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RÉNYI ENTROPIES OF DYNAMICAL SYSTEMS: A GENERALIZATION APPROACH

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Abstract. Entropy measures have received considerable attention in quantifying the structural complexity of real-world systems and are also used as measures of information obtained from a realization of the considered experiments. In the present study, new notions of entropy for a dynamical system are introduced. The Rényi entropy of measurable partitions of order $q \in (0, 1) \cup (1, \infty)$ and its conditional version are defined, and some important properties of these concepts are studied. It is shown that the Shannon entropy and its conditional version for measurable partitions can be obtained as the limit of their Rényi entropy and conditional Rényi entropy. In addition, using the suggested concept of Rényi entropy for measurable partitions, the Rényi entropy for dynamical systems is introduced. It is also proved that the Rényi entropy for dynamical systems is invariant under isomorphism.

Keywords: Measurable partition, Rényi entropy, Conditional Rényi entropy, Dynamical system.
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1. Introduction

In 1865, the concept of entropy was initially introduced by Clausius in the context of equilibrium thermodynamics. A few years later, an important paper was published by Boltzmann, presenting two famous results, which are currently known as the Boltzmann equation and the Boltzmann H-theorem [7]. The Boltzmann H-theorem had the statistical interpretation of the thermodynamic entropy, which is described by the formula:

$$S = k\, logW,$$

where k is a constant and W is the number of possible microstates corresponding to the macroscopic state of a given thermodynamic system [8].
In 1948, Shannon extended the notion of entropy to the information theory [41] and expressed it by the formula:

$$H\,(X) \;=\; -\sum_{i=1}^{n} p_i \log_2 p_i,$$

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in which X is a discrete random variable and p_i is the probability of the event $X = \{x_i\}$. Shannon entropy mathematically quantified the statistical nature of lost information in phone-line signals. From a formal point of view, the Boltzmann entropy was introduced for the continuous cases, but the Shannon entropy was defined for the discrete cases.

Inspired by Shannon entropy, Kolmogorov and Sinai offered the concept of entropy to the ergodic theory concerning the problem of isomorphisms of dynamical systems [23,43]. They proved the existence of non-isomorphic Bernoulli shifts and showed that the two isomorphisms dynamical systems had the same Kolmogorov-Sinai entropy. Later on, Adler et al. introduced the concept of topological entropy as an invariant of topological conjugate relation for continuous maps [1]. This concept classifies the dynamical systems according to conjugate relation and measures the complex behavior of the orbits in a dynamical system as well. The concept of a fuzzy dynamical system and its entropy was introduced by Markechová [27]. The main idea of the fuzzy entropy is that the partitions are replaced by fuzzy partitions. Some researchers have defined fuzzy entropy considering algebraic structures like MV-algebra and effect algebra as a probability space. The suitable types of entropy theories of Shannon and Kolmogorov-Sinai for the case of a product MV-algebra, MV-algebras, and hyper MV-algebras were defined in [28, 29, 34, 38], respectively. A comprehensive exploration of the algebraic Shannon entropies for hypergroupoids and commutative hypergroups was conducted by Mehrpooya et al. in [30]. Di Nola developed research on the notion of entropy on effect algebra in [13], and then it was extended by Eslami Giski and et al. [16–18].

Gradually, the notion of entropy was applied to other fields of science such as physics, biology, economy, and pattern recognition. Corda et al. studied the entropy of iterated function systems and explored their relations with Black Holes and Bohr-Like Black Holes entropy [11]. In the literature, there exist numerous papers about the use of entropy in Urban systems. We can see a comprehensive investigation of entropy and its application to Urban systems in [35]. Flexibility is of strategic importance for many organizations in order to survive in highly competitive and dynamic markets. Shuiabi et al. applied entropy as a measure of the flexibility of production operations [42]. In the information field, entropy represents the loss of information of a physical system observed by an outsider [41]. In Biology, statistical entropy has been used in some new approaches for describing Biosystems [5]. Also, Schug et al. have introduced a definition of tissue specificity based on Shannon entropy to rank human genes according to their overall tissue specificity and by their specificity to particular tissues [40]. In finance, Shannon entropy was used as a measure of risk, capturing risk without using any information about the market. It is capable of measuring the risk reduction effect of diversification [31, 47]. Camesasca et al. have introduced a methodology to quantify the quality of mixing in polymeric by adapting the Shannon information entropy [10]. Agop et al. have shown that Onicescu's informational energy can be correlated by means of

Shannon's maximum informational entropy variational principle with $SL(2,R)$ invariance of the Kepler type motions [3].
The theory of Rényi entropy, which is a generalization of the Shannon entropy was first offered by Rényi in 1961 [37]. Rényi entropy was defined by the formula:

$$H_q(X) = \frac{1}{1-q}\log_2 \sum_{i=1}^{n} (P_i)^q ,$$

where X is a discrete random variable, p_i is the probability of the event $X = \{x_i\}$, and $q \in (0,1) \cup (1,\infty)$.
The main reason for Rényi to define his new entropy was to use it in an information-theoretic proof of the central limit theorem. Rényi entropy was more flexible than Shannon entropy and has been extensively studied in the following decades [2, 19–21, 36]. This notion was not just a mere mathematical generalization, and there is comprehensive literature on its applications in many fields. In the information theory, it has for long been known that the Rényi entropy is related to so-called cut-off rates [12]. A new thresholding technique in pattern recognition based on two-dimensional Rényi entropy was offered in [39]. One of the most important properties of a cryptographic system is proof of its security. The Rényi entropy has also presented a method used to prove the security of unconditionally secure cryptosystems [9,44]. The problem of bounding the expected number of guesses in terms of Rényi entropies was investigated by Arikan in the context of sequential decoding [4]. Peccarelli and Ebrahimi discused Rényi entropy measures from a statistical perspective and explored an algorithm for variable selection can be applicable across many different disciplines, especially machine learning [33]. Li et al. in [26] have used the evolution of the Rényi entropy to deal with chaos. They proved that Rényi entropy decreases linearly in the regular case and exponentially in the chaotic case. Based on Rényi entropy, Lenzi et al. have shown that it is possible to obtain generalized statistical mechanics, which can be maximized with adequate constraints [25]. Entropy was used for studing the black holes. Dong [14] has provided the first holographic calculation of mutual Rényi information between two disks of arbitrary dimension
In the original article, Shannon also defined the notion of conditional entropy, and there was a relationship between this notion and the join Shannon entropy of random variables. However, a similar and generalized concept was not presented in the paper of Rényi. Later some scholars introduced different definitions of conditional Rényi entropy, and there is as of now no commonly accepted definition for the conditional Rényi entropy [2, 22, 36].

