cap B^c$ is directed. From Lemma 2.1 one can have that $\sup(D) = \sup(D \cap B^c)$. Now, $D \cap B^c$ is S-directed (Indeed, suppose that $l \in D \cap B^c$. Since $D$ is S-directed, there exists $m \in \sup(\{l\})$ such that $m \in D$ . Also, since $B^c$ is d*-closed, $\sup(\{l\}) \subseteq B^c$. Thus, $m \in D \cap B^c$. Therefore, $(D \cap B^c) \cap \sup(\{l\}) \neq \phi$. Hence, $D \cap B^c$ is S-directed).Then $\sup(D) = \sup(D \cap B^c) \subseteq B^c \subseteq A^c \cup B^c$. Also, if $x \in A^c \cup B^c$, then $\sup(\{x\}) \subseteq A^c \cup B^c$. Hence, $A^c \cup B^c$ is d*-closed. Therefore, $A \cap B$ is d*-open.

(3) Let $\{A_j : j \in J\}$ be a family of d*-open subset of $X$ and $D$ be an S-directed subset of $\bigcap_{j \in J} A_j^c$. So, $D \subseteq A_j^c$ for every $j \in J$. Then $\sup(D) \subseteq A_j^c$ for every $j \in J$. So, $\sup(D) \subseteq \bigcap_{j \in J} A_j^c$. Suppose $l \in \bigcap_{j \in J} A_j^c$. So, $l \in A_j^c$ for every $j \in J$. Thus $\sup(\{l\}) \subseteq A_j^c$ for every $j \in J$ and $\sup(\{l\}) \subseteq \bigcap_{j \in J} A_j^c$. Hence, $\bigcap_{j \in J} A_j^c$ is d*-closed. Therefore, $\bigcup_{j \in J} A_j$ is d*-open.

□

**Example 2.11.** *Let $X = \{a, b, c\}$. Define the binary relation "$\preccurlyeq$" on $X$ as follows:*
*$a \preccurlyeq c, b \preccurlyeq c$ and $c \preccurlyeq c$. Then $(X, \preccurlyeq)$ is a t-set and for every $x \in X, \sup(\{x\}) = \{c\} \neq \phi$. It is obvious that $\tau_{d^*} = P(X)$.*

**Proposition 2.12.** *Let $(X, \preccurlyeq)$ be a t-set. Then $\{A \subseteq X : A$ is an upper subset$\}$ is a topology on $X$.*

*Proof.* (1) It is obvious that $X$ and $\phi$ are upper subsets.

(2) From Proposition 1.2 (4), we have that $\uparrow (\bigcap_{j \in J} A_j) \subseteq \bigcap_{j \in J} \uparrow (A_j)$. So, $\uparrow (\bigcap_{j \in J} A_j) \subseteq \bigcap_{j \in J} A_j$. Hence, $\bigcap_{j \in J} A_j$ is an upper subset.

(3) Suppose $x \in \uparrow (\bigcup_{j \in J} A_j)$. Then there exists $j \in J$ such that $y \in A_j$ and $y \preccurlyeq x$. Thus $x \in \uparrow (A_j)$. So, $x \in A_j$. Therefore $x \in \bigcup_{j \in J} A_j$ and $\bigcup_{j \in J} A_j$ is an upper subset of $X$.

□

**Proposition 2.13.** *Let $(X, \preccurlyeq)$ be a t-set. Then $\{A \subseteq X : A$ is a lower subset$\}$ is a topology on $X$.*

*Proof.* From Proposition 1.2 the proof is similar to the proof of Proposition 2.4. □

**Theorem 2.14.** *Let $(X, \preccurlyeq)$ be a t-set. If for every $x \in X, \sup(\{x\}) \neq \phi$, then $\tau_{S^*} = \{A \subseteq X : A$ is Scott*-open$\}$ is a topology on $X$ (called the Scott*-topology on $X$).*

*Proof.* Follows from Proposition 2.4 and Theorem 2.1. □

**Example 2.15.** *Let $(X, \preccurlyeq)$ be the t-set defined in Example 2.2. Then the set of all upper subsets of $X$ is $\{\phi, \{b\}, \{c\}, \{d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, X\}$ and $\tau_{S^*} = \{\phi, \{c\}, X\}$. Also, if $(X, \preccurlyeq)$ is the t-set defined in Example 2.3, then $\tau_{S^*} = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$.*

