Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

ON LOWER BOUNDS FOR THE METRIC DIMENSION OF GRAPHS

M. Jannesari

Article type: Research Article
(Received: 03 March 2022, Received in revised form: 25 May 2022)
(Accepted: 30 May 2022, Published Online: 16 June 2022)

Abstract. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the ordered k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the (metric) representation of v with respect to W, where $d(x, y)$ is the distance between the vertices x and y. A set W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W. The minimum cardinality of a resolving set for G is its metric dimension, and a resolving set of minimum cardinality is a basis of G. Lower bounds for metric dimension are important. In this paper, we investigate lower bounds for metric dimension. Motivated by a lower bound for the metric dimension k of a graph of order n with diameter d in [S. Khuller, B. Raghavachari, and A. Rosenfeld, Landmarks in graphs, Discrete Applied Mathematics 70(3)(1996)217 − 229], which states that $k \geq n - d^k$, we characterize all graphs with this lower bound and obtain a new lower bound. This new bound is better than the previous one, for graphs with diameter more than 3.

Keywords: Resolving set; Metric dimension; Metric basis; Lower bound; Diameter.
2020 MSC: 05C12

1. Introduction

Throughout this paper $G = (V, E)$ is a finite simple connected graph of order $n(G)$. The distance between two vertices u and v, denoted $d(u, v)$, is the length of a shortest path between u and v in G. The diameter of G, denoted $d(G)$, is $max_{u,v \in V(G)} d(u, v)$. For all i such that $1 \leq i \leq d(G)$, $\Gamma_i(v)$ is the set of all vertices $x \in V(G)$ with $d(v, x) = i$. We use notation P_n and K_n for a path and a complete graph on n vertices, respectively.

The vertices of a connected graph can be represented by different ways. For example, the vectors which theirs components are the distances between the vertex and the vertices in a given subset of vertices. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) = (d(v, w_1), \ldots, d(v, w_k))$$

m.jannesari@shr.ui.ac.ir, ORCID: 0000-0002-4178-6973
DOI: 10.22103/jmmrc.2022.19121.1211
Publisher: Shahid Bahonar University of Kerman
How to cite: M. Jannesari, *On lower bounds for the metric dimension of graphs*, J. Mahani Math. Res. 2023; 12(1): 35-41.
© the Authors

is the *(metric) representation* of v with respect to W. The set W is a *resolving set (locating set)* for G if distinct vertices have different representations in this case it is said the set W resolves G. A resolving set W for G with minimum cardinality called a *basis* of G. The *metric dimension* of G, denoted $\dim(G)$, is the cardinality of a basis. Elements in a basis are *landmarks*.

The subject of (metric) representation is introduced by Slater [10] (see [5]). He described the usefulness of these ideas when working with U.S. Sonar and Coast Guard Loran stations [10]. Also, these concepts have some applications in chemistry for representing chemical compounds [7] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [9]. It was noted in [4,8] that the problem of finding the metric dimension of a graph is NP-hard. For more applications and results in these concepts see [1, 2, 6, 8].

When determining whether a given set W of vertices of a graph G resolves G, it is suffices to check the representations of vertices in $V(G)\backslash W$ because $w \in W$ is the only vertex of G for which $d(w, w) = 0$. It is obvious that every graph G of order n satisfies $1 \leq \dim(G) \leq n - 1$. Khuller et al. [8] and Chartrand et al. [3] independently proved that $\dim(G) = 1$ if and only if G is a path. Also, Chartrand et al. [3] proved that the only graph of order n, $n \geq 2$, with metric dimension $n - 1$ is the complete graph K_n.

As mentioned, the metric dimension problem is an NP-hard problem, and for many graph families it is difficult or impossible to obtain. Therefore, obtaining bounds for metric dimension is valuable. In this Paper, we investigate lower bounds for metric dimension of graphs. Motivated by the following lower bound we characterize all graphs with this bound.

Theorem 1.1. [8] Let G be a graph with metric dimension k and order n. Let d be the diameter of G. Then

$$n \leq k + d^k.$$

By the next theorem all complete graphs, K_n and paths, P_n attain this lower bound.

Theorem 1.2. [3] Let G be a connected graph of order n. Then

(a) $\dim(G) = 1$ if and only if $G = P_n$;

(b) $\dim(G) = n - 1$ if and only if $G = K_n$.

The second aim of this paper is to find a new lower bound for metric dimension in terms of diameter and order of a graph. This new bound is better than the bound in Theorem 1.1 for graphs with diameter greater than 3.