The application of the conditional Rényi entropy can be found in many fields [6, 15, 24, 32, 45].
This paper hopes to shed some light on Rényi entropies by defining them on classical dynamical systems. In the present paper, we first briefly review the definitions and facts in Section 2 that will especially serve as a useful guideline

for our extension to the Rényi entropy of dynamical systems. Also, Section 2 briefly reviews some basic properties of the Rényi entropy of measurable partitions with a special focus on refinement of a measurable partition, interior subset, and common refinement of two measurable partitions. Next, in Section 3, we introduce conditional Rényi entropy and examine the behavior of this new notion, and also we investigate its correlation with Rényi entropy of join refinement of two partitions. Finally, in Section 4, we introduce the Rényi entropy of dynamical systems, and we prove that this new entropy is invariant under isomorphism.

2. Rényi Entropy of Measurable partitions

This section starts with introducing Rényi entropy on measurable partitions of measurable space $(\Omega,\ S)$, then several basic properties of this measure are given. In particular, it is proved that the Rényi entropy is an extension of the Shannon entropy.

Definition 2.1. Let $A = \{A_1, ..., A_n\}$ be a measurable partition of a measurable space (Ω, S). The Rényi entropy of A of order $q \in (0, 1) \cup (1, \infty)$ is defined as the number:

$$H_q(A) := \frac{1}{1-q}\log_2 \sum_{i=1}^{n} (\mu(A_i))^q.$$

In the rest of this paper we use "log" instead of "$\log_2$ ".
Using the above notion, we have the following result.

Lemma 2.2. *If $A = \{A_1, ..., A_n\}$ is a measurable partition of a measurable space (Ω, S), then*

$$0 \le H_q(A) \le \log n.$$

Proof. Consider the uniform distribution $p_i = \mu(A_i) = \frac{1}{n}$, $i = 1, 2, ..., n$, over $A = \{A_1, ..., A_n\}$. Rényi entropy $H_q(A)$ has its maximum value, and we have $H_q(A) = \frac{1}{1-q}\log \sum_{i=1}^{n} \left(\frac{1}{n}\right)^q = \log n$. On the other hand, we know $0 \le \mu(A_i) \le 1$ and for $1 < q$, we get $\sum_{i=1}^{n} \mu(A_i)^q \le \sum_{i=1}^{n} \mu(A_i) = \mu\left(\bigcup_{i=1}^{n} A_i\right) = \mu(\Omega) = 1.$

But this means that
$\log \sum_{i=1}^{n} (\mu(A_i))^q \le 0$, i.e., $H_q(A) = \frac{1}{1-q}\log \sum_{i=1}^{n} (\mu(A_i))^q \ge 0.$
Furthermore if $0 < q < 1$. Then

$$\sum_{i=1}^{n} \mu(A_i)^q \ge \sum_{i=1}^{n} \mu(A_i) = \mu\left(\bigcup_{i=1}^{n} A_i\right) = \mu(\Omega) = 1$$

and $H_q(A) = \frac{1}{1-q}\log \sum_{i=1}^{n} (\mu(A_i))^q \ge 0.$

□

In the following proposition, it is proved that the Rényi entropy is an extension of the Shannon entropy.

Proposition 2.3. *Suppose that* $A = \{A_1, ..., A_n\}$ *is a measurable partition of a measurable space* (Ω, S)*. Then*

$$H_1(A) := \lim_{q \to 1} H_q(A).$$

Proof. Let $f(q) = \log \sum\limits_{i=1}^{n} \mu(A_i)^q$ and $g(q) = 1 - q$ be two differentiable functions. We have $\lim\limits_{q \to 1} f(q) = \lim\limits_{q \to 1} g(q) = 0$. Using L'Hospital's rule, we obtain

$$\lim_{q \to 1} H_q(A) = \lim_{q \to 1} \frac{1}{1 - q} \log \sum_{i=1}^{n} \mu(A_i)^q = \lim_{q \to 1} \frac{-\sum\limits_{i=1}^{n} \mu(A_i)^q \ln \mu(A_i)}{\sum\limits_{i=1}^{n} \mu(A_i)^q \ln 2}$$

$$= \frac{-\sum\limits_{i=1}^{n} \mu(A_i) \ln \mu(A_i)}{\sum\limits_{i=1}^{n} \mu(A_i) \ln 2} = -\sum_{i=1}^{n} \mu(A_i) \log_2 \mu(A_i) = H_1(A).$$

□

Proposition 2.4. *Let* $A = \{A_1, ..., A_n\}$ *and* $B = \{B_1, ..., B_m\}$ *be measurable partitions of a measurable space* (Ω, S)*. Then*
(*i*) $A \prec B$ *implies* $H_q(A) \leq H_q(B)$;
(*ii*) $\max(H_q(A);\ H_q(B)) \leq H_q(A \vee B)$.

Proof. (*i*) if $A \prec B$ then for every $A_i \in A$ there exists a subset $\alpha_i \subset \{1, ..., m\}$ such that $A_i = \cup_{j \in \alpha_i} B_j$, $\alpha_i \cap \alpha_j = \emptyset$ for $i \neq j$, and $\cup_{i=1}^{n} \alpha_i = \{1, ..., m\}$. If $0 < q < 1$, then: $\sum\limits_{i=1}^{n} \mu(A_i)^q = \sum\limits_{i=1}^{n} \mu\left(\bigcup\limits_{j \in \alpha_i} B_j\right)^q = \sum\limits_{i=1}^{n} \left(\sum\limits_{j \in \alpha_i} \mu(B_j)\right)^q \leq \sum\limits_{i=1}^{n} \sum\limits_{j \in \alpha_i} \mu(B_j)^q = \sum\limits_{j=1}^{m} \mu(B_j)^q$.
Thus, we get

$$H_q(A) = \frac{1}{1 - q} \log \sum_{i=1}^{n} (\mu(A_i))^q \leq \frac{1}{1 - q} \log \sum_{j=1}^{m} \mu(B_j)^q = H_q(B).$$

Also if $1 < q$, then $\sum\limits_{i=1}^{n} \mu(A_i)^q = \sum\limits_{i=1}^{n} \mu\left(\bigcup\limits_{j \in I_i} B_j\right)^q = \sum\limits_{i=1}^{n} \left(\sum\limits_{j \in I_i} \mu(B_j)\right)^q \geq \sum\limits_{i=1}^{n} \sum\limits_{j \in I_i} \mu(B_j)^q = \sum\limits_{j=1}^{m} \mu(B_j)^q$, and this implies $H_q(A) \leq H_q(B)$.
(*ii*) Considering the first part of this proposition and the facts that $A \prec A \vee B$ and $B \prec A \vee B$, the proof is evident.

□

Proposition 2.5. *Let $A = \{A_1, ..., A_n\}$ and $B = \{B_1, ..., B_m\}$ be measurable partitions of a measurable space (Ω, S). Then*
(i) if $A \overset{\circ}{\subseteq} B$ and $0 < q < 1$, then $H_q(A) \leq H_q(B)$;
(ii) if $A \overset{\circ}{\subseteq} B$ and $1 < q$, then $H_q(A) \geq H_q(B)$.

