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ON HYPERIDEALS OF KRASNER HYPERRINGS BASED ON DERIVED UNITARY RINGS

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Dedicated to sincere professor Mashaallah Mashinchi

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Abstract. In this paper first, we introduce and analyze the strongly regular relations λ_e^* and Λ_e^* on a hyperring such that the derived quotient ring is unitary and unitary commutative, respectively. Next, we define and study the notion of λ_e-parts in a hyperring and characterize the λ_e-parts in a λ_e-strong hyperring R. Finally, we introduce the notion of λ_e-closed hyperideal in a hyperring and study some of its fundamental properties in Krasner hyperrings.

Keywords: (Krasner) hyperring, strongly regular relation, λ_e-closed hyperideal.
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1. Introduction and Preliminaries

The study of equivalence relations on hyperrings, that help us to obtain corresponding quotient rings satisfying special conditions, has made a considerable progress in the first decade of the twentieth century, but it still represents an interesting research subject, as proved by the publications appeared in the last years [1, 22]. The starting point of the general approach to these relations is [3, Theorem 13], where the authors represented relations of a multialgebra $\mathbf{A}$ using polynomial functions of a universal algebra $P^*(\mathbf{A})$ of non-empty subsets of $\mathbf{A}$. Then a characterization for these relations was provided in [25, Proposition 4.1]. The fact that term function is an important tool in multialgebras is clearly explained also by other authors, for example see [6,7,10,13,18–20,27,28]. It is already confirmed [24, 25] that the general approach to multialgebra gives widely available results and even stronger results than they currently have. In particular, Theorem 6 in [26] states that, for each multialgebra $\mathbf{A} =< A, F >$ and $\mathcal{I}$ the set of some identities, there exists the smallest equivalence relation on $\mathbf{A}$ for which the corresponding factor multialgebra is a universal algebra satisfying all the identities from $\mathcal{I}$. Following the same idea, in this article we introduce the smallest equivalence relation on a hyperring such that its

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corresponding quotient ring becomes a unitary ring. This relation leads us to investigate new particular complete parts and hyperideals in hyperrings.

First we will recall some basic definitions and results, that are fundamental in the following sections.

A *hypergroupoid* $(H, \circ)$ is a non-empty set H together with a hyperoperation $\circ$ defined on H, that is a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$ and sometimes, for simplicity, by xy. If A, B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup_{x \in A, y \in B} x \circ y$. $x \circ A$ is used for $\{x\} \circ A$ (respectively $A \circ x$ for $A \circ \{x\}$). A hypergroupoid $(H, \circ)$ is called a *hypergroup* in the sense of [15] if for all $x, y, z \in H$ the following two conditions hold: (i) $x \circ (y \circ z) = (x \circ y) \circ z$, (ii) $x \circ H = H \circ x = H$, meaning that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in x \circ u$ and $y \in v \circ x$. If $(H, \circ)$ satisfies only the first axiom, then it is called a *semi-hypergroup*. An exhaustive review updated to 1992 of hypergroup theory appears in [4], while the book [5] contains several applications of this theory.

If $(H, \circ)$ is a semi-hypergroup (respectively hypergroup) and $R \subseteq H \times H$ is an equivalence, we set

$$A \overset{=}{R} B \Leftrightarrow a \, R \, b, \qquad \forall a \in A, \forall b \in B,$$

for all pairs (A, B) of non-empty subsets of H.

The relation R is called *strongly regular on the left side* (*on the right side*) if $x \, R \, y \Rightarrow a \circ x \overset{=}{R} a \circ y$ ($x \, R \, y \Rightarrow x \circ a \overset{=}{R} y \circ a$, respectively), for all $(x, y, a) \in H^3$. Moreover, R is called *strongly regular* if it is strongly regular on the right and on the left side.

Proposition 1.1. *Let $(H, \circ)$ be a semi-hypergroup and R and S be two strongly regular relations on H. Then, $(R \cup S)^*$, which is the transitive closure of $R \cup S$, is strongly regular.*

Proof. Suppose that x and y are two elements in H such that $x(R \cup S)^* y$. We show that for all $a \in H$, $a \circ x \overline{\overline{(R \cup S)^*}} a \circ y$. For this reason consider the arbitrary elements $s \in a \circ x$ and $t \in a \circ y$. Since $x(R \cup S)^* y$, there exist $z_1, \ldots, z_n$ in H such that

$$x(R \cup S) z_1 (R \cup S) z_2 \ldots (R \cup S) z_n (R \cup S) y.$$

Since R and S are strongly regular, we have

$$a \circ x \overline{\overline{(R \cup S)}} a \circ z_1 \overline{\overline{(R \cup S)}} a \circ z_2 \ldots \overline{\overline{(R \cup S)}} a \circ z_n \overline{\overline{(R \cup S)}} a \circ y.$$

Therefore for all $i \in \{1, \ldots, n\}$ and $l_i \in a \circ z_i$, we have

$$s(R \cup S) l_1 (R \cup S) l_2 \ldots (R \cup S) l_n (R \cup S) t.$$

So $s(R \cup S)^* t$ and hence $(R \cup S)^*$ is strongly regular on the left side. Similarly we can show that $(R \cup S)^*$ is strongly regular on the right and the proof is then complete. □

If $(H, \circ)$ is a semi-hypergroup (respectively hypergroup) and R is strongly regular, then the quotient H/R is a semigroup (respectively group) under the operation:

$$R(x) \otimes R(y) = R(z), \qquad \forall z \in x \circ y,$$

where $R(a)$ is the equivalence class of $a \in H$ with respect to R [4].

For every $n \in \mathbb{N}$, the fundamental relation β_n is defined as follows:

$$\forall (x, y) \in H^2, x \ \beta_n \ y \Longleftrightarrow \exists (z_1, z_2, \ldots, z_n) \in H^n, \{x, y\} \subseteq \prod_{i=1}^{n} z_i.$$

Besides, $\beta_1 = \{(x, x) \mid x \in H\}$ and $\beta = \bigcup_{n \in \mathbb{N}} \beta_n$. It is known that in every hypergroup $\beta = \beta^*$ [11], where β^* is the transitive closure of the relation β. Moreover, it is the smallest strongly regular relation on H such that the quotient H/β^* is a group.

In general, if R is an equivalence relation on a set A, then $\forall S \in P^*(A)$, we write $R(S) = \bigcup_{x \in S} R(x)$.

Let $(H, \circ)$ be a hypergroupoid and A a non-empty subset of H. A is called *invertible on the left* in H if, $\forall (x, y) \in H^2$, the following implication holds: $y \in A \circ x \Longrightarrow x \in A \circ y$. Similarly the invertibility to the right is defined. We say that A is *invertible* if it is invertible on the right and on the left side [5]. An element e of a hypergroup H is called an *identity* if $a \in a \circ e \cap e \circ a$ for all $a \in H$, while an element u of H is called a *scalar identity* if $u \circ x = x \circ u = \{x\}$ for all $x \in H$ [5].

The hyperrings were introduced and studied by Krasner [14], Nakasis [21], Massouros [16] and especially reviewed by Davvaz and Leoreanu-Fotea [8]. A well-known type of a hyperring, called *Krasner hyperrings*, is essentially rings, with approximately modified axioms in which addition is a hyperoperation, while the multiplication is an operation. Some principal notions of hyperring theory can be found in the book [8]. Another type of hyperrings was introduced by Rota [29] in 1982, where the multiplication is a hyperoperation, while the addition is an operation, and it is called a *multiplicative hyperring*, that was subsequently investigated by Olson and Ward [23]. Vougiouklis [30] introduced the *general hyperrings*, in which the addition and the multiplication are both hyperoperations.

According to [30] a *general hyperring*, called by short only hyperring, is a triple $(R, +, \cdot)$ satisfying the following ring-like axioms (i) $(R, +)$ is a hypergroup, (ii) $(R, \cdot)$ is a semi-hypergroup, (iii) the multiplication is distributive with respect to the hyperaddition. A hyperring $(R, +, \cdot)$ is said to be *unitary hyperring* if relating to the multiplication, $(R, \cdot)$ is a semi-hypergroup with the scalar identity e. The element e is called the *unit* of R.

A Krasner hyperring (see [14]) is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup, that is:

i) for every $x, y, z \in R$, $x + (y + z) = (x + y) + z$;
ii) for every $x, y \in R$, $x + y = y + x$;
iii) there exists $0 \in R$ such that $0 + x = x = x + 0$ for all $x \in R$;
iv) for every $x \in R$ there exists a unique element $x' \in R$ such that $0 \in x + x'$ (we shall write $-x$ for x' and we call it the *opposite* of x);
v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, that is, $x \cdot 0 = 0 \cdot x = 0$.
(3) The multiplication is distributive with respect to the hyperaddition $+$.

In [31] Vougiouklis defined the relation Γ on a hyperring as follows: $x\Gamma y$ if and only if $x, y \subseteq u$, where u is a finite sum of finite products of elements of R, in fact there exist $n, k_i \in \mathbb{N}$ and $x_{ij} \in R$ such that $u = \sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} x_{ij}$. He proved that the quotient R/Γ^*, where Γ^* is the transitive closure of Γ, is a ring and also that Γ^* is the smallest equivalent relation on R such that the quotient R/Γ^* is a fundamental ring. Both operations $\oplus$ and $\odot$ on R/Γ^* are defined as follows:

$$\forall z \in \Gamma^*(x) + \Gamma^*(y), \;\; \Gamma^*(x) \oplus \Gamma^*(y) = \Gamma^*(z);$$
$$\forall z \in \Gamma^*(x) \cdot \Gamma^*(y), \;\; \Gamma^*(x) \odot \Gamma^*(y) = \Gamma^*(z).$$

The commutativity of the addition in rings can be related with the existence of the unit for the multiplication. When we speak about a general hyperring $(R, +, \cdot)$, the hyperaddition is not commutative and there is no unit with respect to the multiplication. So the commutativity, as well as the existence of the unit, are not assumed in the fundamental ring. Besides, we know that there exist many rings (where the hyperoperation $+$ is commutative) without unit. Having this idea in mind and try to analyse it for hyperrings, Davvaz and Vougiouklis [9] introduced the α-relation as a new strongly regular equivalence relation on a hyperring such that the quotient is an ordinary commutative ring. Let's recall here its definition.