2. Lower bounds

The first goal of this section is to characterize all connected graphs that attain the lower bound in Theorem 1.1. In fact, if G is a graph with diameter d, metric dimension k, order $k + d^k$, and W is a basis of G, then for each

k-vector $R = (a_1, a_2, \dots, a_k), 1 \leq a_i \leq d$, there exists a vertex in G with metric representation R, with respect to W. To get a characterization some results are needed.

Lemma 2.1. *Let G be a graph with metric dimension k, diameter d and order $k + d^k$. If W is a basis of G, then for each $w \in W$, there exists a vertex $v \in V(G) \setminus W$ with $d(v, w) = d$.*

Proof. Since order of G is $k+d^k$, there exists a vertex $v \in V(G)$ with $r(v|W) = (d, d, \dots, d)$. Therefore v is in $V(G) \setminus W$ and for each $w \in W$, $d(v, w) = d$. □

The next two lemmas present the maximum value of the number of neighbours of a landmark in a graph G with metric dimension k, diameter d and order $k + d^k$.

Lemma 2.2. *Let G be a graph with metric dimension k, diameter d and order $k + d^k$. If W is a basis of G, then for each $w \in W$, $|\Gamma_i(w)| = d^{k-1}, 1 \leq i \leq d$.*

Proof. Let $W = \{w_1, w_2, \dots, w_k\}$ be a basis of G. Note that, for every vertex $v \in \Gamma_i(w_j)$, $1 \leq j \leq k$, the i-th entry of $r(v|W)$ is i. Clearly, there are d^{k-1} k-vectors with entries in $\{1, 2, \dots, d\}$, where the j-th entry is fixed. That is $|\Gamma_i(w_j)| \leq d^{k-1}$, for $1 \leq i \leq d$ and $1 \leq j \leq k$. On the other hand every vertex v with i in the j-th entry of $r(v|W)$ is in $\Gamma_i(w_j)$. Since order of G is $k + d^k$, each k-vector with entries in $\{1, 2, \dots, d\}$ is metric representation of a vertex of G. Therefore $|\Gamma_i(w_j)| \geq d^{k-1}$. □

Lemma 2.3. *Let G be a graph and $W = \{w_1, w_2, \dots, w_k\}$ be a basis of G. Then each $w_i, 1 \leq i \leq k$ can has at most 3^{k-1} neighbours.*

Proof. Let $w_i \in W$. If $u, v \in \Gamma_1(w_i)$, then

$$d(u, v) \leq d(u, w_i) + d(w_i, v) = 2.$$

Now let $1 \leq i \neq j \leq k$ and $d(v, w_j) = \min\{d(x, w_j)|x \in \Gamma_1(w_i)\}$. Hence for each $u \in \Gamma_1(w_i)$,

$$d(u, w_j) \in \{d(v, w_j), d(v, w_j) + 1, d(v, w_j) + 2\}.$$

That means, there are at most three possibilities $d(v, w_j), d(v, w_j) + 1$, and $d(v, w_j)+2$ for the j-th entry of metric representation of every vertex in $\Gamma_1(w_i)$. This implies that W can produce 3^{k-1} distinct metric representations with 1 in the i-th entry for vertices in graph G. Since W is a basis for G, metric representations of all vertices in $\Gamma_1(w_i)$ with respect to W are distinct. Therefore $\Gamma_1(w_i)$ has at most 3^{k-1} members. □

This lemma yields a lower bound for metric dimension in terms of minimum degree of graphs.

Corollary 2.1. Let G be a graph with minimum degree δ. If $\dim(G) \geq 2$, then $\dim(G) \geq 1 + \log_3 \delta$.

Proof. Let G be a graph with metric dimension k and w be a landmark in G. By Lemma 2.3, $\delta \leq \deg(w) \leq 3^{k-1}$. Therefore $k \geq 1 + \log_3 \delta$. □

The next definition is needed to characterize all graphs with metric dimension k, diameter d and order $k + d^k$.

Definition 2.2. For integers $d \leq 3$ and $k \geq 1$ let $\mathcal{F}_{d,k}$ be a family of graphs G with the following properties.

(a) $V(G) = U \cup W$, where $W = \{w_1, w_2, \ldots, w_k\}$ and U is the set of all k-vectors with entries in $\{1, 2, \ldots, d\}$;

(b) For $w_i \in W$, $1 \leq i \leq k$, a vertex $u \in U$ is adjacent to w_i if and only if the i-th coordinate of u is 1;

(c) The adjacency of vertices in U is such that for $w_i \in W$, $1 \leq i \leq k$, a vertex $u \in U$ is belong to $\Gamma_j(w_i)$, $1 \leq j \leq d$, if the i-th coordinate of u is j. Existence of other edges between vertices in U is such that $d(G) = d$;

(c) Existence of each edge between two vertices in W is arbitrary.