Proof. (i) if $A \overset{\circ}{\subseteq} B$ then for every $A_i \in A$ there exists $B_j \in B$ such that $\mu((A_i - B_j) \cup (B_j - A_i)) = 0$. That means $\mu(A_i - B_j) + \mu(B_j - A_i) = 0$ and $\mu(A_i) = \mu(A_i - B_j) + \mu(B_j \cap A_i) = \mu(B_j \cap A_i)$.
In the same way, we get $\mu(B_j) = \mu(B_j \cap A_i)$.

Without loss of generality, we sort measurable partition B in a way that for $i = 1, 2, ..., n$, B_i is a member such that $\mu(A_i \,\Delta\, B_i) = 0$. Hence,

$$H_q(A) = \frac{1}{1-q} \log \sum_{i=1}^{n} (\mu(A_i))^q = \frac{1}{1-q} \log \sum_{i=1}^{n} \mu(A_i \cap B_i)^q$$

$$\leq \frac{1}{1-q} \log \sum_{i=1}^{n} \mu(A_i \cap B_i)^q + \frac{1}{1-q} \log \sum_{i=n+1}^{m} \mu(B_i)^q$$

$$= \frac{1}{1-q} \log \sum_{i=1}^{n} \mu(B_i)^q + \frac{1}{1-q} \log \sum_{i=n+1}^{m} \mu(B_i)^q = H_q(B).$$

(ii) The proof is similar to the proof of part "(i)". □

We have an immediate corollary of the two propositions.

Corollary 2.6. *If A is a measurable partition of a measurable space (Ω, S) and $0 < q < 1$, then $H_q(A \vee A) = H_q(A)$.*

Proposition 2.7. *If measurable partitions $A = \{A_1, ..., A_n\}$ and $B = \{B_1, ..., B_m\}$ are independent, then $H_q(A \vee B) = H_q(A) + H_q(B)$.*

Proof. $H_q(A \vee B) = \frac{1}{1-q} \log \sum_{i=1}^{n} \sum_{j=1}^{m} (\mu(A_i \cap B_j))^q =$
$\frac{1}{1-q} \log \sum_{i=1}^{n} \sum_{j=1}^{m} (\mu(A_i)\, \mu(B_j))^q = \frac{1}{1-q} \log \sum_{i=1}^{n} (\mu(A_i))^q \sum_{j=1}^{m} (\mu(B_j))^q =$
$H_q(A) + H_q(B)$.

□

Proposition 2.8. *If a partition $A = \{A_1, ..., A_n\}$ is a measurable partition of a measurable space (Ω, S) and $0 < q_1 < q_2$, then $H_{q_2}(A) \leq H_{q_1}(A)$.*

Proof. Let $1 < q_1 < q_2$ and $\lambda(x) = x^{\frac{q_1 - 1}{q_2 - 1}}$ for $x \in [0, \infty)$. The assumption $q_2 \geq q_1$ implies that $\frac{q_1 - 1}{q_2 - 1} \leq 1$, so the function λ is concave function. Putting

$a_i = \mu(A_i)$, $x_i = (\mu(A_i))^{q_2 - 1}$, $i = 1, 2, ..., n$. Hence we get:

$$\left(\sum_{i=1}^{n} (\mu(A_i))^{q_2}\right)^{\frac{1}{q_2 - 1}} = \left(\sum_{i=1}^{n} \mu(A_i)(\mu(A_i))^{q_2 - 1}\right)^{\frac{q_1 - 1}{(q_2 - 1)(q_1 - 1)}}$$

$$= \left(\left[\sum_{i=1}^{n} \mu(A_i)(\mu(A_i))^{q_2 - 1}\right]^{\frac{q_1 - 1}{q_2 - 1}}\right)^{\frac{1}{q_1 - 1}} \geq$$

$$\left(\sum_{i=1}^{n} \mu(A_i)(\mu(A_i))^{q_1 - 1}\right)^{\frac{1}{q_1 - 1}} = \left(\sum_{i=1}^{n} (\mu(A_i))^{q_1}\right)^{\frac{1}{q_1 - 1}}.$$

The case of $q_1, q_2 \in (0, 1)$ is obtained by similar arguments. Finally, the case where $q_2 \in (1, \infty)$ and $q_1 \in (0, 1)$ is followed by transitivity. □

Example 2.9. *Suppose that* $\Omega = [0, 1)$.
$A = \left\{\left[0, \frac{1}{5}\right), \left[\frac{1}{5}, \frac{2}{5}\right), \left[\frac{2}{5}, \frac{3}{5}\right), \left[\frac{3}{5}, \frac{4}{5}\right), \left[\frac{4}{5}, 1\right)\right\}$ *and*
$B = \left\{\left[0, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{4}{5}\right) \left[\frac{4}{5}, 1\right)\right\}$ *are measurable partitions of* Ω. *We have*

$$A \vee B = \left\{\left[0, \frac{1}{5}\right), \left[\frac{1}{5}, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{2}{5}\right), \left[\frac{2}{5}, \frac{2}{4}\right), \left[\frac{2}{4}, \frac{3}{5}\right), \left[\frac{3}{5}, \frac{3}{4}\right), \left[\frac{3}{4}, \frac{4}{5}\right), \left[\frac{4}{5}, 1\right)\right\},$$

$$H_q(A) = \frac{1}{1 - q} \log \sum_{i=1}^{5} \left(\frac{1}{5}\right)^q = \frac{1}{1 - q} \log\left(5^{1 - q}\right) = \log 5,$$

and

$$H_q(B) = \frac{1}{1 - q} \log \sum_{i=1}^{4} \left(\frac{1}{4}\right)^q = \frac{1}{1 - q} \log\left(4^{1 - q}\right) = \log 4,$$

also

$$H_q(A \vee B) = \frac{1}{1 - q} \log\left(2\left(\frac{1}{20}\right)^q + 2\left(\frac{2}{20}\right)^q + 2\left(\frac{3}{20}\right)^q + 2\left(\frac{4}{20}\right)^q\right)$$
$$= \frac{1}{1 - q} \log\left(2\left(\frac{1}{20}\right)^q (1 + 2^q + 3^q + 4^q)\right).$$

If $q = 2, 3$, *then*

$$H_2(A \vee B) = \frac{1}{1 - 2} \log\left(2\left(\frac{1}{20}\right)^2 \left(1 + 2^2 + 3^2 + 4^2\right)\right) = \log\left(\frac{20}{3}\right)$$

$$H_3(A \vee B) = \frac{1}{1 - 3} \log\left(2\left(\frac{1}{20}\right)^3 \left(1 + 2^3 + 3^3 + 4^3\right)\right) = \frac{1}{2} \log(40),$$

and we have

$$H_2(A \vee B) = \log\left(\frac{20}{3}\right) \geq \frac{1}{2} \log(40) = H_3(A \vee B).$$