Definition 1.2. If R is a hyperring, we set
$$\alpha_0 = \{(x, x) \mid x \in R\}$$
and, for every integer $n \geq 1$, α_n is the relation defined as follows:
$$x\alpha_n y \Longleftrightarrow \exists (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n, \exists \sigma \in S_n \quad \text{and}$$
$$[\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}, \exists \sigma_i \in S_{k_i}, (i = 1, \ldots, n)] \quad \text{such that}$$
$$x \in \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \qquad \text{and} \qquad y \in \sum_{i=1}^{n} A_{\sigma(i)},$$
where $A_i = \prod\limits_{j=1}^{k_i} x_{i\sigma_i(j)}$.

Obviously, for every $n \geq 1$, the relation α_n is symmetric, while the relation $\alpha = \bigcup\limits_{n\geq 0} \alpha_n$ is reflexive and symmetric.

Theorem 1.3. *[9] Let $(R, +, \cdot)$ be a hyperring and α^* be the transitive closure of α.*

(1) *α^* is a strongly regular relation on both $(R, +)$ and $(R, \cdot)$.*
(2) *The quotient R/α^* is a commutative ring.*
(3) *The relation α^* is the smallest equivalence relation such that the quotient R/α^* is a commutative ring.*

Note that the relation α in Theorem 1.2 consists of two relations α_+ and $\alpha_\times$ defined as follows:

(1) $x \ \alpha_+ \ y \Leftrightarrow \exists n \in \mathbb{N}, \exists (k_1, \ldots, k_n) \in \mathbb{N}^n, \exists \sigma \in S_n,$

$$x \in \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \text{ and } y \in \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod\limits_{j=1}^{k_i} x_{ij}$. In fact, if in Theorem 1.2 we set $\sigma_i = id$, then we obtain α_+.

(2) $x \, \alpha_\times \, y \Leftrightarrow \exists n \in \mathbb{N}, \exists (k_1, \ldots, k_n) \in \mathbb{N}^n,$ and $\forall i = 1, \ldots, n, \exists (x_{i1}, \ldots, x_{ik_i}), \exists \sigma_i \in S_{k_i}$ such that $x \in \sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} x_{ij}$ and $y \in \sum\limits_{i=1}^{n} A_i$, where $A_i = \prod\limits_{j=1}^{k_i} x_{i\sigma_i(j)}$. In fact, if in Theorem 1.2 we set $\sigma = id$, then we obtain $\alpha_\times$.

Then immediately we obtain the new result.

Proposition 1.4. *If $(R, +, \cdot)$ is a hyperring, then*

(i) *α_+^* and $\alpha_\times^*$ are strongly regular relations on both $(R, +)$ and $(R, \cdot)$;*
(ii) *$\alpha^* = (\alpha_\times \ \cup \ \alpha_+)^*$.*

Proof. (i) The proof follows directly from the definition.

(ii) By Theorem 1.1 the proof is obvious. □

Example 1.5. *Let $R = \{0, 1, 2\}$ be a set with hyperoperation $+$ and the binary operation $\cdot$ defined as follows:*

$+$	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

and

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Then $(R, +, \cdot)$ is a hyperring. Thus $\alpha_+^* = R \times R$ and $\alpha_\times^* = \Delta = \{(x, x) \mid x \in R\}$ the diagonal relation on R.

In this paper first we show that the relation α^* produces with two subrelations and then we introduce and analyze some new binary relations λ_e^* and Λ_e^* on hyperrings such that the derived rings are unitary and unitary commutative, respectively. We also investigate λ_e-parts on hyperrings. By introducing the notion of λ_e-closed hyperideal I in a hyperring R, we construct the ring R/I. Then we study the relationship between ideals in the ring R/I with hyperideals in the Krasner hyperring R containing I.

2. The relation λ_e

In this section, by replacing $\alpha_\times$ with a suitable relation $\lambda_\times^e$, we introduce the relations λ_e and Λ_e in order to obtain a new characterization of the derived hyperring.

Definition 2.1. Suppose that $(R, +, \cdot)$ is a hyperring and $e \in R$. We say that the pair

$$(\sum_{i=1}^{m} \prod_{j=1}^{k_i} x_{_{ij}}, \sum_{i=1}^{m} \prod_{t=1}^{k'_i} y_{_{it}})$$

satisfies the condition $\mathfrak{P}_\mathfrak{e}$ whenever one of the following situations occurs:

(1) $\forall i \in \{1, \ldots, m\}$, $k_i = k'_i$ and $\forall j \in \{1, \ldots, k_i\}$, $x_{ij} = y_{ij}$;

(2) there exist $i_1, \ldots, i_d \in \{1, \ldots, m\}$ such that
- $\forall 1 \leqslant i \leqslant m$ with $i \notin \{i_1, \ldots, i_d\}$ and $\forall 1 \leqslant t \leqslant k'_i$, we have $k_i = k'_i$ and $x_{_{it}} = y_{_{it}}$;
- $\forall 1 \leqslant j \leqslant d$ there exist p_{i_j} and $l_{i_j} \in \mathbb{N}$ such that $1 \leqslant p_{i_j} \leqslant k'_{i_j}$, $p_{i_j} \leqslant l_{i_j} \leqslant k'_{i_j}$ and $k'_{i_j} = k_{i_j} + (l_{i_j} - p_{i_j} + 1)$
- $\forall 1 \leqslant j \leqslant d$ we have

$$y_{_{i_jt}} \overset{\text{def}}{:=} \begin{cases} x_{_{i_jt}}, & \text{if } 1 \leqslant t < p_{i_j}; \\ e, & \text{if } p_{i_j} \leqslant t \leqslant l_{i_j}; \\ x_{_{i_j(t+p_{i_j}-l_{i_j}-1)}}, & \text{if } l_{i_j} < t \leqslant k'_{i_j}. \end{cases} \tag{1}$$

Remark 2.2. The condition $\mathfrak{P}_\mathfrak{e}$ is as follows:
The pair $\big(\sum\limits_{i=1}^{m} \prod\limits_{j=1}^{k_i} x_{_{ij}}, \sum\limits_{i=1}^{m} \prod\limits_{t=1}^{k'_i} y_{_{it}}\big)$ is in $\mathfrak{P}_\mathfrak{e}$ if the difference of elements in finite products $\prod\limits_{j=1}^{k_i} x_{_{ij}}$ and $\prod\limits_{t=1}^{k'_i} y_{_{it}}$ can be seen in finite products of e's for every i. In other words if we omit the element e from products $\prod\limits_{j=1}^{k_i} x_{_{ij}}$ and $\prod\limits_{t=1}^{k'_i} y_{_{it}}$ then the remain products are the same. For example $(x_{11}x_{12} + x_{21}x_{22}x_{23} + x_{31}x_{32}, x_{11}eex_{12} + x_{21}x_{22}x_{23} + x_{31}x_{32}e)$ is in $\mathfrak{P}_\mathfrak{e}$.

Definition 2.3. Suppose that the hyperring R and the element $e \in R$ are given. For all $m \geqslant 1$, define:

$$\Re_m^e := \Big\{ \big(\sum_{i=1}^{m} \prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{m} \prod_{t=1}^{k'_i} y_{it}\big), \big(\sum_{i=1}^{m} \prod_{t=1}^{k'_i} y_{it}, \sum_{i=1}^{m} \prod_{j=1}^{k_i} x_{ij}\big) \mid \text{ the pair } \big(\sum_{i=1}^{m} \prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{m} \prod_{t=1}^{k'_i} y_{it}\big) \text{ satisfies condition } \mathfrak{P}_{\mathfrak{e}} \Big\};$$

and $\Re^e := \bigcup_{m \geqslant 1} \Re_m^e$.

Example 2.4. *Let R be a ring with identity $e = 1_R$. Then for every $m \geq 1$, we have $\Re_m^e = \Delta$ and so $\Re^e = \Delta$.*

Example 2.5. *Let $(A, +, \cdot)$ be a ring and N be a subgroup of its multiplicative semigroup such that for each $a \in A$ we have $aN = Na$. Then the multiplicative classes $\bar{x} = xN (x \in A)$ form a partition of A, and let $\bar{A} = A/N$ be the set of these classes. If for all $\bar{x}, \bar{y} \in \bar{A}$, we define*

$$\bar{x} \oplus \bar{y} = \{\bar{z} \mid z \in \bar{x} + \bar{y}\}, \quad and \quad \bar{x} * \bar{y} = \overline{x \cdot y},$$

then the obtained structure is a hypergroup, see [8, Example 3.1.3(3)].
Now let $(A, +, \cdot) = (\mathbb{Z} \times 2\mathbb{Z}, +, \cdot)$, $N = \{-1, 1\} \times \{0\} = \{(-1, 0), (1, 0)\}$. Thus for each $(n, 2m) \in \mathbb{Z} \times 2\mathbb{Z}$ we have $\overline{(n, 2m)} = \{(-n, 0), (n, 0)\} = \overline{(n, 0)}$ and so $(\mathbb{Z} \times 2\mathbb{Z})/N = \{\overline{(n, 0)} \mid n \in \mathbb{Z}\}$. Put $e := \overline{(3, 0)}$. Therefore,

$$\Re^e = \{\big(\overline{(2k+1, 0)}, \overline{(2k'+1, 0)}\big), \big(\overline{(2k, 0)}, \overline{(2k', 0)}\big) \mid k, k' \in \mathbb{Z}\}.$$