**Example 2.16.** *Consider $\mathbb{R}$ with its usual order and let $\tau_{S^*}$ denote the right ray topology on $\mathbb{R}$, that is; $\tau_{S^*} = \{\phi, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$. Clearly, the right ray topology on $\mathbb{R}$ is the Scott*-topology of usual order.*

*Remark* 2.17. A pre-ordered set $(X, \preccurlyeq)$ with $\sup(\{x\}) \neq \phi$ for every $x \in X$ is a t-set. But, a t-set with $\sup(\{x\}) \neq \phi$ for every $x \in X$ is not a pre-ordered set in general.

**Example 2.18.** *Let $X = \{a, b, c\}$. Define the t-set $(X, \preccurlyeq)$ as in Example 2.3. Then $(X, \preccurlyeq)$ is a t-set and for every $x \in X, \sup(\{x\}) = \{c\} \neq \phi$. It is obvious that $(X, \preccurlyeq)$ is not a pre-ordered set because "$\preccurlyeq$" is not reflexive.*

**Theorem 2.19.** *Let $(X, \preccurlyeq)$ be a t-set and $x \in X$. Then $\downarrow x$ is Scott-closed.*

*Proof.* First, suppose $y \in \downarrow (\downarrow x))$. Then there $z \in \downarrow x$ such that $y \preccurlyeq z$. So, $y \in \downarrow x$. Hence $\downarrow x$ is a lower subset of $X$. Second, suppose that $D$ is a directed subset of $\downarrow x$ and $m \in \sup(D)$. Now, $x \in ub(D)$ and $m \preccurlyeq x$. Hence $m \in \downarrow x$. Thus $\sup(D) \subseteq \downarrow x$ and $\downarrow x$ is d-closed. Therefore $\downarrow x$ is Scott-closed. □

**Corollary 2.20.** *Let $(X, \preccurlyeq)$ be a t-set such that for every $x \in X, \sup(\{x\}) \neq \phi$. Then $X - \downarrow x \in \tau_S$, where $\tau_S$ is the set of all Scott-open subsets of $X$.*

**Theorem 2.21.** *Let $(X, \preccurlyeq)$ be a pre-ordered set. Then $\downarrow x = cl_{\tau_S}(\{x\})$.*

*Proof.* From Corollary 2.2 we have $X - \downarrow x \in \tau_S$. Since $x \preccurlyeq x$, then $cl_{\tau_S}(\{x\}) = \bigcap\{F \subseteq X : F$ is $\tau_S$-closed and $\{x\} \subseteq F\} \subseteq \downarrow x \subseteq \bigcap\{\downarrow (F) \subseteq X : F$ is $\tau_S$-closed and $\{x\} \subseteq F\} = \bigcap\{F \subseteq X : F$ is $\tau_S$-closed and $\{x\} \subseteq F\} = cl_{\tau_S}(\{x\})$. □

**Proposition 2.22.** *Let $F$ be a finite set in a pre-ordered set $(X, \preccurlyeq)$. Then $X - \downarrow F \in \tau_S$.*

*Proof.* Since $z \in \downarrow F$ if and only if there exists $x \in F$ such that $z \preccurlyeq x$ if and only if $z \in \bigcup\{\downarrow x : x \in F\}$, $\downarrow F = \bigcup\{\downarrow x : x \in F\}$. Then from Theorem 2.3, we have for every $x \in X, \downarrow x$ is Scott-closed. Since the union of finitely number of a d-closed subsets is Scott-closed, $\downarrow F$ is a Scott-closed subset so that $X - \downarrow F \in \tau_S$. □

**Definition 2.23.** A triple $(X, \preccurlyeq, \tau)$ is called a topological t-set where $(X, \preccurlyeq)$ is a t-set and $(X, \tau)$ is a topological space.

**Definition 2.24.** Let $(X, \preccurlyeq)$ be a t-set and $\tau$ be a topology on $X$. The topological space $(X, \tau)$ is a transitive-$T_\circ$($t - T_\circ$ for short) if and only if for all $x, y \in X$ such that either $x \not\preccurlyeq y$ or $y \not\preccurlyeq x$ imply that there exists $u \in \tau$ such that $x \in u, y \notin u$ or $y \in u, x \notin u$.