Let $q = \frac{1}{2}, \frac{1}{3}$. *We have*

$H_{\frac{1}{2}}(A \vee B) = \frac{1}{1-\frac{1}{2}} \log\left(2\left(\frac{1}{20}\right)^{\frac{1}{2}}\left(1 + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + 4^{\frac{1}{2}}\right)\right) =$
$2\log\left(3 + \sqrt{2} + \sqrt{3}\right) - \log 5,$
$H_{\frac{1}{3}}(A \vee B) = \frac{1}{1-\frac{1}{3}} \log\left(2\left(\frac{1}{20}\right)^{\frac{1}{3}}\left(1 + 2^{\frac{1}{3}} + 3^{\frac{1}{3}} + 4^{\frac{1}{3}}\right)\right) =$
$\frac{3}{2}\log\left(2\left(1 + 2^{\frac{1}{3}} + 3^{\frac{1}{3}} + 4^{\frac{1}{3}}\right)\right) - \frac{1}{2}\log(20),$

and we get

$$H_{\frac{1}{2}}(A \vee B) \leq H_{\frac{1}{3}}(A \vee B).$$

Also

$$H_2(A \vee B) = \log(20) - \log(3) \geq H_2(A) = log 5,$$
$$H_2(A \vee B) = \log(20) - \log(3) \geq H_2(B) = log 4.$$

3. Conditional Rényi Entropy

This section aims to propose a definition of the conditional Rényi entropy on the measurable space (Ω, S). For this purpose, we consider $\left\|\mu_{A|B=B_j}\right\|_q = \left(\sum_{i=1}^{n} \mu(A_i \mid B_j)^q\right)^{\frac{1}{q}}$ and $\mu(A \mid B) = \frac{\mu(A \cap B)}{\mu(B)}$ for measurable partitions $A = \{A_1, ..., A_n\}$ and $B = \{B_1, ..., B_m\}$.

Definition 3.1. Let $A = \{A_1, ..., A_n\}$ and $B = \{B_1, ..., B_m\}$ be measurable partitions of a measurable space Ω. Then, the conditional Rényi entropy H_q of A given B of order $q \in (0, 1) \cup (1, \infty)$ is defined as:

$$H_q(A \mid B) := \frac{q}{1-q} \log\left(\sum_{j=1}^{m} \mu(B_j)\left\|\mu_{A|B=B_j}\right\|_q\right).$$

We have $\sum_{i=1}^{n} \mu(A_i \cap B_j) = \mu\left(\bigcup_{i=1}^{n}(A_i \cap B_j)\right) = \mu\left(\left(\bigcup_{i=1}^{n} A_i\right) \cap B_j\right) = \mu(B_j)$
For $1 < q$

$$\sum_{j=1}^{m} \mu(B_j)\left\|\mu_{A|B=B_j}\right\|_q = \sum_{j=1}^{m} \mu(B_j)\left(\sum_{i=1}^{n} \mu_{A|B=B_j}{}^q\right)^{\frac{1}{q}}$$
$$\leq \sum_{j=1}^{m} \mu(B_j)\left(\sum_{i=1}^{n} \mu(A_i \mid B_j)\right) = \sum_{j=1}^{m} \mu(B_j) = 1$$

and for $q < 1$

$$\sum_{j=1}^{m} \mu(B_j)\left\|\mu_{A|B=B_j}\right\|_q = \sum_{j=1}^{m} \mu(B_j)\left(\sum_{i=1}^{n} \mu_{A|B=B_j}{}^q\right)^{\frac{1}{q}}$$

$$\leq \sum_{j=1}^{m} \mu(B_j) \left(\sum_{i=1}^{n} \mu\left(A_i \mid B_j\right) \right) = 1.$$

Therefore, $H_q\left(A \mid B\right)$ is a positive measure.

Proposition 3.2. *Let* $A = \{A_1, ..., A_n\}$, $B = \{B_1, ..., B_m\}$ *and* $C = \{C_1, ..., C_l\}$ *be measurable partitions of* Ω. *Then*
(*i*) $H_q\left(A \mid B\right) = q\, H_q\left(A \vee B\right)$
(*ii*) *if* $A \overset{\circ}{\subseteq} B$ *and* $0 < q < 1$, *then* $H_q(A \mid C) \leq H_q(B \mid C)$;
(*iii*) *if* $A \overset{\circ}{\subseteq} B$ *and* $1 < q$, *then* $H_q(A \mid C) \geq H_q(B \mid C)$.

Proof. (*i*) $H_q\left(A \mid B\right) = \frac{q}{1-q} \log \left(\sum_{j=1}^{m} \mu(B_j) \left\| \mu_{A|B=B_j} \right\|_q \right) =$
$\frac{q}{1-q} \log \left(\sum_{j=1}^{m} \mu(B_j) \left(\sum_{i=1}^{n} \mu_{A|B=B_j}{}^{q} \right)^{\frac{1}{q}} \right)$
$= \frac{q}{1-q} \log \left(\sum_{j=1}^{m} \mu(B_j) \left(\sum_{i=1}^{n} \left(\frac{\mu(A_i \cap B_j)}{\mu(B_j)} \right)^{q} \right)^{\frac{1}{q}} \right) = q\, H_q\left(A \vee B\right).$
(*ii*) if $A \overset{\circ}{\subseteq} B$ then for every $A_i \in A$ there exists $B_j \in B$ such that $\mu(A_i \,\Delta\, B_i) = 0$.
Without loss of generality, we sort measurable partition B in a way that for $i = 1, 2, ..., n$, B_i is a member such that $\mu(A_i \,\Delta\, B_i) = 0$. In the proposition 2.5, it is proved that $\mu(B_i) = \mu(B_i \cap A_i) = \mu(A_i)$. Also, we have $\mu\left(A_i\right) = \mu\left(A_i \cap B_i\right) + \mu\left(A_i \cap B_i{}^{c}\right)$. Therefore $\mu\left(A_i \cap B_i{}^{c}\right) = 0$ and $\mu\left(A_i \cap C_k\right) = \mu\left(A_i \cap C_k \cap B_i\right) + \mu\left(A_i \cap C_k \cap B_i{}^{c}\right)$
$\leq \mu\left(C_k \cap B_i\right) + \mu\left(A_i \cap B_i{}^{c}\right) = \mu\left(C_k \cap B_i\right)$. So

$$H_q\left(A \mid C\right) = \frac{q}{1-q} \log \left(\sum_{k=1}^{l} \mu(C_k) \left(\sum_{i=1}^{n} \left(\frac{\mu\left(A_i \cap C_k\right)}{\mu\left(C_k\right)} \right)^{q} \right)^{\frac{1}{q}} \right)$$

$$\leq \frac{q}{1-q} \log_2 \left(\sum_{k=1}^{l} \mu(C_k) \left(\sum_{j=1}^{m} \left(\frac{\mu\left(B_i \cap C_k\right)}{\mu\left(C_k\right)} \right)^{q} \right)^{\frac{1}{q}} \right) = H_q\left(A \mid C\right).$$