Because multiplying the number 3 by every integer does not change whether the number is even or odd. For instance $\big(\overline{(1, 0)}, \overline{(1, 0)}e\big) \in \Re_1^e$, so $\big(\overline{(1, 0)}, \overline{(3, 0)}\big) \in \Re^e$. Also $\big(\overline{(1, 0)} + \overline{(2, 0)}, \overline{(1, 0)}e + \overline{(2, 0)}\big) \in \Re_2^e$ and $\big(\overline{(1, 0)} + \overline{(2, 0)}, \overline{(1, 0)} + \overline{(2, 0)}e\big) \in \Re_2^e$, this implies that

$$\{\big(\overline{(3, 0)}, \overline{(5, 0)}\big), \big(\overline{(3, 0)}, \overline{(1, 0)}\big), \big(\overline{(1, 0)}, \overline{(5, 0)}\big), \big(\overline{(1, 0)}, \overline{(1, 0)}\big)\} \subset \Re^e$$

and

$$\{\big(\overline{(1, 0)}, \overline{(5, 0)}\big), \big(\overline{(1, 0)}, \overline{(7, 0)}\big), \big(\overline{(3, 0)}, \overline{(5, 0)}\big), \big(\overline{(3, 0)}, \overline{(7, 0)}\big)\} \subset \Re^e.$$

By $\big(\overline{(1, 0)} + \overline{(3, 0)}, \overline{(1, 0)}e + \overline{(3, 0)}\big) \in \Re_2^e$ and $\big(\overline{(1, 0)} + \overline{(3, 0)}, \overline{(1, 0)} + \overline{(3, 0)}e\big) \in \Re_2^e$ we have

$$\{\big(\overline{(2, 0)}, \overline{(0, 0)}\big), \big(\overline{(2, 0)}, \overline{(6, 0)}\big), \big(\overline{(4, 0)}, \overline{(0, 0)}\big), \big(\overline{(4, 0)}, \overline{(6, 0)}\big)\} \subset \Re^e$$

and

$$\{\big(\overline{(2, 0)}, \overline{(8, 0)}\big), \big(\overline{(2, 0)}, \overline{(10, 0)}\big), \big(\overline{(4, 0)}, \overline{(8, 0)}\big), \big(\overline{(4, 0)}, \overline{(10, 0)}\big)\} \subset \Re^e.$$

Suppose that $(R,+,\cdot)$ is a hyperring. For every integer $m \geq 1$ and $x, y \in R$ define:

(2) $$x\ (\lambda^e_\times)_m\ y \Leftrightarrow \exists (A,B) \in \Re^e_m \ \text{ such that } x \in A \ \text{ and } y \in B.$$

Take $\lambda^e_\times := \bigcup\limits_{m\geq 1} (\lambda^e_\times)_m$. It follows that $\lambda^e_\times$ is reflexive and transitive. Let $(\lambda^e_\times)^*$ denote as usually the transitive closure of $\lambda^e_\times$.

Proposition 2.6. *If $(R,+,\cdot)$ is a hyperring, then $(\lambda^e_\times)_m \subseteq (\lambda^e_\times)_{m+1}$.*

Proof. If $x\ (\lambda^e_\times)_m\ y$, then there exists $(A,B) \in \Re^e_m$ such that $x \in A$ and $y \in B$. Without losing the generality suppose that $A = \sum\limits_{i=1}^{m} \prod\limits_{j=1}^{k_i} x_{ij}$ and $y = \sum\limits_{i=1}^{m} \prod\limits_{t=1}^{k'_i} y_{it}$. Since $(R,+)$ is a hypergroup, there exist $u, w \in R$ such that $x_{mk_m} \in u + w$. Therefore

$$\sum_{i=1}^{m} \prod_{j=1}^{k_i} x_{ij} \subseteq (x_{11} \ldots x_{1k_1}) + \ldots + (x_{(m-1)1} \ldots x_{(m-1)k_{(m-1)}}) +$$
$$(x_{m1} \ldots x_{m(k_m-1)} u) + (x_{m1} \ldots x_{m(k_m-1)} w).$$

Set $k_{m+1} := k_m$ and

$$x'_{ij} \overset{\text{def}}{:=} \begin{cases} x_{ij}, & \text{if } 1 \leqslant i \leqslant m-1, 1 \leqslant j \leqslant k_i; \\ x_{mj}, & \text{if } i = m, 1 \leqslant j \leqslant k_m - 1; \\ u, & \text{if } i = m, j = k_m; \\ x_{mj}, & \text{if } i = m+1, 1 \leqslant j \leqslant k_{(m+1)} - 1; \\ w, & \text{if } i = m+1, j = k_{m+1}. \end{cases}$$

Similarly define y'_{it} from y_{it}. Therefore

$$x \in \sum_{i=1}^{m+1} \prod_{j=1}^{k_i} x'_{ij} \quad \text{and} \quad y \in \sum_{i=1}^{m+1} \prod_{t=1}^{k'_i} y'_{it},$$

where $\Big(\sum\limits_{i=1}^{m+1} \prod\limits_{j=1}^{k_i} x'_{ij}, \sum\limits_{i=1}^{m+1} \prod\limits_{t=1}^{k'_i} y'_{it} \Big) \in \Re^e_{m+1}$. Thus $x\ (\lambda^e_\times)_{m+1}\ y$. □

Definition 2.7. A hyperring $(R,+,\cdot)$ is called $(\lambda_e)_n$-complete if $\forall (k_1, \ldots, k_n) \in \mathbb{N}^n$, $\forall (x_{i1}, \ldots x_{ik_i}) \in R^{k_i}$, $i = 1, \ldots n$, there is

$$\lambda_e \Big(\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \Big) = \sum_{i=1}^{n} \prod_{t=1}^{k'_i} y_{it},$$

where $\Big(\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} x_{ij}, \sum\limits_{i=1}^{n} \prod\limits_{t=1}^{k'_i} y_{it} \Big) \in \Re^e_n$.

We recall from [17] that a hyperring R is called *n-complete*, if $\forall(k_1,\dots,k_n)\in\mathbb{N}^n$, $\forall(x_{i1},\dots,x_{ik_i})\in R^{k_i}$, there is

$$\Gamma\Big(\sum_{i=1}^{n}\prod_{j=1}^{k_i}x_{ij}\Big)=\sum_{i=1}^{n}\prod_{j=1}^{k'_i}y_{ij}.$$

Proposition 2.8. *If $(R,+,\cdot)$ is a unitary hyperring with the unit e, then R is a $(\lambda_e)_n$-complete hyperring if and only if R is an n-complete hyperring.*

Proof. Since $(R,\cdot)$ is a semi-hypergroup with the scalar identity e, it follows that $\lambda^e_\times=\alpha_+$ and therefore $\Gamma=\lambda_e$. □

Theorem 2.9. *If $(R,+,\cdot)$ is a $(\lambda_e)_n$-complete hyperring, then*

$$(\lambda_e)_n=\lambda_e.$$

Proof. Suppose that $x\ \lambda_e\ y$. Thus there exists $(A,B)\in\Re^e$ such that $x\in A$ and $y\in B$. Without losing the generality suppose that

$$A=\sum_{i=1}^{m}\prod_{j=1}^{k_i}x_{ij}\quad\text{and}\quad B=\sum_{i=1}^{m}\prod_{t=1}^{k'_i}y_{it}.$$

If A and B satisfy condition (1) of Theorem 2.1, then the proof is obvious. Now suppose that A and B satisfy condition (2) of Theorem 2.1. If $m\leqslant n$, then by Theorem 2.6, we have $(\lambda^e_\times)_m\subseteq(\lambda^e_\times)_n$. If $m>n$, since $(R,+)$ is a hypergroup, it follows that there exists $s\in R$ such that $s\in\sum\limits_{i=n}^{m}\prod\limits_{j=1}^{k_i}x_{ij}$.
Set $l_i:=\begin{cases}k_i, & \text{if } 1\leqslant i\leqslant n-1;\\ 1, & \text{if } i=n.\end{cases}$ and $z_{ij}:=\begin{cases}x_{ij}, & \text{if } 1\leqslant i\leqslant n-1 \text{ and } 1\leqslant j\leqslant l_i;\\ s, & \text{if } i=n\end{cases}$.
Therefore $x\in\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{l_i}z_{ij}$ and since R is a $(\lambda_e)_n$-complete hyperring, it follows that

$$\lambda_e(x)\subseteq\lambda_e\Big(\sum_{i=1}^{n}\prod_{j=1}^{l_i}z_{ij}\Big)=\sum_{i=1}^{n}\prod_{j=1}^{l'_i}z'_{it},$$

where $\Big(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{l_i}z_{ij},\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{l_i}z_{ij}\Big)\in\Re^e_n$. By $x\ \lambda_e\ y$ we have $y\in\lambda_e(x)$ and hence $x\ (\lambda_e)_n\ y$, concluding the proof. □

Example 2.10. *Let R be a ring with identity $e=1_R$. Then R is a $(\lambda_e)_n$-complete hyperring, so $(\lambda_e)_n=\lambda_e=\Delta$.*

Example 2.11. *Let $(\bar{A},\oplus,*)$ be the hyperring defined in the Theorem 2.5 and put $e=\overline{(3,0)}$. It is not difficult to see that*

$$\lambda_e(\overline{(0,0)})=\{\overline{(2k,0)}\mid k\in\mathbb{Z}\}\ \textit{and}\ \lambda_e(\overline{(1,0)})=\{\overline{(2k+1,0)}\mid k\in\mathbb{Z}\}.$$

Proposition 2.12. *If $(R,+,\cdot)$ is a hyperring, then $(\lambda^e_\times)^*$ is a strongly regular relation both on $(R,+)$ and $(R,\cdot)$.*

Proof. We can see very easily that $(\lambda^e_\times)^*$ is an equivalence relation. In order to prove that it is strongly regular, we show that if $x\ \lambda^e_\times\ y$, then

$$\begin{cases} x+a\ \overline{\overline{\lambda^e_\times}}\ y+a, & a+x\ \overline{\overline{\lambda^e_\times}}\ a+y \\ x\cdot a\ \overline{\overline{\lambda^e_\times}}\ y\cdot a, & a\cdot x\ \overline{\overline{\lambda^e_\times}}\ a\cdot y, \end{cases}$$

for every $a\in R$. If $x\ \lambda^e_\times\ y$, then there exists a pair $(A,B)\in\Re^e$ such that $x\in A$ and $y\in B$. So there exists $m\geqslant 1$ such that $A=\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{k_i}x_{ij}$, $B=\sum\limits_{i=1}^{m}\prod\limits_{t=1}^{k'_i}y_{it}$ and the pair (A,B) satisfies condition $\mathfrak{P}_\mathfrak{e}$. Thus,

$$x+a\subseteq\Big(\sum_{i=1}^{m}\prod_{j=1}^{k_i}x_{ij}\Big)+a \quad\text{and}\quad y+a\subseteq\Big(\sum_{i=1}^{m}\prod_{t=1}^{k'_i}y_{it}\Big)+a.$$

Now, let $k_{m+1}=1,\ k'_{m+1}=1,\ x_{(m+1)1}=a,\ y_{(m+1)1}=a$. So

$$x+a\subseteq\sum_{i=1}^{m+1}\prod_{j=1}^{k_i}x_{ij} \quad\text{and}\quad y+a\subseteq\sum_{i=1}^{m+1}\prod_{t=1}^{k'_i}y_{it}.$$

Therefore $\Big(\sum\limits_{i=1}^{m+1}\prod\limits_{j=1}^{k_i}x_{ij},\sum\limits_{i=1}^{m+1}\prod\limits_{t=1}^{k'_i}y_{it}\Big)\in\Re^e_{m+1}$. This implies that $(A+a,B+a)\in\Re^e$. Thus, for all $u\in x+a$ and $v\in y+a$, we have $u\ \lambda^e_\times\ v$ and hence $x+a\ \overline{\overline{\lambda^e_\times}}\ y+a$. Similarly we can show that $a+x\ \overline{\overline{\lambda^e_\times}}\ a+y$.