**Lemma 2.25.** *Let $(X, \preccurlyeq)$ be a t-set and $\tau$ be a topology on $X$. The topological space $(X, \tau)$ is $t - T_\circ$ if and only if for all $x, y \in X$ such that either $x \not\preccurlyeq y$ or $y \not\preccurlyeq x$, we have either $x \notin cl_\tau(\{y\})$ or $y \notin cl_\tau(\{x\})$.*

*Proof.* The proofs are straightforward from Definition 2.4. □

**Proposition 2.26.** *Let $(X, \preccurlyeq)$ be a t-set and $\tau$ be a topology on $X$.*

(1) *If "$\preccurlyeq$" is symmetric and $(X, \tau)$ is $t - T_\circ$ space, then $(X, \tau)$ is $T_\circ$ space.*
(2) *If "$\preccurlyeq$" is reflexive and $(X, \tau)$ is $T_\circ$ space, then $(X, \tau)$ is $t - T_\circ$ space.*

*Proof.* The proofs are straightforward from Definition 2.4. □

**Theorem 2.27.** *If $(X, \preccurlyeq)$ is a pre-ordered set, then $(X, \tau_S)$ is $t - T_\circ$ space.*

*Proof.* Suppose $x, y \in X$ such that either $x \not\preccurlyeq y$ or $y \not\preccurlyeq x$. Then $x \notin \downarrow y$ or $y \notin \downarrow x$. From Theorem 2.4 we have $x \notin cl_{\tau_S}(\{y\})$ or $y \notin cl_{\tau_S}(\{x\})$. Therefore, from Lemma 2.2 we have $(X, \tau_S)$ is $t - T_\circ$ space. □

**Theorem 2.28.** *A subspace of a Scott*-topology is a Scott*-subspace.*

*Proof.* Let $(X, \tau_{S^*})$ be a Scott*-topological space and let $A \subseteq X$. Then, $A$ has the relative topology $\tau_A = \{A \cap U : U \in \tau_{S^*}\}$. Let $B \in \tau_A$. So, $\exists U \in \tau_{S^*}$ such that $B = A \cap U$. Let $x \in B$ and $y \in A$ such that $x \leq y$. Since $x \in U \in \tau_{S^*}$, $y \in U$ and hence $y \in A \cap U = B$. Thus, $B$ is an up set with respect to $A$. Now, let $S$ be any directed subset of $A$ such that $\bigvee S \cap B \neq \phi$. Then, $S \cap A = S \neq \phi$. Since $U$ is Scott*-open, then $S \cap U \neq \phi$. Therefore $S \cap B = S \cap (A \cap U) = (S \cap A) \cap U = S \cap U \neq \phi$. Hence $B$ is a Scott*-open with respect to $A$. □

## 3. Continuous t-set and its Scott-topology

In this section we introduce and study interaction between a continuous t-set and its Scott-topology and Scott*-topology. Also, we give some new types of topologies on a t-set.

**Theorem 3.1.** *Let $(X, \preccurlyeq)$ be a t-set.*

(1) *If $x \in X$ is an isolated point, then $\uparrow x$ is a Scott-open subset of X;*
(2) *If for some $x \in X$, $\uparrow x$ is a d-open subset of $X$, then $x$ is an isolated point.*

*Proof.* (1) Suppose $D$ is a directed subset of $X$ and $\sup(D) \cap \uparrow x \neq \phi$. Then there exists $y \in X$ such that $y \in \uparrow x$ and $y \in \sup(D)$. Thus $x \in \downarrow (\sup(D))$. Since $x \ll x$, there exists $d \in D$ such that $d \in \uparrow x$ which implies $D \cap \uparrow x \neq \phi$. So, $\uparrow x$ is d-open. Furthermore, since $\uparrow x$ is an upper subset of $X$, $\uparrow x$ is Scott-open.

(2) Suppose $D$ is a directed subset of $X$ and $x \in \downarrow (\sup(D))$. Then there exists $m \in \sup(D)$ such that $x \preccurlyeq m$. So, $m \in \sup(D) \cap \uparrow x$. Since $\uparrow x$ is d-open, $D \cap \uparrow x \neq \phi$. Hence there exists $d \in D$ such that $x \preccurlyeq d$. Therefore $x \ll x$.

