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ON KRONECKER PRODUCT OF TWO RL-GRAPHS AND SOME RELATED RESULTS

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Dedicated to sincere professor Mashaallah Mashinchi

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Abstract. Using the kronecker product definition of two simple graphs, the kronecker product of two RL-graphs was defined and is defined and it is further shown to be an RL-graph. Consequently, it is demonstrated that the kronecker product of two RL-graphs is a commutative property (i.e $G \otimes H = H \otimes G$). It is also stated that the kronecker product of two strong RL-graphs is a strong RL-graph but not necessarily vice-versa. It is bounded α and β of the kronecker product of two RL-graphs by α and β of its constituent graphs, respectively. Moreover, if H is an RL-graph, and G and G' are two isomorphic RL-graphs, then the kronecker product of G and H and the kronecker product of G' and H are isomorphic RL-graphs. In addition, some notions such as regular RL-graphs, α-regular RL-graphs, and totally regular RL-graphs are proposed and explicated. An application of this operation, which has calculated work efficiency of two companies when they work together by the kronecker product is also suggested. Finally, it is brought one application of this operation that is determined and estimated the group that has the maximum interact among its members. Ultimately, in light of the above, some related theorems are proved and several examples are provided to illustrate these new notions.

Keywords: RL-graph, Strong RL-graph, Kronecker Product of two RL-graphs.
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1. Introduction

Ever since Euler stated the graph concept in 1965 to solve Königsberg Bridge [3], many researchers in this field have used this notion to solve various problems [1,2]. Every year novel ideas are introduced in graph theory to develop it, some of which have applications in multiple fields and help solve human beings' problems [9]. One of these concepts is the notion of a graph constructed on a residuated lattice (called L-graph), presented by Zahedi et al. They used this type of graph to model books in a library or to choose the least medicine to treat a particular disease. They have discussed this issue in detail in their

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papers [11, 13, 18]. Further, a novel operation of two L-graphs, called maximal product, was introduced by considering a residuated lattice; based on that, we have identified some relationships between L-graph and L-graph automaton as well [12]. Now in this study, we decide to present a new operation on this type of graph and use it to select the best group for teamwork. We also suggest that researchers in different fields use this concept to solve various problems by fully expressing the modeling method and using this operation. Also, we tend to find a deep connection between graphs and automata so that researchers in both theories can take advantage of both.

After introducing the concept of fuzzy set by Zadeh in 1965, it was used to model uncertain and ambiguous natural events [15, 16], [17]. Kaufman took advantage of this concept and subsequently suggested the fuzzy concept of graphs [8]. From then until today, this concept has been considered by many writers and researchers for modeling complex topics [10]. When discussing the use of various sciences to solve daily problems, we can say that graph and fuzzy graph theories have significantly contributed to solving human problems. Many companies today have used these concepts to grow and generate revenue. For instance, many companies such as Google maps apply graph theory, using graphs for building transportation systems, where intersection of two(or more) roads are considered to be a vertex and the road connecting two vertices is considered to be an edge. In this case, their navigation system is thus based on the algorithm to calculate the shortest path between two vertices. Facebook users are also considered to be the vertices and if they are friends then there is an edge running between them. Facebook friend suggestion algorithm employs graph theory as well. Facebook is an example of undirected graph. In World Wide Web, web pages are considered to be the vertices. As an example of directed graph, there is an edge from a page u to other page v if there is a link of page v on page u. It has been the basic idea behind Google page ranking algorithm.

In this study, we used the notion of the tensor of two matrices and introduced the Kronecker product of two L-graphs. This operator creates a connection between two unrelated structures and relates the effect of these two structures to each other. At the end of this study, we show that this notion has many applications, and we have mentioned only two of them. Therefore, aims at introducing the kronecker product of two RL-graphs using a comprehensive well-defined operation. Additionally, the notions as strong RL-graph, regular RL-graph, and totally regular RL-graph are explicated in details and further the relationships between these graphs and their operations are investigated. Finally, two applications of this operation are presented and elucidated. Accordingly, some examples and theorems are proposed for clarification of suggested notions.

2. Preliminaries

In this section, some definitions of the graph theory [4, 7, 14], the residuated lattice [6], and the L-graph [11, 12, 18] are notified.

Definition 2.1. [4] The degree of a vertex v in a simple graph $G = (V, E)$, denoted by $d_G(v)$, is the number of edges of G incident with v.

Definition 2.2. [4] A simple graph $G = (V, E)$ is k-regular if $d_G(v) = k$ for all $v \in V$; a regular graph is one that is k-regular for some k.

Definition 2.3. [14] A graph $G = (V, E)$ is disconnected if its vertex set can be partitioned into two nonempty subsets X and Y so that no edge has one end in X and one end in Y.

Definition 2.4. [14] The adjacency matrix of $G = (V, E)$ is the $n \times n$ matrix $A_G := (a_{uv})$, where a_{uv} is the number of edges joining vertices u and v, each loop counting as two edges.

Definition 2.5. [6] A residuated lattice is an algebra $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that

(1) $L = (L, \wedge, \vee, 0, 1)$ is a lattice (the corresponding order will be denoted by $\leq$) with the smallest element 0 and the greatest element 1,
(2) $L = (L, \otimes, 1)$ is a commutative monoid (i.e., $\otimes$ is commutative, associative, and $x \otimes 1 = x$ holds),
(3) $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ holds (adjointness condition).

Proposition 2.6. *[6] Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. Then the following properties hold:*
(R_1) $1 * x = x$, *where* $* \in \{\wedge, \otimes, \rightarrow\}$,
(R_2) $x \otimes 0 = 0, 1' = 0, 0' = 1$,
(R_3) $x \otimes y \leq x \wedge y \leq x, y$, *and* $y \leq (x \rightarrow y)$,
(R_4) $(x \rightarrow y) \otimes x \leq y$,
(R_5) $x \leq y$ *implies* $x * z \leq y * z$, *where* $* \in \wedge, \vee, \otimes$,
(R_6) $z \otimes (x \wedge y) \leq (z \otimes x) \wedge (z \otimes y)$,
(R_7) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$,
(R_8) $(x \vee y) \rightarrow z = (x \rightarrow y) \wedge (x \rightarrow z)$,
(R_9) *if* $x \vee y = 1$, *then* $x \rightarrow y = y$ *and* $x \otimes y = x \wedge y$.