(*iii*) The proof of this part is similar to the proof of Part "(*ii*)" and hence is omitted. □

Let measurable partitions $A = \{A_1, ..., A_n\}$ and $B = \{B_1, ..., B_m\}$ be independent. A simple calculation establishes that

$$H_q(A\,|B) = \frac{q}{1-q}\log\left(\sum_{j=1}^{m}\mu(B_j)\left(\sum_{i=1}^{n}\left(\frac{\mu(A_i\cap B_j)}{\mu(B_j)}\right)^q\right)^{\frac{1}{q}}\right)$$
$$= \frac{q}{1-q}\log\left(\sum_{j=1}^{m}\mu(B_j)\left(\sum_{i=1}^{n}\left(\frac{\mu(A_i)\mu(B_j)}{\mu(B_j)}\right)^q\right)^{\frac{1}{q}}\right)$$
$$= \frac{q}{1-q}\log\left(\sum_{j=1}^{m}\mu(B_j)\left(\sum_{i=1}^{n}(\mu(A_i))^q\right)^{\frac{1}{q}}\right) = \frac{q}{1-q}\log\left(\left(\sum_{i=1}^{n}(\mu(A_i))^q\right)^{\frac{1}{q}}\right)$$
$$= H_q(A).$$

Furthermore, if $A=\{A_1,...,A_n\}$ and $B=\{\Omega\}$, we get $\mu(A_i\cap\Omega) = \mu(A_i) = \mu(A_i)\,\mu(\Omega)$. But this means that A and B are independent, and in consequence $H_q(A\,|\Omega) = H_q(A)$.

Proposition 3.3. *Let $A=\{A_1,...,A_n\}$, $B=\{B_1,...,B_m\}$ and $C=\{C_1,...,C_l\}$ be measurable partitions of Ω and $A\prec B$. Then*
(i) $H_q(A\,|\,C)\le H_q(B\,|\,C)$;
(ii)$H_q(A\,|\,B) = 0$.

Proof. (i)If $A\prec B$ then for every $A_i\in A$, there exists a subset $\alpha_i\subset\{1,...,m\}$ such that $A_i=\cup_{j\in\alpha_i}B_j, \alpha_i\cap\alpha_j=\emptyset$ for $i\ne j$, and $\cup_{i=1}^{n}\alpha_i=\{1,...,m\}$. Without loss of generality, suppose that $\alpha_1=\{1,...,m_1\}$, ..., $\alpha_n=\{m_{n-1},...,m_n\}$ which $m_n=m$.
We have $\mu^q(A_i\cap C_k) = \mu^q\left(\bigcup_{j=m_{i-1}}^{m_i}B_j\cap C_k\right) = \left(\sum_{j=m_{i-1}}^{m_i}\mu(B_j\cap C_k)\right)^q$.
If $0<q<1$, then $\left(\sum_{j=m_{i-1}}^{m_i}\mu(B_j\cap C_k)\right)^q \le \sum_{j=m_{i-1}}^{m_i}\mu^q(B_j\cap C_k)$. Therefore

$$H_q(A\,|\,C) = \frac{q}{1-q}\log\left(\sum_{k=1}^{l}\mu(C_k)\left(\sum_{i=1}^{n}\left(\frac{\mu(A_i\cap C_k)}{\mu(C_k)}\right)^q\right)^{\frac{1}{q}}\right)$$
$$\le \frac{q}{1-q}\log\left(\sum_{k=1}^{l}\mu(C_k)\left(\sum_{i=1}^{n}\sum_{j=m_{i-1}}^{m_i}\left(\frac{\mu(B_j\cap C_k)}{\mu(C_k)}\right)^q\right)^{\frac{1}{q}}\right)$$
$$= \frac{q}{1-q}\log\left(\sum_{k=1}^{l}\mu(C_k)\left(\sum_{j=1}^{m}\left(\frac{\mu(B_j\cap C_k)}{\mu(C_k)}\right)^q\right)^{\frac{1}{q}}\right) = H_q(B\,|\,C).$$

In the case of $1<q$, the proof is similar to the proof of the case $0<q<1$.
(ii) If $j\notin\alpha_i=\{m_{i-1},...,m_i\}$, then $\mu(A_i\cap B_j) = \mu\left(\bigcup_{k=m_{i-1}}^{m_i}B_k\cap B_j\right) = \mu(\emptyset) = 0$.

If $j \in \alpha_i = \{m_{i-1}, ..., m_i\}$, then $\mu(A_i \cap B_j) = \mu(B_j)$. Therefore

$$H_q(A\,|\,B) = \frac{q}{1-q}\log\left(\sum_{j=1}^{m}\mu(B_j)\left(\sum_{i=1}^{n}\left(\frac{\mu(A_i\cap B_j)}{\mu(B_j)}\right)^q\right)^{\frac{1}{q}}\right)$$

$$= \frac{q}{1-q}\log\left(\sum_{i=1}^{n}\sum_{j\in\alpha_i}\mu(B_j)\left(\sum_{i=1}^{n}\left(\frac{\mu(A_i\cap B_j)}{\mu(B_j)}\right)^q\right)^{\frac{1}{q}}\right)$$

$$= \frac{q}{1-q}\log\left(\sum_{j=1}^{m}\mu(B_j)\right) = \frac{q}{1-q}\log_2(1) = 0.$$

□

Now we present some examples of the above results.