Now we prove that $(\lambda^e_\times)^*$ is a strongly regular relation on $(R,\cdot)$. If $x\ \lambda^e_\times\ y$, then there exists $(A,B)\in\Re^e$ such that $x\in A$ and $y\in B$. Therefore there exists $m\geqslant 1$ such that $A=\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{k_i}x_{ij}$, $B=\sum\limits_{i=1}^{m}\prod\limits_{t=1}^{k'_i}y_{it}$ and the pair $\Big(\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{k_i}x_{ij},\sum\limits_{i=1}^{m}\prod\limits_{t=1}^{k'_i}y_{it}\Big)$ satisfies condition $\mathfrak{P}_\mathfrak{e}$. Thus

$$x\cdot a\subseteq\Big(\sum_{i=1}^{m}\prod_{j=1}^{k_i}x_{ij}\Big)\cdot a \quad\text{and}\quad y\cdot a\subseteq\Big(\sum_{i=1}^{m}\prod_{t=1}^{k'_i}y_{it}\Big)\cdot a.$$

Now let for all $1\leqslant i\leqslant m,\ t_i=k_i+1, t'_i=k'_i+1,\ x_{it_i}=a$ and $y_{it'_i}=a$. So $x\cdot a\subseteq\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i}x_{ij},\ y\cdot a\subseteq\sum\limits_{i=1}^{m}\prod\limits_{t=1}^{t'_i}y_{it}$ and $\Big(\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i}x_{ij},\sum\limits_{i=1}^{m}\prod\limits_{t=1}^{t'_i}y_{it}\Big)\in\Re^e_m$. Therefore $(A\cdot a,B\cdot a)\in\Re^e$ and hence for all $u\in x\cdot a$ and $v\in y\cdot a$, we have $u\ \lambda^e_\times\ v$. Thus $x\cdot a\ \overline{\overline{\lambda^e_\times}}\ y\cdot a$. Similarly we can show that $a\cdot x\ \overline{\overline{\lambda^e_\times}}\ a\cdot y$. $\square$

Definition 2.13. For any hyperring R, define the relation λ_e as $\lambda_e=\lambda^e_\times\cup\alpha_+$.

Obviously, the relation λ_e is reflexive and symmetric. Let λ_e^* be the transitive closure of λ_e. We have the following result.

Proposition 2.14. *If $(R, +, \cdot)$ is a hyperring, then λ_e^* is a strongly regular relation both on $(R, +)$ and $(R, \cdot)$.*

Proof. The proof follows from Theorem 1.1, Theorem 1.4 and Theorem 2.12. □

Theorem 2.15. *If $(R, +, \cdot)$ is a hyperring, then R/λ_e^* is a ring with the unit $\lambda_e^*(e)$.*

Proof. By Theorem 2.14 we conclude that R/λ_e^* is a ring with the following operations:

$$\lambda_e^*(a) \oplus \lambda_e^*(b) = \lambda_e^*(c), \quad \text{where } c \in \lambda_e^*(a) + \lambda_e^*(b);$$
$$\lambda_e^*(a) \odot \lambda_e^*(b) = \lambda_e^*(d), \quad \text{where } d \in \lambda_e^*(a) \cdot \lambda_e^*(b).$$

It remains to prove that $\lambda_e^*(e)$ is the unitary element of the ring R/λ_e^*. By Theorem 2.3, for all $x \in R$ we know that (x, xe) and (x, ex) are in $\Re_1^e$. So, for all $y \in xe \cup ex$, there is $x \ \lambda_\times^e \ y$ and hence $\lambda_e^*(x) = \lambda_e^*(y)$. Now suppose that $z \in \lambda_e^*(x) \cdot \lambda_e^*(e)$, so there exist $a \in \lambda_e^*(x)$ and $b \in \lambda_e^*(e)$ such that $z \in ab$. Thus we have

$$a = x_1 \ \lambda_e \ x_2 \ \lambda_e \ ... \ \lambda_e \ x_n = x \ \text{ and } \ b = y_1 \ \lambda_e \ y_2 \ \lambda_e \ ... \ \lambda_e \ y_m = e.$$

Therefore

$$ab = x_1 b \ \overline{\overline{\lambda_e}} \ x_2 b \ \overline{\overline{\lambda_e}} \ ... \ \overline{\overline{\lambda_e}} \ x_n b = xb \ \text{ and } \ xb = xy_1 \ \overline{\overline{\lambda_e}} \ xy_2 \ \overline{\overline{\lambda_e}} \ ... \ \overline{\overline{\lambda_e}} \ xy_m = xe$$

and so, for all $y \in xe$ we have $z \ \lambda_e \ y$. Thus $\lambda_e^*(z) = \lambda_e^*(y) = \lambda_e^*(x)$ and hence $\lambda_e^*(x) \odot \lambda_e^*(e) = \lambda_e^*(x)$. Similarly we can prove that $\lambda_e^*(e) \odot \lambda_e^*(x) = \lambda_e^*(x)$. Hence $\lambda_e^*(e)$ is the unitary element of the ring R/λ_e^*. □

Example 2.16. *Consider the ring $(2\mathbb{Z}, +, \cdot)$ of even integer numbers. Put $e = 4$ and so 6 is the smallest positive element in $2\mathbb{Z}$ in which $\lambda_e^*(0) = \lambda_e^*(6)$ (indeed since $(0, -2+2) \in \lambda_e^*$, $(0, 6) = (0, -2 + 4 \times 2) \in \lambda_e^*$). Let $a, b \in 2\mathbb{Z}$ such that $a \ \lambda_e^* \ b$. Thus $\lambda_e^*(0) = \lambda_e^*(a - b)$. By division algorithm there is a unique pair of integers q and r such that $a - b = 6q + r$ and $0 \leq r < 6$. Since $a - b$ is even, $r = 0, 2$ or 4. Suppose that $r = 2$ or 4. Because $\lambda_e^*(0) = \lambda_e^*(6q)$, hence $\lambda_e^*(0) = \lambda_e^*(r)$ and it is a contradiction. Thus $r = 0$ and so $a - b \in 6\mathbb{Z}$. This implies that $a' - b' \in 3\mathbb{Z}$, where $a = 2a'$ and $b = 2b'$. Therefore $2\mathbb{Z}/\lambda_e^*$ is isomorphic to the ring $\mathbb{Z}_3$.*

Theorem 2.17. *Suppose that $(R, +, \cdot)$ is a hyperring. Then the relation λ_e^* is the smallest equivalence relation such that the quotient R/λ_e^* is a ring with the unit $\lambda_e^*(e)$.*

Proof. Let μ be an equivalence relation on R such that R/μ is a ring with the unit $\mu(e) = \lambda_e^*(e)$ and let $\phi : R \longrightarrow R/\mu$ be the canonical projection. Suppose that $x \ \lambda_e \ y$. Thus we have the following two cases.

Case 1. $x\ \alpha_+\ y$, so there exists $\sigma \in S_n$ such that

$$x \in \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \quad \text{and} \quad y \in \sum_{i=1}^{n} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)j}.$$

So $\phi(x) = \bigoplus_{i=1}^{n} \Big(\bigodot_{j=1}^{k_i} \mu(x_{ij}) \Big)$ and $\phi(y) = \bigoplus_{i=1}^{n} \Big(\bigodot_{j=1}^{k_{\sigma(i)}} \mu(x_{\sigma(i)j}) \Big)$. By commutativity of the group $(R/\mu, \bigoplus)$, it follows that $\phi(x) = \phi(y)$ and hence $x\ \mu\ y$.