Definition 2.7. [11] $G = (\alpha, \beta)$ is called an L-graph on $G^* = (V, E)$ that is a simple graph if $\alpha : V \rightarrow L$ and $\beta : E \rightarrow L$ are functions, with $\beta(st) \leq \alpha(s) \otimes \alpha(t)$ for every $st \in E$. Besides, if G^* is a path (cycle, bipartite, complete, complete bipartite) graph, then G is called a path (cycle, bipartite, complete, complete bipartite) L-graph on G^*.

Definition 2.8. [18] Let $G = (\alpha, \beta)$ be an L-graph on $G^* = (V, E)$ such that $\beta(st) = \alpha(s) \otimes \alpha(t)$, for every $st \in E$. Then G is a strong L-graph.

Notation 2.9. Through this paper we used RL-graph instead of L-graph.

Definition 2.10. [11] Let $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ be two RL-graphs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $c \in L \backslash \{1\}$. Then G_1 and G_2 are isomorphic with threshold c, denoted by $G_1 \cong_c G_2$ if there exists a bijection h from V_1 into V_2 such that the following conditions hold for all $u, v \in V_1$:

(i) $uv \in E_1$ if and only if $h(u)h(v) \in E_2$,
(ii) $\alpha_1(u) > c$ if and only if $\alpha_2(h(u)) > c$,
(iii) $\beta_1(uv) > c$ if and only if $\beta_2(h(u)h(v)) > c$.

h is an isomorphism ($\cong$) if and only if h is an isomorphism with threshold c for every $c \in L \backslash \{1\}$.

Definition 2.11. [5] The tensor product of two matrices is the same as their kronecker product. Consider you have an $m \times n$ matrix A, and a $p \times q$ matrix B. Their kronecker product $A \otimes B$ is an $mp \times nq$ matrix. In general,

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}.$$

3. The kronecker product of two RL-graphs

Throughout this study, we consider that L is a residuated lattice, and G^* is a simple graph.

In this section, through using the definition of the kronecker product of two matrices, a novel operation on two matrices and their arrays belonging to a residuated lattice are defined and it is noted by $\odot$. The related notion is therefore clarified by an example. In addition, the adjacency matrix of RL-graph G, the matrix of membership of its vertices, and the matrix of membership of its edges are introduced and explicated through appropriate examples. Subsequently, the notion of a kronecker product of two RL-graphs is proposed using a comprehensive well-defined operation. An example expresses the related issue.

Definition 3.1. Suppose $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, the $m \times n$ matrix $A := (a_{ij})$ and the $l \times k$ matrix $B := (b_{ij})$, where $a_{ij} \in L$ and $b_{ij} \in L$. Then

$$A \odot B = \begin{bmatrix} a_{11} \otimes B & a_{12} \otimes B & \dots & a_{1n} \otimes B \\ a_{21} \otimes B & a_{22} \otimes B & \dots & a_{2n} \otimes B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} \otimes B & a_{m2} \otimes B & \dots & a_{mn} \otimes B \end{bmatrix}.$$

Example 3.2. *Consider* $L = (P(X), \cap, \cup, \otimes, \rightarrow, \emptyset, X)$, *where* $X = \{a, b, c, d\}$, $M \otimes N = M \cap N$ *and* $M \rightarrow N = \begin{cases} X & if\ M \subseteq N, \\ N & if\ otherwise, \end{cases}$ *for every*

$M, N \in P(X)$ *and two matrices* $A = \begin{bmatrix} \{a\} & \{b,c\} & \{a,d\} \\ \{c\} & \{c,d\} & \{a,d\} \end{bmatrix}$ *and* $B = \begin{bmatrix} \{c\} & \{d\} \end{bmatrix}$.
Then $A \odot B = \begin{bmatrix} \{\} & \{\} & \{c\} & \{\} & \{\} & \{d\} \\ \{c\} & \{\} & \{c\} & \{d\} & \{\} & \{d\} \end{bmatrix}$.

Definition 3.3. Let $G = (\alpha, \beta)$ on $G^* = (V, E)$ be an RL-graph. Then the adjacency matrix of G is equal to the adjacency matrix of G^*. Also, the matrix of membership of vertices of G is the $n \times 1$ matrix $\alpha_G := (\alpha(u))$, where α is the membership of vertex u of G. Besides, the matrix of membership of edges of G is equal to the $n \times n$ matrix $\beta_G := (\beta(uv))$, where $\beta(uv)$ is the membership of edge joins vertices u and v.

Example 3.4. *Suppose* $L = ([0,1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *and a path RL-graph* $G = (\alpha_1, \beta_1)$ *on* $G^* = (V_1, E_1)$, *as in Figure 1, where* $a \otimes b = \begin{cases} (a+b-1) & if\ a+b \geq 1, \\ 0 & if\ a+b < 1, \end{cases}$ *and* $a \rightarrow b = \begin{cases} 1 & if\ b-a \geq 0, \\ (1-a+b) & if\ b-a < 0, \end{cases}$ $V_1 = \{q_1, q_2, q_3, q_4\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4\}$, $\beta_1(q_iq_j) = \alpha_1(q_i) \otimes \alpha_1(q_j)$, *for every* $q_iq_j \in E_1$, $\alpha_1(q_1) = 0.9$, $\alpha_1(q_2) = 0.7$, $\alpha_1(q_3) = 0.8$, $\alpha_1(q_4) = 1$, $\beta_1(q_1q_2) = 0.6$, $\beta_1(q_2q_3) = 0.5$ *and* $\beta_1(q_3q_4) = 0.8$.
Hence, $A_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\alpha_G = \begin{bmatrix} 0.9 \\ 0.7 \\ 0.8 \\ 1 \end{bmatrix}$ *and* $\beta_G = \begin{bmatrix} 0 & 0.6 & 0 & 0 \\ 0.6 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.8 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$.