□

**Corollary 3.2.** (1) *Let $(X, \preccurlyeq)$ be a t-set such that for all $x \in X, \sup(\{x\}) \neq \phi$. If $x$ is an isolated point, then $\uparrow x \in \tau_S$;*

(2) *If $(X, \preccurlyeq)$ is a pre-ordered set and for some $x \in X$, $\uparrow x \in \tau_S$, then $x$ is an isolated point.*

The interpolation property plays an important role in the theory of continuous t-sets. Some important applications of the interpolation property are stated in the next results.

**Theorem 3.3.** *Let $(X, \preccurlyeq)$ be a t-set. If "$\ll$" is interpolative, then for all $x \in X$ we have $\Uparrow x$ is Scott-open.*

*Proof.* First, suppose $z \in \uparrow (\Uparrow x)$. Then there exists $y \in \Uparrow x$ such that $y \preccurlyeq z$. From Proposition 1.1(2) $z \in \Uparrow x$. Hence $\Uparrow x$ is an upper subset of $X$. Second, suppose $D$ is a directed subset of $X$ with $\sup(D) \cap \Uparrow x \neq \phi$. Then there exists $z \in X$ such that $z \in \downarrow (\sup(D))$ (because $z \preccurlyeq z$ for each $z \in \sup(D)$) and $z \in \Uparrow x$. So, there exists $y \in X$ such that $x \ll y \ll z$. Hence there exists $d \in D$ such that $y \preccurlyeq d$. From Proposition 1.1 (2) $d \in \Uparrow x$. Therefore $D \cap \Uparrow x \neq \phi$. □

**Corollary 3.4.** *If $(X, \preccurlyeq)$ is a continuous t-set, then for all $x \in X, \Uparrow x \in \tau_{S^*}$.*

**Theorem 3.5.** *Let $(X, \preccurlyeq)$ be a pre-ordered set and $x, y \in X$. If there exists $O \in \tau_S$ such that $y \in O \subseteq \uparrow x$, then $x \ll y$.*

*Proof.* Let $D$ be a directed subset of $X$ such that $y \in \downarrow (\sup(D))$. Then there exists $m \in \sup(D)$ such that $y \ll m$. Since $y \in \uparrow y \subseteq \uparrow (O) \subseteq O$, then $m \in \sup(D) \cap O$. Hence $\sup(D) \cap O \neq \phi$. Then there exists $d \in D$ such that $d \in O \subseteq \uparrow x$. Thus $x \preccurlyeq d$. Therefore $x \ll y$. □

**Theorem 3.6.** *Let $(X, \preccurlyeq)$ be a t-set such that for all $z \in X, \sup(\{z\}) \neq \phi$ and "$\preccurlyeq$" is interpolative. If $x, y \in X$ such that $x \preccurlyeq y$, then there exists a Scott-open subset* O *of* X *such that $y \in O \subseteq \uparrow x$.*

*Proof.* From Theorem 3.2, $\Uparrow x$ is a Scott-open set. Suppose that $y \in \Uparrow x$. Hence $x \preccurlyeq y$. From Proposition 1.1 (3), $y \in \Uparrow x$. Therefore $y \in \Uparrow x \subseteq \uparrow x$. □

**Theorem 3.7.** *Let $(X, \preccurlyeq)$ be a continuous pre-ordered set. Then for all $x \in X$ and for all $O \in \tau_S$ with $z \in O$ for some $z \in \sup(\Downarrow x)$ we have that there exists $O' \in \tau_S$ and $x' \in X$ such that $x \in O' \subseteq \uparrow (x') \subseteq O$.*

*Proof.* Suppose $x \in X$ and $O \in \tau_S$ with $z \in O$ for some $z \in \sup(\Downarrow x)$. Then $\sup(\Downarrow x) \cap O \neq \phi$. Hence there exists $x' \in (\Downarrow x) \cap O$ so that $\uparrow (x') \subseteq \uparrow (O) \subseteq O$. Since $x' \ll x$, from Theorem 3.4 there exists $O' \in \tau_S$ such that $x \in O' \subseteq \uparrow (x') \subseteq (O) \subseteq O$. □