Case 2. $x\ \lambda^e_\times\ y$, so there exists $(A,B) \in \Re^e$ such that $x \in A$ and $y \in B$. Thus there exists $m \geqslant 1$ such that $A = \sum\limits_{i=1}^{m} \prod\limits_{j=1}^{k_i} x_{ij}$, $B = \sum\limits_{i=1}^{m} \prod\limits_{t=1}^{k'_i} y_{it}$ and the pair $\big(\sum\limits_{i=1}^{m} \prod\limits_{j=1}^{k_i} x_{ij}, \sum\limits_{i=1}^{m} \prod\limits_{t=1}^{k'_i} y_{it}\big)$ satisfies condition $\mathfrak{P}_\mathfrak{e}$. If (A,B) satisfies part (1) of Theorem 2.1, then $x\ \mu\ y$. If (A,B) satisfies part (2) of Theorem 2.1, then we have

$$\begin{array}{ccccccc}
\bigodot_{j=1}^{k_1} \mu(x_{1j}) \bigoplus & \cdots & \bigoplus \bigodot_{j=1}^{k_{i_1}} \mu(x_{i_1 j}) & \cdots & \bigoplus \bigodot_{j=1}^{k_{i_d}} \mu(x_{i_d j}) \bigoplus & \cdots & \bigoplus \bigodot_{j=1}^{k_m} \mu(x_{mj}) \\
\Big\| & & & & & & \Big\| \\
\bigodot_{t=1}^{k'_1} \mu(y_{1t}) \bigoplus & \cdots & \bigoplus \bigodot_{t=1}^{k'_{i_1}} \mu(y_{i_1 t}) & \cdots & \bigoplus \bigodot_{t=1}^{k'_{i_d}} \mu(y_{i_d t}) \bigoplus & \cdots & \bigoplus \bigodot_{t=1}^{k'_m} \mu(y_{mt}).
\end{array}$$

By Theorem 2.1 for each $1 \leqslant n \leqslant d$, we have $\bigodot\limits_{j=1}^{k_{i_n}} \mu(x_{i_n j}) = \bigodot\limits_{t=1}^{k'_{i_n}} \mu(y_{i_n t})$. Therefore $\phi(x) = \phi(y)$ and hence $x\ \mu\ y$. Thus $x\ \lambda_e\ y$ implies that $x\ \mu\ y$. Finally, let $x\ \lambda^*_e\ y$. Since μ is transitively closed, we obtain

$$x \in \lambda^*_e(y) \Rightarrow x \in \mu(y).$$

Therefore $\lambda^*_e \subseteq \mu$. □

Definition 2.18. Let $(R, +, \cdot)$ be a hyperring. Define the relation Λ_e by $\Lambda_e = \lambda^e_\times\ \cup\ \alpha$.

Obviously, the relation Λ_e is reflexive and symmetric. Also, if R is a unitary hyperring with unit e, then $\Lambda_e = \alpha$. Let Λ^*_e be the transitive closure of Λ_e.

Example 2.19. *Suppose that $(R, +, \cdot)$ is a unitary hyperring. According to theorem 2.8 we have $\lambda^e_\times = \alpha_+$. Thus $\Lambda_e = \lambda^e_\times\ \cup\ \alpha = \alpha_+ \cup\ \alpha = \alpha$*

Example 2.20. *Let $(\bar{A}, \oplus, *)$ be the hyperring defined in the Theorem 2.5 and put $e = \overline{(3,0)}$. Thus $\Lambda^*_e = \lambda^*_e$. By Theorem 2.11 we have $\lambda_e(\overline{(0,0)}) = \{\overline{(2k,0)} \mid k \in \mathbb{Z}\}$ and $\lambda_e(\overline{(1,0)}) = \{\overline{(2k+1,0)} \mid k \in \mathbb{Z}\}$. We claim the ring $\bar{A}/\Lambda^*_e$ is isomorphic to the ring $(\mathbb{Z}_2, +, \cdot)$. For the proof, it is enough to show that*

$\big(\overline{(1,0)},\overline{(0,0)}\big) \notin \Re^e$. *Otherwise, there exists* $\Big(\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{k_i}\overline{(x_{ij},0)},\sum\limits_{i=1}^{m}\prod\limits_{t=1}^{k'_i}\overline{(y_{it},0)}\Big) \in \Re^e_m$ *such that*

$$\overline{(1,0)} \in \sum_{i=1}^{m}\prod_{j=1}^{k_i}\overline{(x_{ij},0)} \qquad \textit{and,} \tag{3}$$

$$\overline{(0,0)} \in \sum_{i=1}^{m}\prod_{t=1}^{k'_i}\overline{(y_{it},0)}. \tag{4}$$

The relation (2.3) implies that just odd summands, such as $\prod\limits_{j=1}^{k_i}\overline{(x_{ij},0)}$, *exist in the sum* $\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{k_i}\overline{(x_{ij},0)}$, *where every* x_{ij} *is an odd number and in the other summands at least one of* x_{ij} *is an even number. Multiplying* e *in each element* $\overline{(n,0)}$ *of* $\bar{A}$ *yields the first component of the equivalence class representative to be multiplied by 3. Since multiplying the number* 3 *by every integer does not change whether the number is even or odd, we have* $\overline{(0,0)} \notin \sum\limits_{i=1}^{m}\prod\limits_{t=1}^{k'_i}\overline{(y_{it},0)}$. *This is in contrast to (2.4). Therefore* $\bar{A}/\Lambda^*_e = \{\Lambda^*_e(\overline{(0,0)}),\Lambda^*_e(\overline{(1,0)})\}$ *which is isomorphic to the ring* $(\mathbb{Z}_2,+,\cdot)$.

Theorem 2.21. *Let* $(R,+,\cdot)$ *be a hyperring. Then,*

(i) Λ^*_e *is a strongly regular relation both on* $(R,+)$ *and* $(R,\cdot)$ *and the quotient* R/Λ^*_e *is a commutative ring with the unit* $\Lambda^*_e(e)$.
(ii) *The relation* Λ^*_e *is the smallest equivalence relation such that the quotient* R/Λ^*_e *is a ring with the unit* $\Lambda^*_e(e)$.

Proof. (i) The proof follows from Theorem 1.1, Theorem 2.12 and Theorem 2.15. Notice that commutativity follows from Theorem 1.3.

(ii) The proof is similar to the proof of Theorem 2.17. □

3. The transitivity conditions of λ_e and λ_e-strong hyperrings

In this section, first we state the conditions that are equivalent to the transitivity of the relation λ_e. Then we introduce the notion of λ_e-strong hyperring and study its relationship with the transitivity of the relation λ_e.

Definition 3.1. Let M be a non-empty subset of a hyperring R. We say that M is a λ_e-part if the following conditions are satisfied:

(P1) for every $n \in \mathbb{N}$, $i = 1,\dots,n$, $k_i \in \mathbb{N}$, and any permutation $\sigma \in \mathbb{S}_n$, we have

$$\sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij} \bigcap M \neq \emptyset \Longrightarrow \sum_{i=1}^{n}\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)j} \subseteq M,$$

(P2) for every $m \in \mathbb{N}$, $i = 1, \ldots, m$, $k_i \in \mathbb{N}$, we have

$$\sum_{i=1}^{m} \prod_{j=1}^{k_i} x_{ij} \bigcap M \neq \emptyset \Longrightarrow \sum_{i=1}^{m} \prod_{t=1}^{k'_i} y_{it} \subseteq M,$$

whenever the pair $\big(\sum_{i=1}^{m} \prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{m} \prod_{t=1}^{k'_i} y_{it}\big)$ satisfies condition $\mathfrak{P}_{\mathfrak{e}}$.

Proposition 3.2. *Let M be a non-empty subset of a hyperring R. The following conditions are equivalent:*

(1) *M is a λ_e-part of R.*
(2) *For any $x \in M$ such that $x\lambda_e y$, it follows that $y \in M$.*
(3) *For any $x \in M$ such that $x\lambda_e^* y$, it follows that $y \in M$.*

Proof. ($1 \Longrightarrow 2$): Let $(x, y) \in R^2$ be a pair such that $x \in M$ and $x\lambda_e y$. Then we have the following two possibilities.

Case 1: If $x\alpha_+ y$, then there exist $n \in \mathbb{N}$ and the permutation $\sigma \in \mathbb{S}_n$ such that $x \in \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}$ and $y \in \sum_{i=1}^{n} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)j}$. Since M is a λ_e-part, by Theorem 3.1(P1), we have $\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{\sigma(i)j} \subseteq M$ and thus $y \in M$.

Case 2: If $x\lambda_\times^e y$, then there exists a pair $\big(\sum_{i=1}^{m} \prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{m} \prod_{t=1}^{k'_i} y_{it}\big)$ satisfying $\mathfrak{P}_{\mathfrak{e}}$ such that $x \in \sum_{i=1}^{m} \prod_{j=1}^{k_i} x_{ij}$ and $y \in \sum_{i=1}^{m} \prod_{t=1}^{k'_i} y_{it}$. Since M is a λ_e-part, by Theorem 3.1(P2), we have $\sum_{i=1}^{m} \prod_{t=1}^{k'_i} y_{it} \subseteq M$ and therefore $y \in M$.

($2 \Longrightarrow 3$): Let $x \in M$ and $y \in R$ such that $x\lambda_e^* y$. Obviously, there exist $m \in \mathbb{N}$ and $(x = w_0, w_1, \ldots, w_{m-1}, w_m = y) \in R^{m+1}$ such that $x = w_0\lambda_e w_1\lambda_e \ldots \lambda_e w_{m-1}\lambda_e w_m = y$. Since $x \in M$ we obtain $y \in M$, by applying the hypothesis (2) m times.

($3 \Longrightarrow 1$): First let $\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \bigcap M \neq \emptyset$ and $x \in \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \bigcap M$. For every permutation $\sigma \in \mathbb{S}_n$ and for every $y \in \sum_{i=1}^{n} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)j}$ we have $x\ \alpha_+\ y$ and hence $x\ \lambda_e\ y$. Thus $x \in M$ and $x\lambda_e^* y$ and so by (3) we obtain $y \in M$, meaning that $\sum_{i=1}^{n} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)j} \subseteq M$, as requested in the part $(P1)$ of Definition 3.1.

Now let $\sum_{i=1}^{m}\prod_{j=1}^{k_i} x_{ij} \bigcap M \neq \emptyset$ and $x \in \sum_{i=1}^{m}\prod_{j=1}^{k_i} x_{ij} \bigcap M$. For every $y \in \sum_{i=1}^{m}\prod_{t=1}^{k'_i} y_{it}$ we have $x\lambda_{\times}^{e} y$ and hence $x\ \lambda_e\ y$. Similarly as before we have $y \in M$ and thus $\sum_{i=1}^{m}\prod_{t=1}^{k'_i} y_{it} \subseteq M$, as requested in the part $(P2)$ of Definition 3.1. Now the proof is complete. □

Before proving the next theorem, we introduce the following notations.