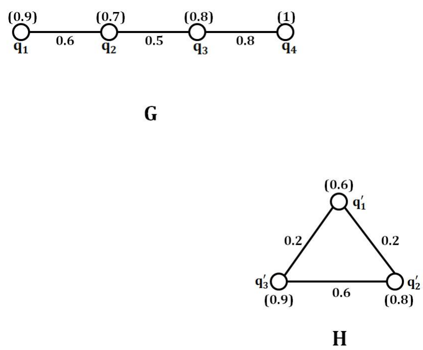

FIGURE 1. The path RL-graph G and the cycle RL-graph H

Definition 3.5. Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be two RL-graphs. Then a kronecker product of two RL-graphs G and H is defined by $K = G \otimes H = (\alpha, \beta)$ on $K^* = (V, E)$, where $A_K = A_G \otimes A_H$, $\alpha_K = \alpha_G \odot \alpha_H$ and $\beta_K = \beta_G \odot \beta_H$.

Theorem 3.6. *Let* $G = (\alpha_1, \beta_1)$ *and* $H = (\alpha_2, \beta_2)$ *be two RL-graphs on* $G^* = (V_1, E_1)$ *and* $H^* = (V_2, E_2)$, *respectively. Also, let* $K = (\alpha, \beta)$ *on*

$K^* = (V, E)$ be their kronecker product. Then K on K^*, is an RL-graph and $|V| = |V_1| \times |V_2|$.

Proof. Suppose $V_1 = \{q_1, q_2, \ldots, q_n\}$ and $V_2 = \{q'_1, q'_2, \ldots, q'_m\}$. So, we know $\alpha_G := (\alpha_1(q_i))$, where $q_i \in V_1$, for all $i \in \{1, 2, \ldots, n\}$, and $\alpha_H := (\alpha_2(q'_i))$, where $q'_i \in V_2$ for all $i \in \{1, 2, \ldots, m\}$. By using the definition of α_K, hence, $(\alpha_K)_{1,j+i} = \alpha_1(q_{\frac{i}{m}+1}) \otimes \alpha_2(q'_j)$ for all $j \in \{1, 2, \ldots, m\}$ and $i \in \{0, m, \ldots, nm\}$. Also, consider $\beta_G := (\beta_1(q_i q_j))$, where $q_i, q_j \in V_1$ for all $i, j \in \{1, 2, \ldots, n\}$ and $\beta_H := (\beta_2(q'_i q'_j))$, where $q'_i, q'_j \in V_2$ for all $i, j \in \{1, 2, \ldots, m\}$. Thus, $\beta_K := \beta_1(q_l q_s) \otimes \beta_H$, where $q_l, q_s \in V_1$ for all $l, s \in \{1, 2, \ldots, n\}$. As $\beta_1(q_l q_s) \otimes \beta_H := \beta_1(q_l q_s) \otimes \beta_2(q'_i q'_j)$, we know that we need to prove $\beta_1(q_l q_s) \otimes \beta_2(q'_i q'_j) \leq \alpha_1(q_l) \otimes \alpha_2(q'_i)$. Hence, by using the definitions of β_1 and β_2,

$$\begin{aligned}\beta_1(q_l q_s) \otimes \beta_2(q'_i q'_j) &\leq \alpha_1(q_l) \otimes \alpha_1(q_s) \otimes \alpha_2(q'_i) \otimes \alpha_2(q'_j) \\ &\leq \alpha_1(q_l) \otimes \alpha_2(q'_i) \textit{ by Propsition } 2.6(R_3).\end{aligned}$$

Therefore, K is the RL-graph on K^*, and $|V| = |V_1| \times |V_2|$. □

Example 3.7. *Consider the residuated lattice L, and the path RL-graph G in Example 3.4, and suppose the cycle RL-graph H, as in Figure 1, where $V_2 = \{q'_1, q'_2, q'_3\}$, $E_2 = \{q'_1 q'_2, q'_1 q'_3, q'_2 q'_3\}$, $\alpha_2(q'_1) = 0.6$, $\alpha_2(q'_2) = 0.8$, $\alpha_2(q'_3) = 0.9$, $\beta_2(q'_i q'_j) = (\alpha_2(q'_i) \wedge \alpha_2(q'_j)) \otimes (\alpha_2(q'_i) \wedge \alpha_2(q'_j))$ for every $q'_i q'_j \in E_2$, $\beta_2(q'_1 q'_2) = 0.2$, $\beta_2(q'_1 q'_3) = 0.2$ and $\beta_2(q'_2 q'_3) = 0.6$. So, $A_H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\alpha_H = \begin{bmatrix} 0.6 \\ 0.8 \\ 0.9 \end{bmatrix}$ and $\beta_H = \begin{bmatrix} 0 & 0.2 & 0.2 \\ 0.2 & 0 & 0.6 \\ 0.2 & 0.6 & 0 \end{bmatrix}$. Then $K = (\alpha, \beta)$ on $K^* = (V, E)$ is their kronecker product, as in Figure 2, where $V = \{q''_1, q''_2, \ldots, q''_{12}\}$, $E = \{q''_1 q''_5, q''_1 q''_6, q''_2 q''_4, q''_2 q''_6, q''_3 q''_4, q''_3 q''_5, q''_4 q''_8, q''_4 q''_9, q''_5 q''_7, q''_5 q''_9, q''_6 q''_7, q''_6 q''_8, q''_7 q''_{11}, q''_7 q''_{12}, q''_8 q''_{10}, q''_8 q''_{12}, q''_9 q''_{10}, q''_9 q''_{11}\}$, $\alpha(q''_1) = 0.5$, $\alpha(q''_2) = 0.7$, $\alpha(q''_3) = 0.8$, $\alpha(q''_4) = 0.3$, $\alpha(q''_5) = 0.5$, $\alpha(q''_6) = 0.6$, $\alpha(q''_7) = 0.4$, $\alpha(q''_8) = 0.6$, $\alpha(q''_9) = 0.7$, $\alpha(q''_{10}) = 0.6$, $\alpha(q''_{11}) = 0.8$, $\alpha(q''_{12}) = 0.9$, $\beta(q''_1 q''_5) = 0$, $\beta(q''_1 q''_6) = 0$, $\beta(q''_2 q''_4) = 0$, $\beta(q''_2 q''_6) = 0.2$, $\beta(q''_3 q''_4) = 0$, $\beta(q''_3 q''_5) = 0.2$, $\beta(q''_4 q''_8) = 0$, $\beta(q''_4 q''_9) = 0$, $\beta(q''_5 q''_7) = 0$, $\beta(q''_5 q''_9) = 0.1$, $\beta(q''_6 q''_7) = 0$, $\beta(q''_6 q''_8) = 0.1$, $\beta(q''_7 q''_{11}) = 0$, $\beta(q''_7 q''_{12}) = 0$, $\beta(q''_8 q''_{10}) = 0$, $\beta(q''_8 q''_{12}) = 0.4$, $\beta(q''_9 q''_{10}) = 0$ and $\beta(q''_9 q''_{11}) = 0.4$.*