Example 3.4. *Suppose that* $\Omega = [0,1)$. $A = \left\{[0,\frac{1}{3}), [\frac{1}{3},\frac{2}{3}), [\frac{2}{3},1)\right\}$, $B = \left\{[0,\frac{1}{5}), [\frac{1}{5},\frac{1}{3}), [\frac{1}{3},\frac{3}{5}), [\frac{3}{5},\frac{2}{3}), [\frac{2}{3},\frac{4}{5}), [\frac{4}{5},1)\right\}$ *and* $C = \left\{[0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), [\frac{1}{2},\frac{4}{5}), [\frac{4}{5},1)\right\}$ *are partitions and* $A \prec B$. *A simple calculation establishes that*

$$H_q(A\,|\,B) = \frac{q}{1-q}\log\left(\left(\frac{1}{5^q}\right)^{\frac{1}{q}} + \left(\frac{2^q}{15^q}\right)^{\frac{1}{q}} + \left(\frac{4^q}{15^q}\right)^{\frac{1}{q}} + \left(\frac{1}{15^q}\right)^{\frac{1}{q}} + \left(\frac{2^q}{15^q}\right)^{\frac{1}{q}} + \left(\frac{1}{5^q}\right)^{\frac{1}{q}}\right) = \frac{q}{1-q}\log(1) = 0.$$

$$H_q(A\,|\,C) = \frac{q}{1-q}\log\left(\left(\frac{1}{4^q}\right)^{\frac{1}{q}} + \left(\frac{1}{12^q} + \frac{1}{6^q}\right)^{\frac{1}{q}} + \left(\frac{1}{6^q} + \frac{2^q}{15^q}\right)^{\frac{1}{q}} + \left(\frac{1}{5^q}\right)^{\frac{1}{q}}\right) = \frac{q}{1-q}\log\left(\frac{9}{20} + \frac{(1+2^q)^{\frac{1}{q}}}{12} + \frac{(5^q+4^q)^{\frac{1}{q}}}{30}\right),$$

and

$$H_q(B\,|\,C) = \frac{q}{1-q}\log\left(\left(\frac{1}{5^q} + \frac{1}{20^q}\right)^{\frac{1}{q}} + \left(\frac{1}{12^q} + \frac{1}{6^q}\right)^{\frac{1}{q}} + \left(\frac{1}{10^q} + \frac{1}{15^q} + \frac{2^q}{15^q}\right)^{\frac{1}{q}} + \left(\frac{1}{5^q}\right)^{\frac{1}{q}}\right)$$
$$= \frac{q}{1-q}\log\left(\frac{(4^q+1)^{\frac{1}{q}}}{20} + \frac{(1+2^q)^{\frac{1}{q}}}{12} + \frac{(3^q+2^q+4^q)^{\frac{1}{q}}}{30} + \frac{1}{5}\right).$$

Let $0 < q < 1$. *We get*

$$\frac{(4^q+1)^{\frac{1}{q}}}{20} + \frac{(3^q+2^q+4^q)^{\frac{1}{q}}}{30} + \frac{1}{5} \geq \frac{5}{20} + \frac{(3^q+2^q+4^q)^{\frac{1}{q}}}{30} + \frac{1}{5} \geq \frac{9}{20} + \frac{(5^q+4^q)^{\frac{1}{q}}}{30}.$$

Hence it can be verified that $H_q(A\,|\,C) \leq H_q(B\,|\,C)$.
If $1 < q$, *we have*

$$\frac{(4^q+1)^{\frac{1}{q}}}{20} + \frac{(3^q+2^q+4^q)^{\frac{1}{q}}}{30} + \frac{1}{5} \leq \frac{5}{20} + \frac{(3^q+2^q+4^q)^{\frac{1}{q}}}{30} + \frac{1}{5} \leq \frac{9}{20} + \frac{(5^q+4^q)^{\frac{1}{q}}}{30}.$$

We also see that $H_q(A\,|\,C) \leq H_q(B\,|\,C)$.

4. Rényi Entropy of Dynamical Systems

Let $(\Omega,\ S,\ \mu)$ denote a probability space and further $\varphi:\Omega\to\Omega$ be a measure preserving transformation (i.e., $E\in S$ implies $\varphi^{-1}(E)\in S$ and $\mu(\varphi^{-1}(E))=\mu(E)$. Then (Ω,S,μ,φ) is called a dynamical system. If $A=\{A_1,...,A_n\}$ is a measurable partition of Ω, then
$\varphi^{-1}(A)=\left\{\ \varphi^{-1}(A_1),\ \varphi^{-1}(A_2),...,\varphi^{-1}(A_n)\right\}$ is a measurable partition.
Indeed $\cup_{i=1}^{n}\varphi^{-1}(A_i)=\varphi^{-1}\left(\cup_{i=1}^{n}A_i\right)=\varphi^{-1}\left(\Omega\right)=\Omega$ and $\varphi^{-1}(A_i)\cap\varphi^{-1}(A_j)=\emptyset$.

Proposition 4.1. *Let (Ω,S,μ,φ) be a dynamical system, $A=\{A_1,...,A_n\}$ and $B=\{B_1,...,B_m\}$ be measurable partitions. Then*
(i) $\varphi^{-1}(A\ \vee B)=\varphi^{-1}(A\ )\vee\varphi^{-1}(B)$;
(ii) $A\prec B$ implies that $\varphi^{-1}\left(A\right)\prec\varphi^{-1}\left(B\right)$;
(iii) $H_q\ \left(\varphi^{-n}\left(A\right)\right)=H_q\ \left(\varphi^{-1}\left(A\right)\right)=\ H_q\ \left(A\right)$.

Proof. (i) Since $\varphi^{-1}(A_i\cap B_j)=\varphi^{-1}(A_i\ )\cap\varphi^{-1}(B_j)$, thus the proof is evident.
(ii) If $A\prec B$ then for every $A_i\in A$ there exists a subset $\alpha_i\subset\{1,...,m\}$ such that $A_i=\cup_{j\in\alpha_i}B_j$, $\alpha_i\cap\alpha_j=\emptyset$ for $i\neq j$, and $\cup_{i=1}^{n}\alpha_i=\{1,...,m\}$. $A_i=\cup_{j\in\alpha_i}B_j$ therefore, $\varphi^{-1}\left(A_i\right)=\varphi^{-1}\left(\underset{j\in I_i}{\cup}B_j\right)=\underset{j\in I_i}{\cup}\varphi^{-1}\left(B_j\right)$ which means that $\varphi^{-1}\left(A\right)\prec\varphi^{-1}\left(B\right)$.
(iii) $H_q\ \left(\varphi^{-n}\left(A\right)\right)=\frac{1}{1-q}\log_2\sum\limits_{i=1}^{n}\left(\mu\left(\varphi^{-n}\left(A_i\right)\right)\right)^q=\frac{1}{1-q}\log_2\sum\limits_{i=1}^{n}\left(\mu\left(A_i\right)\right)^q=H_q\ \left(A\right)$.

□

In the following part, we will begin by introducing the Rényi entropy of a measure-preserving transformation φ relative to a measurable partition A of order $q\in\ (0,\,1)\ \cup\ (1,\,\infty)$. Later, we shall remove the dependence on A to obtain the Rényi entropy of a dynamical system (Ω,S,μ,φ).

Definition 4.2. Let (Ω,S,μ,φ) be a dynamical system and A be a measurable partition of Ω. Rényi entropy of measure-preserving transformation φ relative to a measurable partition A of order $q\in(0,\,1)\cup(1,\,\infty)$ is defined by:

$$h_q\left(\varphi,\ A\right):=\lim_{n\to\infty}\sup\ \frac{1}{n}\,H_q\left(\bigvee_{i=0}^{n-1}\ \varphi^{-i}\ A\right)$$

and Rényi entropy of a dynamical system (Ω,S,μ,φ) of order $q\in\ (0,\,1)\ \cup\ (1,\,\infty)$ is introduced by:

$$h_q\left(\varphi\right):=\sup_{A}\left\{h_q\left(\varphi,\ A\right):\ A\text{ is a measurable partition of }\Omega\right\}.$$

Using Lemma 2.2, we can simply prove that $h_q\left(\varphi\right)\geq\ 0$.