Notation 3.3. For every element x of a hyperring R, set:

(N1) $T_n(x) = \{\sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij} \mid x \in \sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij}\}$;

(N2) $P_n^{\alpha_+}(x) = \bigcup_{n\geqslant 1}\{\sum_{i=1}^{n}\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)j} \mid \sigma \in \mathbb{S}_n, \sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij} \in T_n(x)\}$;

(N3) $P_m^{\lambda_{\times}^{e}}(x) = \bigcup_{m\geqslant 1}\{\sum_{i=1}^{m}\prod_{t=1}^{k'_i} y_{it} \mid \sum_{i=1}^{m}\prod_{j=1}^{k_i} x_{ij} \in T_m(x)$ and the pair

$$(\sum_{i=1}^{m}\prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{m}\prod_{t=1}^{k'_i} y_{it}) \text{ satisfies condition } \mathfrak{P}_{\mathfrak{e}}\};$$

(N4) $P(x) = [\bigcup_{n\geqslant 1} P_n^{\alpha_+}(x)] \bigcup [\bigcup_{m\geqslant 1} P_m^{\lambda_{\times}^{e}}(x)]$.

Lemma 3.4. *Suppose that R is a hyperring.*

(i) *For every $x \in R$, there is $P(x) = \{y \in R \mid x\lambda_e y\}$.*
(ii) *If M is a λ_e-part of R and $x \in M$, then $P(x) \subseteq M$.*

Proof. (i) It is easy to see that $P(x) \subseteq \{y \in R \mid x\lambda_e y\}$.

Conversely, for every pair (x, y) of elements of R we have:

$$\begin{aligned} x\alpha_+ y &\Leftrightarrow \exists n \in \mathbb{N}, \exists \sigma \in \mathbb{S}_n : x \in \sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij} \text{ and } y \in \sum_{i=1}^{n}\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)j} \\ &\Leftrightarrow \exists n \in \mathbb{N}, y \in P_n^{\alpha_+}(x) \\ &\Rightarrow y \in P(x), \end{aligned}$$

and

$$x\lambda^e_\times y \Leftrightarrow \exists m \in \mathbb{N}, x \in \sum_{i=1}^{m}\prod_{j=1}^{k_i} x_{ij} \text{ and } y \in \sum_{i=1}^{m}\prod_{t=1}^{k'_i} y_{it} \text{ such that the pair } \Big(\sum_{i=1}^{m}\prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{m}\prod_{t=1}^{k'_i} y_{it}\Big) \text{ satisfies condition } \mathfrak{P}_\mathfrak{e}$$
$$\Leftrightarrow \exists m \in \mathbb{N}, y \in P_m(x)$$
$$\Rightarrow y \in P(x).$$

(ii) The proof follows immediately from (i). □

With this preparation, we are now ready to state equivalent conditions to the transitivity of the relation λ_e.

Theorem 3.5. *Suppose that R is an arbitrary hyperring. Then the following conditions are equivalent:*

(1) *The relation λ_e is transitive.*
(2) *For every $x \in R$, there is $\lambda^*_e(x) = P(x)$.*
(3) *For every $x \in R$, the set $P(x)$ is a λ_e-part of R.*

Proof. ($1 \Longrightarrow 2$): By Theorem 3.4(i), for every pair (x, y) of elements of R we have:

$$y \in \lambda^*_e(x) \Leftrightarrow x\lambda^*_e y \Leftrightarrow x\lambda_e y \Leftrightarrow y \in P(x).$$

($2 \Longrightarrow 3$): If M is a non-empty subset of R, then M is an λ_e-part of R if and only if it is a union of equivalence classes modulo λ^*_e. In particular, every equivalence class modulo λ^*_e is a λ_e-part of R.

($3 \Longrightarrow 1$): If $x\lambda_e y$ and $y\lambda_e z$, then by Theorem 3.4(i), we have $x \in P(y)$ and $y \in P(z)$. Since $P(z)$ is a λ_e-part, by Theorem 3.4(ii), we obtain $P(y) \subseteq P(z)$. Thus $x \in P(z)$ and hence by Theorem 3.4(i), we have $x\lambda_e z$. Therefore λ_e is transitive. □

In the second part of this section, we introduce and study a new type of hyperring, called the λ_e-*strong hyperring*. Suppose that $(R, +, \cdot)$ is a hyperring and $\phi_e : R \longrightarrow R/\lambda^*_e$ is the canonical projection, i.e., for any $a \in R$, $\phi_e(a) = \lambda^*_e(a)$. In Theorem 2.15 we proved that R/λ^*_e is a ring with identity such that $1_{R/\lambda^*_e} = \lambda^*_e(e)$. Set now $D_e(R) := {\phi_e}^{-1}(1_{R/\lambda^*_e})$ and we obtain the following result.

Proposition 3.6. *For a non-empty subset M of a hyperring R we have $D_e(R)\cdot M \cup M \cdot D_e(R) \subseteq {\phi_e}^{-1}(\phi_e(M))$.*

Proof. Since $\lambda^*_e(e) = 1_{R/\lambda^*_e}$, for every $x \in D_e(R)\cdot M$ we have $\lambda^*_e(x) = \lambda^*_e(m)$ for some $m \in M$ and hence $x \in {\phi_e}^{-1}(\phi_e(M))$. Thus $D_e(R) \cdot M \subseteq {\phi_e}^{-1}(\phi_e(M))$. Similarly, the other part can be proved. □

Definition 3.7. Suppose that $(R,+,\cdot)$ is a hyperring and $e \in R$. We say that R is a λ_e-strong hyperring, whenever:

(i) for all $x, y \in R$ if $x\lambda_e^* y$, then $x \cdot e \cap y \cdot e \neq \emptyset$ and $e \cdot x \cap e \cdot y \neq \emptyset$.

(ii) {e} is an invertible set in the semi-hypergroup $(R, \cdot)$.

Theorem 3.8. *Suppose that R is a λ_e-strong hyperring.*

(i) *For a non-empty subset M of R we have:*

(i-1) $M \cdot D_e(R) = D_e(R) \cdot M = {\phi_e}^{-1}(\phi_e(M))$.

(i-2) *M is a λ_e-part if and only if* ${\phi_e}^{-1}(\phi_e(M)) = M$.

(ii) *The relation λ_e is transitive.*

Proof. (i-1) By Theorem 3.6 it is enough to prove that

$$ {\phi_e}^{-1}(\phi_e(M)) \subseteq M \cdot D_e(R) \cap D_e(R) \cdot M. $$

For every $x \in {\phi_e}^{-1}(\phi_e(M))$, an element $m \in M$ exists such that $\phi_e(x) = \phi_e(m)$. Since R is a λ_e-strong hyperring, it follows that $x \cdot e \cap m \cdot e \neq \emptyset$. So there exists $z \in x \cdot e \cap m \cdot e$. Since $\{e\}$ is invertible, we have $x \in z \cdot e$ and hence $x \in m \cdot e \cdot e$. Therefore $x \in M \cdot D_e(R)$, because $e \cdot e \subseteq D_e(R)$. Similarly we can prove that $\phi_e^{-1}(\phi_e(M)) \subseteq D_e(R) \cdot M$.

(i-2) Suppose that M is a λ_e-part and take an arbitrary element $x \in \phi_e^{-1}(\phi_e(M))$. Thus there exists $m \in M$ such that $\phi_e(x) = \phi_e(m)$ and hence $m\lambda_e^* x$, so by Theorem 3.2 we have $x \in M$. Therefore $\phi_e^{-1}(\phi_e(M)) \subseteq M$. It is obvious that $M \subseteq \phi_e^{-1}(\phi_e(M))$ and so $\phi_e^{-1}(\phi_e(M)) = M$.

For proving the sufficiency part, suppose that $m\lambda_e^* x$ and $m \in M$. Thus $\phi_e(x) = \phi_e(m) \in \phi_e(M)$ and so $x \in \phi_e^{-1}(\phi_e(M)) = M$. Therefore by Theorem 3.2 it follows that M is an λ_e-part of R.

(ii) By Theorem 3.5, it is enough to show that, for all $x \in R$, $P(x)$ is an λ_e-part of R. For this reason we prove that $\phi_e^{-1}(\phi_e(P(x))) = P(x)$ and by (i) we have $P(x)$ is an λ_e-part of R.

Now take an arbitrary $z \in \phi_e^{-1}(\phi_e(P(x)))$. So there exists $k \in P(x)$ such that $\phi_e(z) = \phi_e(k)$ and hence $\lambda_e^*(z) = \lambda_e^*(k)$. Since $k \in P(x)$, by Theorem 3.4(i) we have $x\lambda_e k$. Thus $\lambda_e^*(k) = \lambda_e^*(x)$ and so $\lambda_e^*(z) = \lambda_e^*(x)$. Since R is a λ_e-strong hyperring, it follows that $x \cdot e \cap z \cdot e \neq \emptyset$ and hence there exists $s \in x \cdot e \cap z \cdot e$. Therefore $x \in z \cdot e \cdot e$ and $z \in x \cdot e \cdot e$, because $\{e\}$ is an invertible set in R. Thus, $z \in z \cdot e \cdot e \cdot e \cdot e$. Since $(z \cdot e \cdot e, z \cdot e \cdot e \cdot e \cdot e) \in \Re^e$, it follows that $x\lambda_e z$ and hence $z \in P(x)$. So $\phi_e^{-1}(\phi_e(P(x))) \subseteq P(x)$, while the other inclusion $P(x) \subseteq \phi_e^{-1}(\phi_e(P(x)))$ is obvious. Therefore $\phi_e^{-1}(\phi_e(P(x))) = P(x)$ and hence the proof is complete. □

Example 3.9. *Let R be a ring with identity $e = 1_R$. In this case we have $R/\lambda_e^* = \{\{a\} \mid a \in R\} \cong R$ and $D_e(R) = \{e\}$. Thus R is a λ_e-strong hyperring and the conditions of theorem 3.8 valid.*

Example 3.10. *Let $(\mathbb{Z}_6, +_{\bar{3}}, \cdot_{\bar{3}})$ be the hyperring by the following hyperoperations, see [2],*

$+_{\bar{3}}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$
$\bar{1}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$
$\bar{2}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$
$\bar{3}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$
$\bar{4}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$
$\bar{5}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$