Here, it is shown that the kronecker product of two RL-graphs is a commutative property (i.e., $G \otimes H = H \otimes G$), and it is expounded by an example. It is stated that the kronecker product of two strong RL-graphs is a strong RL-graph. In contrast, two RL-graphs are not strong while their kronecker product is the strong RL-graph. It is bounded α and β of the kronecker product of two RL-graphs by α of its constituent graphs and β of its constituent graphs, respectively. Besides, these are clarified by an example.

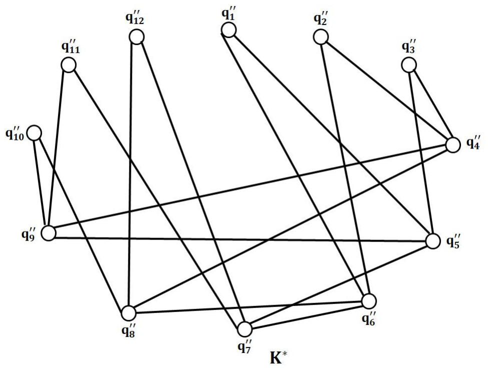

Figure 2. The graph K^*

Theorem 3.8. *Consider two RL-graphs $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$. Then $K = G \otimes H = (\alpha, \beta)$ on $K^* = (V, E)$ and $K' = H \otimes G = (\alpha', \beta')$ on ${K'}^* = (V', E')$ are isomorphic RL-graphs.*

Proof. Consider $V_1 = \{q_i |\, 1 \leq i \leq n\}, V_2 = \{q'_i |\, 1 \leq i \leq m\}, V = \{q''_i |\, 1 \leq i \leq mn\}$ and $V' = \{q'''_i |\, 1 \leq i \leq mn\}$. Let $h : V \longrightarrow V'$ be a map such that $h(q''_{j+i}) = q'''_{\frac{i}{m}+1+nj-n}$ for all $j \in \{1, 2, \ldots, m\}$ and $i \in \{o, m, \ldots, nm\}$. If $q''_{j+i} = q''_{j'+i'}$ so that $j, j' \in \{1, 2, \ldots, m\}$ and $i, i' \in \{0, m, \ldots, nm\}$, then $i = i'$ and $j = j'$. Hence,

$$h(q''_{j+i}) = q'''_{\frac{i}{m}+1+nj-n} = q'''_{\frac{i'}{m}+1+nj'-n} = h(q''_{j'+i'}).$$

It is well defined, and thus, it is a function. If $h(q''_{j+i}) = h(q''_{j'+i'})$ so that $j, j' \in \{1, 2, \ldots, m\}$ and $i, i' \in \{0, m, 2m, \ldots, nm\}$, then $q'''_{\frac{i}{m}+1+nj-n} = q'''_{\frac{i'}{m}+1+nj'-n}$. Hence, $\frac{i}{m}+1+nj-n = \frac{i'}{m}+1+nj'-n$. Thus, $\frac{i}{m}+nj = \frac{i'}{m}+nj'$. Since $\frac{i}{m}, \frac{i'}{m} \in \{0, 1, 2, \ldots, n\}$ and $nj, nj' \in \{n, 2n, \ldots, mn\}$, we have $i = i'$ and $j = j'$. Hence, $q''_{i+j} = q''_{i'+j'}$. Thus, h is a one-one function. We know that $A \otimes A' = A' \otimes A$. So, $q_{i+j} q_{i'+j'} \in E$ if and only if $h(q_{i+j})h(q_{i'+j'}) \in E'$. Obviously, it is also an onto function. We know that $\alpha(q''_{j+i}) = \alpha_1(q_{\frac{i}{m}+1}) \otimes \alpha_2(q'_j)$ such that $j \in \{1, 2, \ldots, m\}$ and $i \in \{o, m, \ldots, nm\}$, and $\alpha'(q'''_{j'+i'}) = \alpha_2(q'_{\frac{i'}{n}+1}) \otimes \alpha_1(q_{j'})$ such that $j' \in \{1, 2, \ldots, n\}$ and $i' \in \{o, n, \ldots, mn\}$. Assume $j' = \frac{i}{m} + 1$, and $i' = nj - n$. So,

$$\begin{aligned}
\alpha'(h(q''_{j+i})) &= \alpha'(q'''_{\frac{i}{m}+1+nj-n}) \\
&= \alpha_2(q'_{\frac{nj-n}{n}+1}) \otimes \alpha_1(q_{\frac{i}{m}+1}) \\
&= \alpha_2(q'_j) \otimes \alpha_1(q_{\frac{i}{m}+1}) \\
&= \alpha_1(q_{\frac{i}{m}+1}) \otimes \alpha_2(q'_j) \ \textit{By communitvty of} \otimes \\
&= \alpha(q''_{j+i}).
\end{aligned}$$