It is easy to see that $\mathcal{F}_{d,1}$ is the family of all path graphs and $\mathcal{F}_{1,k}$ is the family of all complete graphs. Examples for $\mathcal{F}_{2,2}$ and $\mathcal{F}_{3,2}$ are shown in Figure **??**.

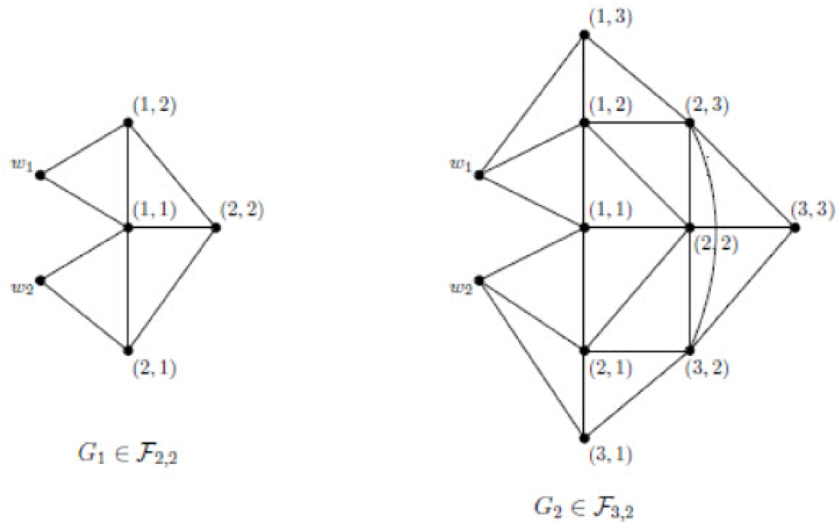

FIGURE 1. Examples for $\mathcal{F}_{2,2}$ and $\mathcal{F}_{3,2}$

Theorem 2.3. For positive integers d and k, a graph G of diameter d and metric dimension k, is of order $k + d^k$ if and only if $k = 1$ or $d \in \{1, 2, 3\}$, and G is isomorphic to a graph in $\mathcal{F}_{d,k}$.

Proof. Let G be a graph of diameter d, metric dimension k and order $k + d^k$. Let $k \geq 2$, W be a basis for G and $w \in W$. Lemma 2.2 concludes that $|\Gamma_1(w)| = d^{k-1}$. On the other hand, by Lemma 2.3, we have $|\Gamma_1(w)| \leq 3^{k-1}$. Thus, for $k \geq 2$ the diameter of G is at most 3. Therefore to find all graphs G with diameter d, metric dimension k and order $k + d^k$, it is suffices to consider four following cases.

Case 1: $k = 1$. By Theorem 1.2, $\dim(G)$ is 1 if and only if $G = P_n$. Clearly $d(P_n) = n - 1$ and $k + d^k = 1 + n - 1 = n$. Therefore a graph G with diameter d and metric dimension 1 is of order $1 + d$ if and only if $G \in \mathcal{F}_{d,1}$.

Case 2: $d = 1$. The only graph G with diameter 1 is K_n. By Theorem 1.2, $\dim(G) = n - 1$ if and only if $G = K_n$. Therefore a graph G with diameter 1 and metric dimension k is of order $1 + k$ if and only if $G \in \mathcal{F}_{1,k}$.

Case 3: $d = 2$. Let G be a graph with diameter 2, metric dimension k and order $k + 2^k$. We prove that G is isomorphic to a graph $H \in \mathcal{F}_{2,k}$. Suppose that $W = \{w_1, w_2, \ldots, w_k\}$ is a basis for G. Let H be a graph with $V(H) = W \cup U(H)$ such that $U(H)$ be the set of all metric representations of vertices $V(G) \setminus W$ with respect to W. Also, for $u, v \in V(G) \setminus W$, $r(u|W)$ is adjacent to $r(v|W)$ in H if and only if u and v are adjacent in G, the adjacency of vertices $w_i, w_j \in W$ in H and G are the same, a vertex $r(v|W) \in U(H)$ is adjacent to a vertex $w_i \in W$ if and only if the i-th coordinate of $r(v|W)$ is 1. Clearly $H \in \mathcal{F}_{2,k}$ and the function $\psi : V(G) \longrightarrow V(H)$ with the following rule is an isomorphism between G and H.

$$\psi(x) = \begin{cases} x & \text{if } x \in W. \\ r(x|W) & \text{if } x \notin W. \end{cases}$$

On the other hand, if $G \in \mathcal{F}_{2,k}$, then W is a basis of G and the metric representation of each vertex of u is itself. Therefore every graph in $\mathcal{F}_{2,k}$ has diameter 2, metric dimension k and order $k + 2^k$.