$\cdot_{\bar{3}}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$
$\bar{1}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{5},\bar{2}\}$
$\bar{2}$	$\{\bar{0},\bar{3}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{1},\bar{4}\}$
$\bar{3}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{0},\bar{3}\}$
$\bar{4}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{2},\bar{5}\}$
$\bar{5}$	$\{\bar{0},\bar{3}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{1},\bar{4}\}$	$\{\bar{0},\bar{3}\}$	$\{\bar{2},\bar{5}\}$	$\{\bar{1},\bar{4}\}$

Put $e := \bar{1}$. Thus $(\mathbb{Z}_6, +_{\bar{3}}, \cdot_{\bar{3}})$ is a λ_e-strong hyperring and $\lambda_e^*(\bar{0}) = \lambda_e^*(\bar{3})$, $\lambda_e^*(\bar{1}) = \lambda_e^*(\bar{4})$ and $\lambda_e^*(\bar{2}) = \lambda_e^*(\bar{5})$. So the ring $\mathbb{Z}_6/\lambda_e^*$ is isomorphic to the ring $(\mathbb{Z}_3, +, \cdot)$. Also the canonical projection $\phi_e : \mathbb{Z}_6 \longrightarrow \mathbb{Z}_6/\lambda_e^*$ is as follows, $\phi_e(\bar{0}) = \phi_e(\bar{3}) = \lambda_e^*(\bar{0})$, $\phi_e(\bar{1}) = \phi_e(\bar{4}) = \lambda_e^*(\bar{1})$ and $\phi_e(\bar{2}) = \phi_e(\bar{5}) = \lambda_e^*(\bar{2})$. This implies that $D_e(\mathbb{Z}_6) = {\phi_e}^{-1}(1_{\mathbb{Z}_6/\lambda_e^*}) = \{\bar{1},\bar{4}\}$, where $1_{\mathbb{Z}_6/\lambda_e^*} = \lambda_e^*(\bar{1})$. It is not difficult to see that for each subset M of $\mathbb{Z}_6$ we have $M \cdot D_e(\mathbb{Z}_6) = D_e(\mathbb{Z}_6) \cdot M = {\phi_e}^{-1}(\phi_e(M))$. For instance if $M = \{\bar{2},\bar{3}\}$, then $\{\bar{2},\bar{3}\} \cdot \{\bar{1},\bar{4}\} = \{\bar{2},\bar{5},\bar{0},\bar{3}\}$ and ${\phi_e}^{-1}(\phi_e(\{\bar{2},\bar{3}\})) = \{\bar{2},\bar{5},\bar{0},\bar{3}\}$.

4. λ_e-closed hyperideals in (Krasner) hyperrings

In this section first we define the notion of λ_e-closed hyperideal I in a hyperring R. By using it, we construct the quotient ring R/I in order to obtain a relationship between hyperideals in the Krasner hyperring R containing I and ideals in the ring R/I.

Let $(R, +, \cdot)$ be a hyperring and $\phi_e : R \longrightarrow R/\lambda_e^*$ be the canonical projection, introduced in the previous section. Set $K(R) = {\phi_e}^{-1}(0_{R/\lambda_e^*})$.

Proposition 4.1. *For a non-empty subset M of a hyperring R, we have $K(R) + M = M + K(R) = {\phi_e}^{-1}(\phi_e(M))$.*

Proof. It is easy to see that $K(R) + M \subseteq {\phi_e}^{-1}(\phi_e(M))$. Conversely, suppose that $x \in {\phi_e}^{-1}(\phi_e(M))$, so there exists $m \in M$ such that $\phi_e(x) = \phi_e(m)$ and hence $\lambda_e^*(x) = \lambda_e^*(m)$. Since $(R, +)$ is a hypergroup, there exists $k \in R$ such that $x \in k+m$ and so $\lambda_e^*(x) \in \lambda_e^*(k)+\lambda_e^*(m)$. Therefore $\lambda_e^*(x) = \lambda_e^*(k) \oplus \lambda_e^*(m)$.

Because $(R/\lambda_e^*, \oplus)$ is a group, we obtain $\lambda_e^*(k) = 0_{R/\lambda_e^*}$ and hence $k \in K(R)$. Thus ${\phi_e}^{-1}(\phi_e(M)) \subseteq K(R) + M$. Similarly, one proves that ${\phi_e}^{-1}(\phi_e(M)) = M + K(R)$. □

Corollary 4.2. *With the notation of Proposition 4.1, we have the following assertions.*

(i) *If R is a hyperring, then $K(R)$ is a hyperideal.*
(ii) *If R is a Krasner hyperring, then*
  (ii-1) $\lambda_e^*(0) = 0_{R/\lambda_e^*}$*;*
  (ii-2) *for all $x \in R$, $\lambda_e^*(-x) = -\lambda_e^*(x)$;*
  (ii-3) *$K(R)$ is a normal hyperideal.*

Proof. The proof of (i) follows from Theorem 4.1. Also part (ii) is straightforward. □

Definition 4.3. Suppose that I is a hyperideal in a hyperring $(R, +, \cdot)$. We say that I is a λ_e*-closed hyperideal*, whenever for every $a \in I$, $\lambda_e^*(a) \subseteq I$.

Remark 4.4. For a Krasner hyperring R, it is obvious that $K(R)$ is a λ_e-closed hyperideal of R.

Theorem 4.5. *Suppose that $(R, +, \cdot)$ is a hyperring.*

(i) *For every ideal $\mathcal{I}$ of R/λ_e^*, ${\phi_e}^{-1}(\mathcal{I})$ is a λ_e-closed hyperideal of R and $K(R) \subseteq {\phi_e}^{-1}(\mathcal{I})$.*

Suppose that I is a λ_e-closed hyperideal of R such that $(I, +)$ is a closed subhypergroup of $(R, +)$ and $K(R) \subseteq I$.

(ii) *$\phi_e(I)$ is an ideal of R/λ_e^*.*
(iii) *If $e \in I$, then $I = R$.*

Denote by $\mathcal{A}$ the set of all λ_e-closed hyperideals of R and by $\mathcal{B}$ the set of all ideals of R/λ_e^*.

(iv) *There is a one-to-one correspondence between $\mathcal{A}$ and $\mathcal{B}$.*
(v) *If $I \in \mathcal{A}$, then I is a maximal element in the partial order set $(\mathcal{A}, \subseteq)$, i.e., I is a maximal hyperideal in R if and only if $\phi_e(I)$ is a maximal ideal in the ring R/λ_e^*.*

Proof. (i) Let $x, y \in {\phi_e}^{-1}(\mathcal{I})$ be given. Then $\lambda_e^*(x) \oplus \lambda_e^*(y) \in \mathcal{I}$. Thus, by Theorem 2.15, it follows that $\lambda_e^*(z) \in \mathcal{I}$, for each $z \in x + y$ and hence $x + y \subseteq {\phi_e}^{-1}(\mathcal{I})$. Besides, there exists $t \in R$ such that $x \in t + y$ and so $\lambda_e^*(t) \in \mathcal{I}$. Therefore ${\phi_e}^{-1}(\mathcal{I}) + x = {\phi_e}^{-1}(\mathcal{I})$, hence $({\phi_e}^{-1}(\mathcal{I}), +)$ is a subhypergroup of $(R, +)$. Since ϕ_e is a surjective map, it follows that ${\phi_e}^{-1}(\mathcal{I})$ is closed under "·". These imply that, ${\phi_e}^{-1}(\mathcal{I})$ is a hyperideal of R and since $0_{R/\lambda_e^*} \in \mathcal{I}$, we conclude that $\lambda_e^*(x) \in \mathcal{I}$ for each $x \in K(R)$. Thus, $K(R) \subseteq {\phi_e}^{-1}(\mathcal{I})$. It is easy to see that ${\phi_e}^{-1}(\mathcal{I})$ is λ_e-closed.

(ii) Let $a, b \in I$ be given such that $\lambda_e^*(x) = \lambda_e^*(a) \oplus \lambda_e^*(b)$. So by Theorem 2.15 we write $\lambda_e^*(x) = \lambda_e^*(c)$, where $c \in a + b \subseteq I$. Thus, $x \in \lambda_e^*(c) \subseteq I$ and hence

$\phi_e(I)$ is closed under "$\oplus$". Because R/λ_e^* is a ring, there exists an element $a' \in R$ such that $\lambda_e^*(a) \oplus \lambda_e^*(a') = 0_{R/\lambda_e^*}$. Therefore $a + a' \subseteq K(R) \subseteq I$ and hence $a' \in I$, because $(I, +)$ is a closed subhypergroup.

(iii) Since $\lambda_e^*(e)$ is the identity of the ring R/λ_e^*, from (ii) we have $\phi_e(I) = R/\lambda_e^*$. Let $r \in R$ be given. Then $\lambda_e^*(r) \in \phi_e(I)$. Thus there exists $a \in I$ such that $\lambda_e^*(r) = \lambda_e^*(a) \subseteq I$. Thus, $r \in \lambda_e^*(r) \subseteq I$.

(iv) Define $\Gamma : \mathcal{A} \longrightarrow \mathcal{B}$ by $\Gamma(I) = \phi_e(I)$ and $\Lambda : \mathcal{B} \longrightarrow \mathcal{A}$ by $\Lambda(\mathcal{I}) = {\phi_e}^{-1}(\mathcal{I})$ for each $I \in \mathcal{A}$ and $\mathcal{I} \in \mathcal{B}$. By parts (i) and (ii), we have that Γ and Λ are mappings and since ϕ_e is surjective, $\Gamma \circ \Lambda = Id_{\mathcal{B}}$. Now let $I \in \mathcal{A}$ and $r \in {\phi_e}^{-1}(\phi_e(I))$.Thus, there exists $a \in I$ such that $\lambda_e^*(r) = \lambda_e^*(a) \subseteq I$ and hence $r \in I$. Therefore $\Lambda \circ \Gamma = Id_{\mathcal{A}}$.

(v) The proof follows from (iv). $\square$

Proposition 4.6. *Suppose that I is a λ_e-closed hyperideal in a Krasner hyperring $(R, +, \cdot)$. Then we have:*

(i) $K(R) \subseteq I$.

(ii) $(I, +)$ *is a complete subhypergroup of* $(H, +)$.