As $\beta(q''_{j+i}q''_{j'+i'}) = \beta_1(q_{\frac{i}{m}+1}q_{\frac{i'}{m}+1}) \otimes \beta_2(q'_jq'_{j'})$ so that $i, i' \in \{o, m, \dots, nm\}$ and $j, j' \in \{1, 2, \dots, m\}$, and $\beta'(q'''_{j+i}q'''_{j'+i'}) = \beta_2(q_{\frac{i}{n}+1}q_{\frac{i'}{n}+1}) \otimes \beta_1(q'_jq'_{j'})$ so that $i, i' \in \{o, n, \dots, mn\}$ and $j, j' \in \{1, 2, \dots, n\}$,

$$\begin{aligned}
\beta'(h(q''_{j+i})h(q''_{j'+i'})) &= \beta'(q'''_{\frac{i}{m}+1+nj-n}q'''_{\frac{i'}{m}+1+nj'-n}) \\
&= \beta_2(q'_{\frac{nj-n}{n}+1}q'_{\frac{nj'-n}{n}+1}) \\
&\otimes \beta_1(q_{\frac{i}{m}+1}q_{\frac{i'}{m}+1}) \\
&= \beta_2(q'_jq'_{j'}) \otimes \beta_1(q_{\frac{i}{m}+1}q_{\frac{i'}{m}+1}) \\
&= \beta(q''_{j+i}q''_{j'+i'}).
\end{aligned}$$

Thus, K and K' are two isomorphic RL-graphs. □

Example 3.9. *Consider the residuated lattice L, and two RL-graphs G and H and their kronecker product K in Example 3.7. Then $K' = (\alpha', \beta')$ on ${K'}^* = (V', E')$ is the kronecker product of two RL-graphs H and G, as shown in Figure 3, where $V' = \{q'''_1, q'''_2, \dots, q'''_{12}\}$, $E' = \{q'''_1q'''_6, q'''_1q'''_{10}, q'''_2q'''_5, q'''_2q'''_7, q'''_2q'''_9, q'''_2q'''_{11}, q'''_3q'''_6, q'''_3q'''_8, q'''_3q'''_{10}, q'''_3q'''_{12}, q'''_4q'''_7, q'''_4q'''_{11}, q'''_5q'''_{10}, q'''_6q'''_9, q'''_6q'''_{11}, q'''_7q'''_{10}, q'''_7q'''_{12}, q'''_8q'''_{11}\}$, $\alpha'(q'''_1) = 0.5$, $\alpha'(q'''_2) = 0.3$, $\alpha'(q'''_3) = 0.4$, $\alpha'(q'''_4) = 0.6$, $\alpha'(q'''_5) = 0.7$, $\alpha'(q'''_6) = 0.5$, $\alpha'(q'''_7) = 0.6$, $\alpha'(q'''_8) = 0.8$, $\alpha'(q'''_9) = 0.8$, $\alpha'(q'''_{10}) = 0.6$, $\alpha'(q'''_{11}) = 0.7$, $\alpha'(q'''_{12}) = 0.9$, $\beta'(q'''_1q'''_6) = 0$, $\beta'(q'''_1q'''_{10}) = 0$, $\beta'(q'''_2q'''_5) = 0$, $\beta'(q'''_2q'''_7) = 0$, $\beta'(q'''_2q'''_9) = 0$, $\beta'(q'''_2q'''_{11}) = 0$, $\beta'(q'''_3q'''_6) = 0$, $\beta'(q'''_3q'''_8) = 0$, $\beta'(q'''_3q'''_{10}) = 0$, $\beta'(q'''_3q'''_{12}) = 0$, $\beta'(q'''_4q'''_7) = 0$, $\beta'(q'''_4q'''_{11}) = 0$, $\beta'(q'''_5q'''_{10}) = 0.2$, $\beta'(q'''_6q'''_9) = 0.2$, $\beta'(q'''_6q'''_{11}) = 0.1$, $\beta'(q'''_7q'''_{10}) = 0.1$, $\beta'(q'''_7q'''_{12}) = 0.4$ and $\beta'(q'''_8q'''_{11}) = 0.4$. So, $h : V \longrightarrow V'$ is the function such that $h(q''_1) = q'''_1$, $h(q''_2) = q'''_5$, $h(q''_3) = q'''_9$, $h(q''_4) = q'''_2$, $h(q''_5) = q'''_6$, $h(q''_6) = q'''_{10}$, $h(q''_7) = q'''_3$, $h(q''_8) = q'''_7$, $h(q''_9) = q'''_{11}$, $h(q''_{10}) = q'''_4$, $h(q''_{11}) = q'''_8$ and $h(q''_{12}) = q'''_{12}$. Thus, K and K' are isomorphic RL-graphs.*

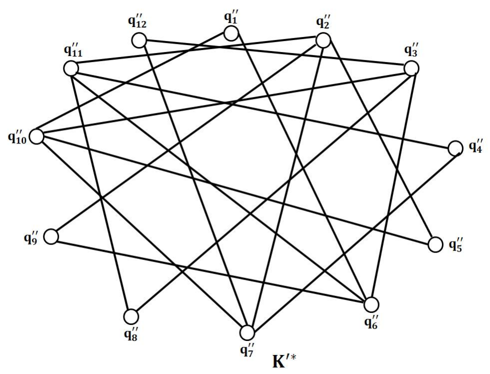

FIGURE 3. The graph ${K'}^*$

Theorem 3.10. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be two strong RL-graphs. Then their kronecker product is a strong RL-graph.*