The following proposition investigates the monotone behaviour of Rényi entropy under refinement.

Proposition 4.3. *Let* $(\Omega, S, \mu, \varphi)$ *be a dynamical system. If* $A = \{A_1, ..., A_n\}$ *and* $B = \{B_1, ..., B_m\}$ *are measurable partitions of* Ω *such that* $A \prec B$, *then* $h_q(\varphi, A) \leq h_q(\varphi, B)$.

Proof. Since $A \prec B$, Proposition 4.1 implies $\varphi^{-1}(A) \prec \varphi^{-1}(B)$ and furthermore we can conclude $\varphi^{-i}(A) \prec \varphi^{-i}(B)$. According to the property (iii) of Lemma 2.3, we have $\bigvee_{i=0}^{n-1} \varphi^{-i}(A) \prec \bigvee_{i=0}^{n-1} \varphi^{-i}(B)$. Also, property (i) of proposition 2.4 implies that $H_q\left(\bigvee_{i=0}^{n-1} \varphi^{-i}(A)\right) \leq H_q\left(\bigvee_{i=0}^{n-1} \varphi^{-i}(B)\right)$. Therefore, $h_q(\varphi, A) \leq h_q(\varphi, B)$.

□

Example 4.4. *Suppose that* $\Omega = [0, 1)$, *A is a measurable partition of* Ω *and* $\varphi : \Omega \to \Omega$ *is the identity function. The operation* $\vee$ *is idempotent, thus* $h_q(\varphi, A) = \lim_{n\to\infty} \sup \frac{1}{n} H_q\left(\bigvee_{i=0}^{n-1} \varphi^{-i} A\right) = \lim_{n\to\infty} \sup \frac{1}{n} H_q(A) = 0$ *for every partition A of* Ω.

Hence, the Rényi entropy of (Ω, S, μ, I) *is the number:*

$$h_q(I) = \sup\{h_q(I,\ A)\ ;\ A \textit{ is a measurable partition of } \Omega\} = 0.$$

In the following, we explore $h_q(\varphi) = 0$, if $\varphi^k = I$ for a $k \in z^+$. First, we need the following proposition.

Proposition 4.5. *Suppose that* $(\Omega, S, \mu, \varphi)$, *is a dynamical system. The relation* $h_q(\varphi^k) = k \cdot h_q(\varphi)$ *holds for any* $k \in z^+$.

Proof. For every measurable partition A of Ω, we have

$$h_q(\varphi^k,\ \vee_{i=0}^{k-1}\varphi^{-i}(A)) = \limsup_{n\to\infty} \tfrac{1}{n} H_q(\vee_{j=0}^{n-1}(\varphi^k)^{-j}(\vee_{i=0}^{k-1}\varphi^{-i}(A)))$$

$$\limsup_{n\to\infty} \tfrac{1}{n} H_q(\vee_{j=0}^{n-1} \vee_{i=0}^{k-1} \varphi^{-(kj+i)}(A)) = \limsup_{n\to\infty} \tfrac{1}{n} H_q(\vee_{i=0}^{nk-1}\varphi^{-i}(A))$$

$$= \limsup_{n\to\infty} \tfrac{nk}{n} \tfrac{1}{nk} H_q(\vee_{i=0}^{nk-1}\varphi^{-i}(A)) = k \cdot h_q(\varphi,\ A).$$

Therefore,

$$\begin{aligned} k \cdot h_q(\varphi) &= k \cdot \sup\{h_q(\varphi,\ A)\ ;\ A \textit{ is a partition of } \Omega\} \\ &= \sup\left\{h_q(\varphi^k,\ \vee_{i=0}^{k-1}\varphi^{-i}(A))\ ; A \textit{ is a partition of } \Omega\right\} \\ &\leq \sup\left\{h_q(\varphi^k,\ B)\ ; B \textit{ is a partition of } \Omega\right\} = h_q\left(\varphi^k\right). \end{aligned}$$

On the other hand $A \prec \vee_{i=0}^{k-1}\varphi^{-i}(A)$, thus

$$h_q(\varphi^k,\ A) \leq h_q(\varphi^k,\ \vee_{i=0}^{k-1}\varphi^{-i}(A)) = k \cdot h_q(\varphi,\ A),$$

which implies

$$h_q\left(\varphi^k\right) = \sup\left\{h_q(\varphi^k,\, A)\,;A\ is\ a\ partition\ of\ \ \Omega\right\}$$
$$\leq k\cdot\sup\left\{h_q(\varphi,\, A)\,;A\ \ is\ a\ partition\ of\ \ \Omega\right\} = k\cdot h_q\left(\varphi\right).$$

□

Proposition 4.6. *Let $(\Omega, S, \mu, \varphi)$ be a dynamical system. If there is a $k \in z^+$ such that $\varphi^k = I$, then $h_q(\varphi) = 0$.*

Proof. At first, we prove that $h_q(I) = 0$.
Indeed $h_q(I,\, A) = \lim\limits_{n\to\infty}\sup\frac{1}{n}H_q\left(\bigvee\limits_{i=0}^{n-1} I^{-i}A\right) = \lim\limits_{n\to\infty}\sup\frac{1}{n}H_q(A) = 0$ for any measurable partition A, which means $h_q(I) = 0$.
According to proposition 4.6, we get $h_q(\varphi) = \frac{1}{k}h_q\left(\varphi^k\right) = \frac{1}{k}h_q(I) = 0$.

□

In the following, we will introduce the notion of isomorphism of dynamical systems.

Definition 4.7. Two dynamical systems $(\Omega_1, S_1, \mu_1, \varphi_1)$ and $(\Omega_2, S_2, \mu_2, \varphi_2)$ are called isomorphic if there exists a bijective mapping $\psi : \Omega_1 \to \Omega_2$ satisfying the following conditions:
(i) the diagram

$$\begin{array}{ccc} \Omega_1 & \underset{\varphi_1}{\rightarrow} & \Omega_1 \\ \psi\ \downarrow & & \downarrow\ \psi \\ \Omega_2 & \underset{\varphi_2}{\rightarrow} & \Omega_2 \end{array}$$

is commutative, i.e., $\psi(\varphi_1(A)) = \varphi_2(\psi(A))$ for every $A \in \Omega_1$;
(ii) $\psi^{-1}(B) \in \Omega_1$ for every $B \in \Omega_2$;
(iii) $\mu_1\left(\psi^{-1}(B)\right) = \mu_2(B)$ for every $B \in \Omega_2$.