Proof. (i) Let $a \in K(R)$ be an arbitrary element. By Theorem 4.2(i), it follows that $\lambda_e^*(a) = \lambda_e^*(0)$. Since $0 \in I$, there is $\lambda_e^*(0) \subseteq I$ and hence $a \in \lambda_e^*(a) \subseteq I$.

(ii) Let $(a_1, \ldots, a_n) \in R^n$ such that $\sum\limits_{i=1}^{n} a_i \cap I \neq \emptyset$. Thus, there exists an element $a \in I$ such that $a \in \sum\limits_{i=1}^{n} a_i$ and hence by Theorem 2.15 we have $\lambda_e^*(a) = \bigoplus\limits_{i=1}^{n} \lambda_e^*(a_i)$. Therefore $\bigoplus\limits_{i=1}^{n} \lambda_e^*(a_i) \subseteq I$. Since for each $b \in \sum\limits_{i=1}^{n} a_i$, we write $\lambda_e^*(b) = \bigoplus\limits_{i=1}^{n} \lambda_e^*(a_i)$, this implies that $b \in \lambda_e^*(b) \subseteq I$. Thus, $\sum\limits_{i=1}^{n} a_i \subseteq I$. $\square$

Let $(R, +, \cdot)$ be a Krasner hyperring and I be a λ_e-closed hyperideal of R. For each $x, y \in R$ define the following relation:

$$x \overset{I}{\equiv}_e y \Longleftrightarrow \exists a \in I, x - y \subseteq \lambda_e^*(a).$$

It is easy to see that $\overset{I}{\equiv}_e$ is an equivalence relation on R. For each $x \in R$ the equivalence class of x is denoted by $[x_I]_e$ and the set of all equivalence classes is denoted by R/I.

Remark 4.7. Let $(R, +, \cdot)$ be a Krasner hyperring and I be a λ_e-closed hyperideal of R.

(i) For each $x \in R$, $\lambda_e^*(x) \subseteq [x_I]_e$. Indeed, if $y \in \lambda_e^*(x)$, then $\lambda_e^*(x) - \lambda_e^*(y) = 0_{R/\lambda_e^*}$. By Theorem 4.2, $\lambda_e^*(0) = 0_{R/\lambda_e^*}$ and so $x - y \subseteq \lambda_e^*(0)$. Thus, $x \overset{I}{\equiv}_e y$ and hence $y \in [x_I]_e$.

(ii) If $I \subseteq K(R)$, then for each $x \in R$, $\lambda_e^*(x) = [x_I]_e$. Thus, $\lambda_e^*(x) \oplus \lambda_e^*(-y) = 0_{R/\lambda_e^*}$ and hence by Theorem 4.2(ii) $\lambda_e^*(x) = -\lambda_e^*(-y) = \lambda_e^*(y)$. Therefore $y \in \lambda_e^*(x)$.

(iii) The relation $\stackrel{I}{\equiv}_e$ is a strongly regular equivalence relation. Indeed, let $x \stackrel{I}{\equiv}_e y$ so there exists $a \in I$ such that $x - y \subseteq \lambda_e^*(a)$. For each $z \in R$ since $z-z \subseteq \lambda_e^*(0)$ and $\lambda_e^*(a)\oplus\lambda_e^*(0) = \lambda_e^*(0)$, it follows that $(x-y)+(z-z) \subseteq \lambda_e^*(a)+\lambda_e^*(0) \subseteq \lambda_e^*(a)$. This implies that for each element $s \in x - z$ and $t \in y - z$, $s \stackrel{I}{\equiv}_e t$.

(iv) It is easy to see that for each $x, y, z, t \in R$ if $z, t \in [x_I]_e + [y_I]_e$, then $[z_I]_e = [t_I]_e$

Based on the previous comments, we have the following result.

Proposition 4.8. *Suppose that I is a λ_e-closed hyperideal in the Krasner hyperring $(R, +, \cdot)$. Then $(R/I, \boxplus, \boxdot)$ is a ring, where*

$$[x_I]_e \boxplus [y_I]_e = [z_I]_e, \quad z \in [x_I]_e + [y_I]_e;$$

$$[x_I]_e \boxdot [y_I]_e = [z_I]_e\ , \quad z \in [x_I]_e \cdot [y_I]_e.$$

Besides $[e_I]_e$ is the identity element of the semigroup $(R/I, \boxdot)$.

Proof. It is not difficult to show that $(R/I, \boxplus, \boxdot)$ is a ring. Note that $[0_I]_e$ is the identity of the group $(R/I, \boxplus)$ and for each $[x_I]_e \in R/I$, $[(-x)_I]_e$ is the inverse of $[x_I]_e$ under the operation $\boxplus$. Now, for each $x \in R$ we have $\lambda_e^*(e)$ is the identity of the ring $(R/\lambda_e^*, \oplus, \odot)$, therefore $\lambda_e^*(x) \odot \lambda_e^*(e) = \lambda_e^*(x) = \lambda_e^*(e) \odot \lambda_e^*(x)$. It follows that $x \cdot e - x \subseteq \lambda_e^*(0)$ and hence $x \cdot e \stackrel{I}{\equiv}_e x$. This implies that $[e_I]_e$ is the identity element of the semigroup $(R/I, \boxdot)$. $\square$

Theorem 4.9. *Suppose that I is a λ_e-closed hyperideal in a Krasner hyperring R.*

(i) *If J is a hyperideal in R containing I, then the set $\mathcal{J} = \{[a_I]_e \mid a \in J\}$ is an ideal in the quotient ring R/I.*
(ii) *If $\mathcal{J}$ is an ideal in the quotient ring R/I, then the set $J = \{a \in R \mid [a_I]_e \in \mathcal{J}\}$ is a hyperideal in R.*
(iii) *There is a one-to-one correspondence between ideals in the quotient ring R/I and hyperideals of R containing I.*

Proof. (i) By Theorem 4.6, there is $K(R) \subseteq I$ and $(I, +)$ is a closed subhypergroup of $(R, +)$ and thus $K(R)$ is closed, too. Thus, by Theorem 4.5, $\phi_e(I)$ is an ideal in the ring R/λ_e^*. Now let $a, b \in J$ and $x \in [a_I]_e + [b_I]_e$ be arbitrary. So there exist $u \in [a_I]_e$ and $v \in [b_I]_e$ such that $x \in u + v$. Therefore $u - a \subseteq \lambda_e^*(a')$ and $v - b \subseteq \lambda_e^*(b')$ for some $a', b' \in I$. This implies that $(u + v) - (a + b) \subseteq \lambda_e^*(a') + \lambda_e^*(b')$. Since $\phi_e(I)$ is an ideal in R/λ_e^*, it follows that $\lambda_e^*(a') \oplus \lambda_e^*(b') \in \phi_e(I)$ and hence there exists $c \in I$ such that $\lambda_e^*(a')+\lambda_e^*(b') \subseteq \lambda_e^*(c)$. Thus, for each $d \in a+b \subseteq J$, $x-d \subseteq \lambda_e^*(c)$. Therefore $x \stackrel{I}{\equiv}_e d$ and hence $[a_I]_e \boxplus [b_I]_e = [d_I]_e \in \mathcal{J}$.

(ii) The proof is straightforward.

(iii) The proof follows directly from (i) and (ii). $\square$

Corollary 4.10. *Suppose that I is a λ_e-closed hyperideal in a Krasner hyperring R. Then I is a maximal hyperideal in R if and only if the derived quotient ring $(R/I, \boxplus, \boxdot)$ is a field.*

Proof. Let I be a maximal hyperideal in R. By Theorem 4.5(v), $\phi_e(I)$ is a maximal ideal in the ring $(R/\lambda_e^*, \oplus, \odot)$. Thus, if $[0_I]_e \neq [x_I]_e \in R/I$, then $\lambda_e^*(x) \notin \phi_e(I)$ and hence there exists $y \in R$ such that $(\lambda_e^*(x) \oplus \phi_e(I))(\lambda_e^*(y) \oplus \phi_e(I)) = \lambda_e^*(e) \oplus \phi_e(I)$. This implies that there exists $a \in I$ such that $xy - e \subseteq \lambda_e^*(a)$ and so $[x_I]_e \boxdot [y_I]_e = [e_I]_e$. Therefore $[x_I]_e$ has an inverse in the ring R/I, meaning that $(R/I, \boxplus, \boxdot)$ is a field. Conversely, let J be an hyperideal in R containing I. By Theorem 4.9, the set $\mathcal{J} = \{[a_I]_e \in R/I \mid a \in J\}$ is an ideal in R/I and so $\mathcal{J} = R/I$ or $\mathcal{J} = \{[0_I]_e\}$. If $\mathcal{J} = R/I$, then $[b_I]_e = [e_I]_e$ for some $b \in J$. Thus, there exists $a \in I$ such that $e - b \subseteq \lambda_e^*(a) \subseteq I$ and so $e \in c + b$ for some $c \in I$. Therefore $e \in J$ and hence by Theorem 4.5(iii), we have $J = R$. Now if $\mathcal{J} = \{[0_I]_e\}$, then for each $x \in J$ there exists $a \in I$ such that $x \in \lambda_e^*(a) \subseteq I$. Thus, $J = I$. These results imply that I is a maximal hyperideal in R. □

5. Conclusions

The study conducted in this research note falls in the area of derived classical structures obtained from hyperstructures having similar properties. In particular, by defining some new strongly regular relations on a general hyperring, namely λ_e^* and Λ_e^* that represent the transitive closures of λ_e and Λ_e, we prove that the corresponding quotient structure is a unitary and unitary commutative ring, respectively. The relation λ_e is not transitive in general, thus here, by using some generalization of the complete parts, we have stated conditions under which λ_e becomes transitive. Finally, we have defined the λ_e-closed hyperideals in hyperrings, proving that they have an important role in Krasner hyperrings. Indeed, if I is a λ_e-closed hyperideal in a Krasner hyperring R, then there exists a one-to-one correspondence between the ideals of the quotient ring R/I and the hyperideals of R containing I. Moreover, such a hyperideal I is maximal in R if and only if the quotient ring R/I is a field.

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