Proof. The proof is similar as Theorem 3.6 with some modifications. □

Example 3.11. *Suppose $L = (\{1, 2, \ldots, 10\}, \vee, \wedge, \otimes, \rightarrow, 1, 10)$, and two strong RL-graph $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ in Figure 4, where $a \otimes b = \begin{cases} (a+b-10) & if\ a+b > 10, \\ 1 & if\ a+b \leq 10, \end{cases}$ and $a \rightarrow\ b = \begin{cases} 1 & if\ b-a \geq 0, \\ (1-a+b) & if\ b-a < 0, \end{cases}$ $V_1 = \{q_1, q_2, q_3\}$, $E_2 = \{q_1q_2, q_2q_3\}$, $\alpha_1(q_1) = 5$, $\alpha_1(q_2) = 7$, $\alpha_1(q_3) = 9$, $\beta_1(q_1q_2) = 2$, $\beta_1(q_2q_3) = 6$, $V_2' = \{q_1', q_2', q_3', q_4'\}$, $E_2' = \{q_1'q_2', q_2'q_3', q_3'q_4', q_1'q_4'\}$, $\alpha_2(q_1') = 7$, $\alpha_2(q_2') = 8$, $\alpha_2(q_3') = 9$, $\alpha_2(q_4') = 5$, $\beta_2(q_1'q_2') = 5$, $\beta_2(q_2'q_3') = 7$, $\beta_2(q_3'q_4') = 4$ and $\beta_2(q_1'q_4') = 2$. Then their kronecker product is $K = (\alpha, \beta)$ on $K^* = (V, E)$, as in Figure 5, where $V = \{q_1'', q_2'', \ldots, q_{12}''\}$, $E = \{q_1''q_6'', q_1''q_8'', q_2''q_5'', q_2''q_7'', q_3''q_6'', q_3''q_8'', q_4''q_5'', q_4''q_7'', q_5''q_{10}'', q_5''q_{12}'', q_6''q_9'', q_6''q_{11}'', q_7''q_{10}'', q_7''q_{12}'', q_8''q_9'', q_8''q_{11}''\}$, $\alpha(q_1'') = 2$, $\alpha(q_2'') = 3$, $\alpha(q_3'') = 4$, $\alpha(q_4'') = 1$, $\alpha(q_5'') = 4$, $\alpha(q_6'') = 5$, $\alpha(q_7'') = 6$, $\alpha(q_8'') = 2$, $\alpha(q_9'') = 6$, $\alpha(q_{10}'') = 7$, $\alpha(q_{11}'') = 8$, $\alpha(q_{12}'') = 4$, $\beta(q_1''q_6'') = 1$, $\beta(q_1''q_8'') = 1$, $\beta(q_2''q_5'') = 1$, $\beta(q_2''q_7'') = 1$, $\beta(q_3''q_6'') = 1$, $\beta(q_3''q_8'') = 1$, $\beta(q_4''q_7'') = 1$, $\beta(q_4''q_5'') = 1$, $\beta(q_5''q_{10}'') = 1$, $\beta(q_5''q_{12}'') = 1$, $\beta(q_6''q_9'') = 1$, $\beta(q_6''q_{11}'') = 3$, $\beta(q_7''q_{10}'') = 3$, $\beta(q_7''q_{12}'') = 1$, $\beta(q_8''q_9'') = 1$ and $\beta(q_8''q_{11}'') = 1$. We can see K is a strong RL-graph.*

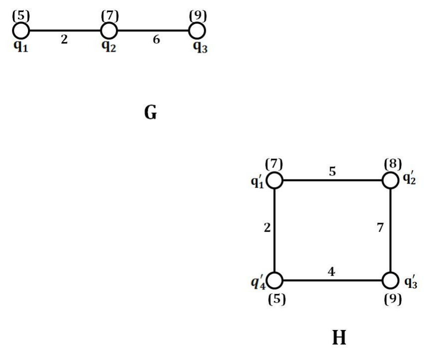

FIGURE 4. The strong RL-graphs G and H

Remark 3.12. The following example indicates that it is possible that kronecker product of two RL-graphs is a strong RL-graphs while they are not strong RL-graphs.

Example 3.13. *Suppose L in Example 3.11, and two RL-graphs $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$, as in Figure 6, where*

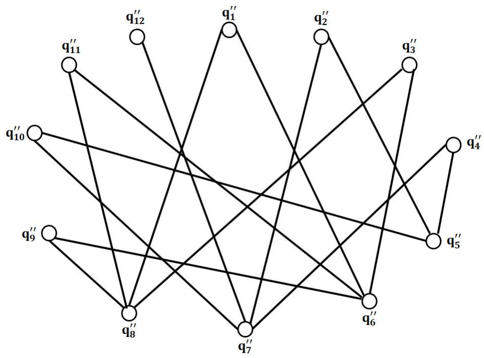

FIGURE 5. The graph K^*

$V_1 = \{q_1, q_2\}$, $E_1 = \{q_1q_2\}$, $\alpha_1(q_1) = 9$, $\alpha_1(q_2) = 6$, $\beta_1(q_1q_2) = 1$, $V_2 = \{q'_1, q'_2\}$, $E_2 = \{q'_1q'_2\}$, $\alpha_2(q'_1) = 7$, $\alpha_2(q'_2) = 3$ and $\beta_2(q'_1q'_2) = 1$. Then their kronecker product is $K = (\alpha, \beta)$ on $K^* = (V, E)$, as in Figure 6, where $V = \{q''_1, q''_2, \ldots, q''_4\}$, $E = \{q''_1q''_4, q''_2q''_3\}$, $\alpha(q''_1) = 6$, $\alpha(q''_2) = 2$, $\alpha(q''_3) = 3$, $\alpha(q''_4) = 1$, $\beta(q''_1q''_4) = 1$ and $\beta(q''_2q''_3) = 1$. Clearly, K is a strong RL-graph but G is not a strong RL-graph.