**Theorem 3.8.** *Let $(X, \preccurlyeq)$ be a pre-ordered set. Then $\preccurlyeq_{\tau_S} \equiv \preccurlyeq$, where $\preccurlyeq_{\tau_S}$ is the specialization pre-ordered relation induced by $\tau_S$.*

*Proof.* $x \preccurlyeq_{\tau_S} y \Leftrightarrow x \in cl_{\tau_S}(\{y\}) = \downarrow y \Leftrightarrow x \preccurlyeq y$. □

**Theorem 3.9.** *Let $(X, \preccurlyeq)$ be a domain pre-ordered set. Then for all $x \in X$ and for all $O \in \tau_S$ there exists $O' \in \tau_S$ and $x' \in X$ such that $x \in O' \subseteq \uparrow (x') \subseteq O$, then we have a directed subset $D$ of $\Downarrow x$ such that $x \in \downarrow (\sup(D))$.*

*Proof.* Suppose $D = \{u \in X : \exists O_u \in \tau_S, x \in O_u \subseteq \uparrow x\}$. From Theorem 3.3, $D \subseteq \Downarrow x$. Since $X$ itself is Scott-open and $x \in X$, there exists $y \in X$ and $O_y \in \tau_S$ such that $x \in O_y \subseteq \uparrow y$. Hence $D \neq \phi$. Let $u, v \in D$ such that $u \neq v$. Then there exist $O_u, O_v \in \tau_S$ such that $x \in O_u \subseteq \uparrow u$ and $x \in O_v \subseteq \uparrow v$. Since $x \in O_u \cap O_v$, there exists $w \in X$ and $O_w \in \tau_S$ such that $x \in O_w \subseteq \uparrow w \subseteq O_u \cap O_v$. Then $w \in D$ and $w \in \uparrow u \cap \uparrow v$. So, $u \preccurlyeq w$ and $v \preccurlyeq w$. Then $D$ is directed subset of $\Downarrow x$. Now there are $y \in X$ and $O_y \in \tau_S$ such that $x \in O_y \subseteq \uparrow y \subseteq O$. Thus $y \in D \cap O$. Assume $m \in \sup(D)$. Hence $y \preccurlyeq m$. Since $O$ is upper subset, then $m \in O$. Now, $x \lesssim_{\tau_S} m$. From Theorem 3.6, $x \preccurlyeq m$. Therefore $x \in \downarrow (\sup(D))$. □

**Theorem 3.10.** *Let $(X, \preccurlyeq)$ be a continuous pre-ordered set. Then for all $x \in X$ and for all $O \in \tau_S$ there exists $O' \in \tau_S$ and $x' \in X$ such that $x \in O' \subseteq \uparrow (x') \subseteq O$, then we have a directed subset $D$ of $\Downarrow x$ such that $x \in \downarrow (\sup(D))$.*

*Proof.* The proof is similar to that of Theorem 3.8. □

**Theorem 3.11.** *Let $(X, \preccurlyeq)$ be a domain pre-ordered set. Assume that for all $x \in X$ and for all $O \in \tau_S$ there exists $z \in \sup(\Downarrow x) \cap O$. Then the following statements are equivalent:*

(1) *$(X, \preccurlyeq)$ is a continuous t-set; and*
(2) *"$\ll$" is interpolative and for all $x \in X$ and for all $O \in \tau_S$ with $x \in O$ there are $O' \in \tau_S$ and $x' \in X$ such that $x \in O' \subseteq \uparrow (x') \subseteq O$.*

*Proof.* Applying Theorems 1.1, 3.5 and 3.7 the result holds. □

Now, one can assign for any topology $\tau$, a new topology $S(\tau)$, where $S(\tau)$ is the Scott-topology induced by specialization pre-ordered relation $\preccurlyeq_\tau$ induced by $\tau$.

**Theorem 3.12.** *Let $(X, \tau)$ be a topological space. Then:*

(1) *$(X, S(\tau))$ is $t - T_\circ$-space with respect to the pre-ordered relation $\preccurlyeq_\tau$ induced by $\tau$.*
(2) *If $(X, \tau)$ is a $T_\circ$-space, then $(X, S(\tau))$ so is.*

*Proof.* (1) The result is a corollary from Theorem 2.5.