The following theorem examines whether two isomorphic dynamical systems have the same Rényi entropy.

Theorem 4.8. *If dynamical systems $(\Omega_1, S_1, \mu_1, \varphi_1)$ and $(\Omega_2, S_2, \mu_2, \varphi_2)$ are isomorphic, then $h_q(\varphi_1) = h_q(\varphi_2)$.*

Proof. Let a mapping $\psi : \Omega_1 \to \Omega_2$ represents an isomorphism of dynamical systems $(\Omega_1, S_1, \mu_1, \varphi_1)$ and $(\Omega_2, S_2, \mu_2, \varphi_2)$. If $B = \{B_i : i = 1, ..., m\}$ is a measurable partition of Ω_2, since ψ is a bijective mapping so we conclude that $\psi^{-1}(B) = \left\{\psi^{-1}(B_i) : i = 1, ..., m\right\}$ is a measurable partition of Ω_1. Indeed, $\bigcup\limits_{i=1}^{n}\psi^{-1}(B_i) = \psi^{-1}\left(\bigcup\limits_{i=1}^{n}B_i\right) = \psi^{-1}(\Omega_2) = \Omega_1$. In addition,

$$H_q\left(\psi^{-1}(B)\right) = \frac{1}{1-q}\log\sum_{i=1}^{m}\left(\mu_1\left(\psi^{-1}(B_i)\right)\right)^q$$
$$= \frac{1}{1-q}\log\sum_{i=1}^{m}\left(\mu_2(B_i)\right)^q = H_q(B).$$

Hence, a simple calculation establishes

$$H_q\left(\vee_{i=0}^{n-1}\varphi_1^{-i}(\psi^{-1}B)\right) = H_q\left(\psi^{-1}\vee_{i=0}^{n-1}\varphi_2^{-i}(B)\right) = H_q\left(\vee_{i=0}^{n-1}\varphi_2^{-i}(B)\right).$$

Therefore,

$$\begin{aligned} h_q(\varphi_2, B) &= \lim_{n\to\infty}\sup \tfrac{1}{n} H_q\left(\bigvee_{i=0}^{n-1}\varphi_2{}^{-i}B\right) \\ &= \lim_{n\to\infty}\sup \tfrac{1}{n} H_q\left(\vee_{i=0}^{n-1}\varphi_1^{-i}(\psi^{-1}B)\right) = h_q\left(\varphi_1, \psi^{-1}B\right). \end{aligned}$$

and $\{h_q(\varphi_2, B) : B$ is a measurable partition of $\Omega_2\} \subset h_q(\varphi_1, A) : A$ is a measurable partition of $\Omega_1\}$, and consequently, $h_q(\varphi_2) \leq h_q(\varphi_1)$. The proof for the second equality is similar. Thus, $h_q(\varphi_2) = h_q(\varphi_1)$.

□

5. Conclusions

This paper studied the Rényi entropy of measurable partitions, conditional Rényi entropy of two measurable partitions, and Rényi entropy of dynamical systems. Some basic properties of these newly introduced notions were obtained, and it was proved that two isomorphic dynamical systems have the same Rényi entropy. In addition, several illustrative examples were provided. The most essential problem of this new entropy was the lack of the property of sub-additivity. It is suggested that a different definition of conditional entropy from the conditional entropy presented in this study might help us fix this problem.

6. Appendix

Definition 6.1. [46] A family S of subsets of a non-empty set Ω is called an $\sigma-$algebra, if (i) $\Omega \in S$; (ii) if $A \in S$, then $\Omega - A \in S$; (iii) if $A_n \in S$ $(n = 1, 2, ...)$, then $\cup_{n=1}^{\infty} A_n \in S$. The couple (Ω, S) is said to be a measurable space, the elements of S are said to be measurable.

Definition 6.2. [46] A function from S to the interval $[0, 1]$ is called a probability measure if it satisfies the following properties:
i. $\mu(\Omega) = 1$;
ii. $\mu(A) \geq 0$, for every $A \in S$;
iii. $\{A_n\}_{n=1}^{\infty} \subset S$ if such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.
The above characterized triplet is called to be a probability space.

The following introduced definitions that are used in different sections of the paper:

Definition 6.3. [46] Let (Ω, S) be a measurable space. A finite sequence $\{A_1, ..., A_n\}$ of pairwise disjoint measurable subsets of Ω is called a measurable partition of Ω, if $\cup_{i=1}^{n} A_n = \Omega$.

Definition 6.4. [46] Let $A = \{A_1, ..., A_n\}$ and $B = \{B_1, ..., B_m\}$ be two measurable partitions of Ω. The measurable partition B is named to be a refinement of A if for every $A_i \in A$ there exists a subset $\alpha_i \subset \{1, ..., m\}$ such that $A_i = \cup_{j\in\alpha_i} B_j, \alpha_i \cap \alpha_j = \emptyset$ for $i \neq j$, and $\cup_{i=1}^{n} \alpha_i = \{1, ..., m\}$. In this case we write $A \prec B$. Furthermore, $A \vee B = \{A_i \cap B_j;\ i = 1, 2, ..., n,\ j = 1, 2, ..., m\}$ is said to be the join refinement of A and B.

Several properties can be derived from the mentioned definitions. We provided them in the following Lemma.

Lemma 6.5. *Let* $A = \{A_1, ..., A_n\}$, $B = \{B_1, ..., B_m\}$ *and* $C = \{C_1, ..., C_l\}$ *be measurable partitions of measurable space* (Ω, S), *then:*
(i) $A \vee B$ *is a measurable partition,*
(ii)$A \prec A \vee B$ *and* $B \prec A \vee B$,
(iii) if $A \prec B$, *then* $A \vee C \prec B \vee C$.

Proof. Refer to [46]. □

Definition 6.6. [46] Let $A = \{A_1, ..., A_n\}$ and $B = \{B_1, ..., B_m\}$ be two measurable partitions of Ω. Then $A \overset{\circ}{\subset} B$ if for each $A_i \in A$ there exists $B_j \in B$ such that $\mu(A_i \,\Delta\, B_j) = 0$, where $A_i \,\Delta\, B_j = (A_i - B_j) \cup (B_j - A_i)$ denotes a symmetric difference of sets $A_i, B_j \in S$. We write $A \approx B$ if $A \overset{\circ}{\subset} B$ and $B \overset{\circ}{\subset} A$. Measurable partitions $A = \{A_1, ..., A_n\}$ and $B = \{B_1, ..., B_m\}$ are named independent, if $\mu(A_i \cap B_j) = \mu(A_i) \cdot \mu(B_j)$.

Remark 6.7. The relation $\approx$ is the equivalence relation in the family of all measurable partitions of Ω.

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