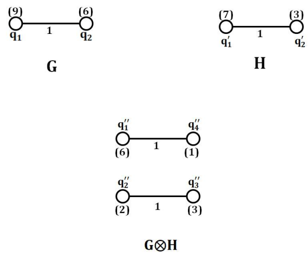

FIGURE 6. The RL-graphs G and H, and their kronecker product $G \otimes H$

Proposition 3.14. *Let $G = (\alpha_1, \beta_1)$ and $H = (\alpha_2, \beta_2)$ be two RL-graphs on $G^* = (V_1, E_1)$ and $H^* = (V_2, E_2)$, respectively. Then the kronecker product of them is the RL-graph $K = (\alpha, \beta)$ on $K^* = (V, E)$ such that*

$$\bigwedge_{q\in V_1} \alpha_1(q) \otimes \bigwedge_{q\in V_2} \alpha_2(q) = \bigwedge_{q\in V} \alpha(q) \leq \alpha(q) \leq \bigvee_{q\in V} \alpha(q) = \bigvee_{q\in V_1} \alpha_1(q) \otimes \bigvee_{q\in V_2} \alpha_2(q),$$

$$\bigwedge_{qq'\in E_1} \beta_1(qq') \otimes \bigwedge_{qq'\in E_2} \beta_2(qq') = \bigwedge_{qq'\in E} \beta(qq') \leq \beta(qq')$$

and

$$\beta(qq') \leq \bigvee_{qq' \in E} \beta(qq') = \bigvee_{qq' \in E_1} \beta_1(qq') \otimes \bigvee_{qq' \in E_2} \beta_2(qq')$$

for every $q \in V$ *and for every* $qq' \in E$.

Example 3.15. *Let* G *and* H *be two RL-graphs, and the kronecker product of them* K *in Example 3.11. Then*

$$\begin{aligned}1 = 5 \otimes 5 = \bigwedge_{q \in V_1} \alpha_1(q) \otimes \bigwedge_{q \in V_2} \alpha_2(q) = \bigwedge_{q \in V} \alpha(q) &\leq 8 \\ &= \bigvee_{q \in V} \alpha(q) \\ &= \bigvee_{q \in V_1} \alpha_1(q) \otimes \bigvee_{q \in V_2} \alpha_2(q) \\ &= 9 \otimes 9 = 8,\end{aligned}$$

and

$$\begin{aligned}1 = 2 \otimes 2 = \bigwedge_{qq' \in E_1} \beta_1(qq') \otimes \bigwedge_{qq' \in E_2} \beta_2(qq') &= \bigwedge_{qq' \in E} \beta(qq') \\ &\leq 3 \\ &= \bigvee_{qq' \in E} \beta(qq') \\ &\leq \bigvee_{qq' \in E_1} \beta_1(qq') \otimes \bigvee_{qq' \in E_2} \beta_2(qq') \\ &= 6 \otimes 7 \\ &= 3.\end{aligned}$$

Here, a disconnected RL-graph is determined. Moreover, if H is an RL-graph, and G and G' are two isomorphic RL-graphs, then the kronecker product of G and H and the kronecker product of G' and H are isomorphic RL-graphs. This theorem is illuminated by an example. It is also stated that if at least one of two RL-graphs is a disconnected RL-graph then their kronecker product is a disconnected RL-graph as well. This issue is explicated through some examples. Additionally, some notions such as regular RL-graphs, α-regular RL-graphs, and totally regular RL-graphs are defined. Then, it is stated that the kronecker product of two totally regular RL-graphs is totally regular RL-graphs.

Definition 3.16. Let $G = (\alpha, \beta)$ on $G^* = (V, E)$ be an RL-graph, while G^* is a disconnected graph. Then G is called the disconnected RL-graph.

Example 3.17. *Let* $L = (\{1, 2, \dots, 10\}, \vee, \wedge, \otimes, \rightarrow, 1, 10)$, *where* $a \otimes b = a \wedge b$ *and* $a \rightarrow b = \begin{cases} 10 & if\ a \leq b, \\ b & if\ b < a, \end{cases}$ *and an RL-graph* $H = (\alpha, \beta)$ *on* $H^* = (V, E)$, *as in Figure 7, where* $V' = \{a_1, a_2, a_3\}$, $E' = \{a_1a_2\}$, $\alpha(a_1) = 7$, $\alpha(a_2) = 8$, $\alpha(a_3) = 1$ *and* $\beta(a_1a_2) = 7$. *Clearly, this RL-graph is disconnected.*

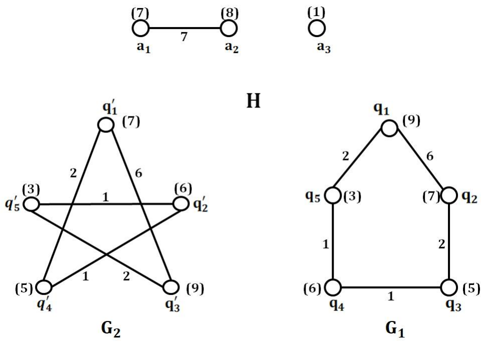

FIGURE 7. The RL-graphs H, G_1 and G_2

Example 3.18. *Let L in Example 3.17, and two RL-graphs $G = (\alpha_1, \beta_1)$ and $H = (\alpha_2, \beta_2)$ on $G^* = (V_1, E_1)$ and $H^* = (V_2, E_2)$, respectively, as in Figure 8, where $V_1 = \{q_1, q_2, q_3, q_4\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_1q_4\}$, $\alpha_1(q_1) = 9$, $\alpha_1(q_2) = 7$, $\alpha_1(q_3) = 6$, $\alpha_1(q_4) = 2$, $\beta_1(q_1q_2) = 6$, $\beta_1(q_2q_3) = 3$, $\beta_1(q_3q_4) = 1$, $\beta_1(q_1q_4) = 2$, $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$, $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_1q'_4\}$, $\alpha_2(q'_1) = 9$, $\alpha_2(q'_2) = 6$, $\alpha_2(q'_3) = 6$, $\alpha_2(q'_4) = 2$, $\beta_2(q'_1q'_2) = 6$, $\beta_2(q'_2q'_3) = 3$, $\beta_2(q'_3q'_4) = 1$ and $\beta_2(q'_1q'_4) = 1$. Also, let a function $h : V_1 \to V_2$ such that $h(q_i) = q'_i$ for every $1 \leq i \leq 4$. Hence, we can see that these two RL-graphs are isomorphic with threshold $c_1 = 7$ but these two RL-graphs are not isomorphic with threshold $c_2 = 6$.*

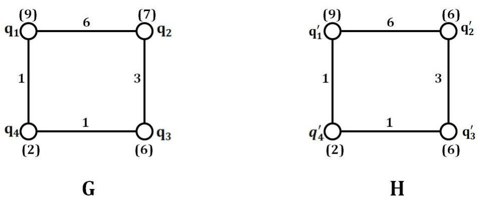

FIGURE 8. The RL-graphs G and H

Remark 3.19. The above example indicated that it is possible that two RL-graphs are isomorphic with threshold c_1, however they are not isomorphic with threshold c_2.