(2) From [8, Proposition 4.3.3], if $(X, \tau)$ is a $T_\circ$-space, then $\preccurlyeq_\tau$ is a partially ordered relation. Therefore, from Theorem 2.5 and Proposition 2.7, we have $(X, S(\tau))$ is a $T_\circ$-space.

□

**Example 3.13.** *Let $(X, \tau)$ be a $T_\circ$-space, where $X = \{a, b, c\}$ and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then, $\preccurlyeq_\tau = \{(a, a), (b, b), (c, c), (b, a), (b, c)\}$. It is obvious that $\preccurlyeq_\tau$ is a pre-ordered. Also, the set of all upper subsets of $X$ induced by $\preccurlyeq_\tau$ is $\{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and also the set of all d-open subsets of $X$ is $\{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Therefore $S(\tau) = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ which is both $T_\circ$-space and $t - T_\circ$-space.*

**Theorem 3.14.** *For any pre-ordered set $(X, \preccurlyeq)$ we have $S(\tau_S) = \tau_S$.*

*Proof.* The result follows from Theorem 2.6. □

Now, to give some new types of topologies on a t-set, we introduce the following obvious theorem.

**Theorem 3.15.** *Let $(X, \preccurlyeq)$ be any t-set and $\bigvee(\{x\}) \neq \phi$ for all $x \in X$. We define the following topologies on X.*

(1) *Alexandroff topology: $\tau_{A|x} = \{A \subseteq X : A \text{ is an upper subset}\}$;*
(2) *The upper topology: $\tau_U$ is the topology generated by the prebasis $\{X - \downarrow x\}$;*
(3) *The lower topology: $\tau_L$ is the topology generated by the prebasis $\{X - \uparrow x\}$;*
(4) *The interval topology: $\tau_I$ is the join of the upper and lower topologies;*
(5) *The Lawson topology: $\tau_{Ls}$ is the join of the lower Scott topologies.*

*Then the following statements are true for any pre-ordered set $(X, \preccurlyeq)$.*

(1) $\tau_U \subseteq \tau_S$;
(2) $\tau_S \subseteq \tau_{A|x}$;
(3) $\tau_S \subseteq \tau_d$;
(4) $\tau_S \subseteq \tau_{LS}$;
(5) $\tau_L \subseteq \tau_{LS}$;
(6) $\tau_U \subseteq \tau_I$;
(7) $\tau_L \subseteq \tau_I$;
(8) $\tau_S = \tau_d \cap \tau_{A|x}$;
(9) $\tau_U \subseteq \tau_{A|x}$;
(10) $\tau_U \subseteq \tau_d$;
(11) $\tau_U \subseteq \tau_{LS}$.

*Proof.* The proof is easy and hence omitted. □

## 4. Conclusion

The aim of this research is to focus on the Scott*-topology and some of its properties. We hope that the results of our paper will be a starting point for a sufficiently general but simple theory of objects that are suitable for modelling various aspects of computation and useful in modern applications of domain theory to general topology and mathematical analysis. We believe that it would be interesting to study this approach if we replace a pre-ordered set by an abstract base, a continuous information system or a t-set. We intend to investigate all these issues in future research works. Moreover, We looking for more studies in the future studying application studies on the Scott*-topology. Further, Looking for a definite definitions of the interior, exterior, boundary and limit points in the Scott*-topology. Furthermore, we will study Theorem 3.12 in more details. Also, a new kind of compactness will be defined using t-sets and a characterization of Alexandroff-continuous functions between t-sets will be given(good willing). Further, we intend to investigate all above issues in fuzzy setting in future research works. We hope to demonstrate some important applications of Scott*-closed sets in convex spaces [18].

## 5. Compliance with ethical standards

**Conflict of Interest:** Authors declare that they have no conflict of interest.
**Ethical approval:** This article does not contain any studies with human participants or animals performed by any of the authors.

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Osama Rashed Sayed
Orcid number: 0000-0001-9162-1862
Department of Mathematics
Faculty of Science
Assiut University
Assiut 71516, Egypt
*Email address*: o_sayed@aun.edu.eg

Nabil Hassan Sayed
Orcid number: 0000-0002-1418-8249
Department of Mathematics
Faculty of Science
The New Valley University
Egypt
*Email address*: nabil.gawed@newvedu.au.edu.eg