Case 4: $d = 3$. Let G be a graph with diameter 3, metric dimension k and order $k + 3^k$. By a similar argument as in case 3 we have, G is isomorphic to a graph $H \in \mathcal{F}_{3,k}$. Moreover, if $G \in \mathcal{F}_{3,k}$, then W is a basis of G and the metric representation of each vertex of u is itself. Therefore every graph in $\mathcal{F}_{3,k}$ has diameter 3, metric dimension k and order $k + 3^k$. □

Example 2.4. Consider the graph G_2 in Figure**??**. It is easy to see that $\{w_1, w_2\}$ is a basis for G_2, hence $\dim(G_2) = 2$. The diameter of G_2 is 3 and it has $3 + 2^3$ vertices.

The following theorem obtains a new lower bound for metric dimension of graphs.

Theorem 2.5. Let G be a graph with metric dimension k and order n. Let d be the diameter of G. Then

$$n \leq k + k3^{k-1} + (d-1)^k.$$

Proof. Let W be a basis for G. The members of W are the only vertices of G such that their representations with respect to W have 0 as a coordinate. The number of this kind of vertices is k. The representation of a vertex of G has 1 as a coordinate if and only if it is adjacent to a member of W. By Lemma 2.3, a vertex of W has at most 3^{k-1} neighbours. Hence there are at most $k3^{k-1}$ vertices that their representation has 1 as a coordinate. Since the diameter of G is d, and every metric representation of vertices of G with respect to W has k coordinate, there are at most $(d-1)^k$ vertices such that there is no 0 and 1 in their metric representation. Therefore the number of vertices in G is at most $k + k3^{k-1} + (d-1)^k$. □

Example 2.6. Let $G = P_5 \square P_2$, cartesian product of graphs P_2 and P_5. Note that $\dim(G) = 2$ and $\mathrm{diam}(G) = 5$. It is easy to see that G has $10 \leq 2 + 2(3^{2-1}) + (5-1)^2 = 24$ vertices.

3. Conclusion

In this section we determine the difference of our lower bound with previous one. If $k = 1$ and $n = k + k3^{k-1} + (d-1)^k$, then $n = d + 1$ and so $G = P_n$. It is easy to see that if $k \geq 2$ and $d \geq 4$ are fixed positive integers, then

$$k + 3^{(k-1)}k + (d-1)^k < k + d^k.$$

Hence for integers n, d, where $d \geq 4$,

$$\{k | k + 3^{(k-1)}k + (d-1)^k \geq n\} \subseteq \{k | k + d^k \geq n\}.$$

That is, for $d \geq 4$ the lower bound in Theorem 2.5 is better than the lower bound in Theorem 1.1.

Example 3.1. Let G be as in Example2.6. Then $n = 10, k + 3^{(k-1)}k + (d-1)^k < k + d^k = 24$ and $k + d^k = 27$.

References

[1] R.C. Brigham, G. Chartrand, R.D. Dutton, and P. Zhang, *On the dimension of trees*, Discrete Mathematics **294** (2005) 279-283.

[2] G.G. Chappell, J. Gimbel, and C. Hartman, *Bounds on the metric and partition dimensions of a graph*, Ars Combinatoria **88** (2008) 349-366.

[3] G. Chartrand, L. Eroh, M.A. Johnson, and O.R. Ollermann, *Resolvability in graphs and the metric dimension of a graph*, Discrete Applied Mathematics **105** (2000) 99-113.

[4] M. Garey and D. Johnson, *Computers and Intractability: A Guide to the Theory of NO-Completeness*, Freeman, New York, 1979.

[5] F. Harary and R.A. Melter, *On the metric dimension of a graph*, Ars Combinatoria **2** (1976) 191-195.

[6] B.L. Hulme, A.W. Shiver, and P.J. Slater, *A Boolean algebraic analysis of fire protection*, Algebraic and Combinatorial Methods in Operations Research, **95** (1984) 215-227.

[7] M.A. Johnson, *Structure-activity maps for visualizing the graph variables arising in drug design*, Journal of Biopharmaceutical Statistics **3** (1993) 217-229.

[8] S. Khuller, B. Raghavachari, and A. Rosenfeld, *Landmarks in graphs*, Discrete Applied Mathematics **70(3)** (1996) 217-229.

[9] R.A. Melter, I. Tomescu, *Metric bases in digital geometry*, Computer Vision, Graphics, and Image Processing, **25** (1984), 113-121.
[10] P.J. Slater, *Leaves of trees*, Congressus Numerantium **14** (1975) 549-559.

Mohsen Jannesari
Orcid number: 0000-0002-4178-6973
Department of Science
Shahreza Campus, University of Isfahan
Isfahan, Iran
Email address: m.jannesari@shr.ui.ac.ir