Theorem 3.20. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be two RL-graphs and let $G' = (\alpha'_1, \beta'_1)$ be an RL-graph on $G'^* = (V'_1, E'_1)$ such that G and G' are two isomorphic RL-graphs. Then $K = G \otimes H = (\alpha, \beta)$ and $K' = G' \otimes H' = (\alpha', \beta')$ are two isomorphic RL-graphs on $K^* = (V, E)$ and $K'^* = (V', E')$, respectively.*

Proof. By using the proof of Theorem 3.6 and the definition of isomorphic two RL-graphs, these RL-graphs are isomorphic. □

Example 3.21. *Suppose L and the RL-graph H in Example 3.17, and two isomorphic RL-graphs $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, as in Figure 7, where $V_1 = \{q_1, q_2, q_3, q_4, q_5\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_5, q_1q_5\}$, $\alpha_1(q_1) = 9$, $\alpha_1(q_2) = 7$, $\alpha_1(q_3) = 5$, $\alpha_1(q_4) = 6$, $\alpha_1(q_5) = 3$, $\beta_1(q_1q_2) = 6$, $\beta_1(q_1q_5) = 2$, $\beta_1(q_3q_4) = 1$, $\beta_1(q_2q_3) = 2$, $\beta_1(q_4q_5) = 1$, $V_2 = \{q'_1, q'_2, q'_3, q'_4, q'_5\}$, $E_2 = \{q'_1q'_3, q'_1q'_4, q'_2q'_4, q'_2q'_5, q'_3q'_5\}$, $\alpha_2(q'_1) = 7$, $\alpha_2(q'_2) = 6$, $\alpha_2(q'_3) = 9$, $\alpha_2(q'_4) = 5$, $\alpha_2(q'_5) = 3$, $\beta_2(q'_1q'_3) = 6$, $\beta_2(q'_1q'_4) = 2$, $\beta_2(q'_2q'_5) = 1$, $\beta_2(q'_2q'_4) = 1$ and $\beta_2(q'_3q'_5) = 2$. Additionally, let h be a function between G_1 and G_2, where $h(q_1) = q'_3$, $h(q_2) = q'_1$, $h(q_3) = q'_4$, $h(q_4) = q'_2$ and $h(q_5) = q'_5$. Clearly G_1 and G_2 are isomorphic RL-graphs. Then the kronecker product of G_1 and H is the RL-graph $G_1 \otimes H = (\alpha', \beta')$ on $(G_1 \otimes H)^* = (V', E')$, as in Figure 9, where $V' = \{u_1, u_2, \ldots, u_{15}\}$, $E' = \{u_1u_5, u_1u_{14}, u_2u_4, u_2u_{13}, u_4u_8, u_5u_7, u_7u_{11}, u_8u_{10}, u_{10}u_{14}, u_{11}u_{13}\}$, $\alpha'(u_1) = 7$, $\alpha'(u_2) = 8$, $\alpha'(u_3) = 1$, $\alpha'(u_4) = 7$, $\alpha'(u_5) = 7$, $\alpha'(u_6) = 1$, $\alpha'(u_7) = 5$, $\alpha'(u_8) = 5$, $\alpha'(u_9) = 1$, $\alpha'(u_{10}) = 6$, $\alpha'(u_{11}) = 6$, $\alpha'(u_{12}) = 1$, $\alpha'(u_{13}) = 3$, $\alpha'(u_{14}) = 3$, $\alpha'(u_{15}) = 1$, $\beta'(u_1u_5) = 6$, $\beta'(u_1u_{14}) = 2$, $\beta'(u_2u_4) = 6$, $\beta'(u_2u_{13}) = 2$, $\beta'(u_4u_8) = 2$, $\beta'(u_5u_7) = 2$, $\beta'(u_7u_{11}) = 1$, $\beta'(u_8u_{10}) = 1$, $\beta'(u_{10}u_{14}) = 1$ and $\beta'(u_{11}u_{13}) = 1$. Also, the kronecker product of G_2 and H is the RL-graph $G_2 \otimes H = (\alpha'', \beta'')$ on $(G_2 \otimes H)^* = (V'', E'')$, as in Figure 10, where $V'' = \{v_1, v_2, \ldots, v_{15}\}$, $E'' = \{v_1v_8, v_1v_{11}, v_2v_7, v_2v_{10}, v_4v_{11}, v_4v_{14}, v_5v_{10}, v_5v_{13}, v_7v_{14}, v_8v_{13}\}$, $\alpha''(v_1) = 7$, $\alpha''(v_2) = 7$, $\alpha''(v_3) = 1$, $\alpha''(v_4) = 6$, $\alpha''(v_5) = 6$, $\alpha''(v_6) = 1$, $\alpha''(v_7) = 7$, $\alpha''(v_8) = 8$, $\alpha''(v_9) = 1$, $\alpha''(v_{10}) = 5$, $\alpha''(v_{11}) = 5$, $\alpha''(v_{12}) = 1$, $\alpha''(v_{13}) = 3$, $\alpha''(v_{14}) = 3$, $\alpha''(v_{15}) = 1$, $\beta''(v_1v_8) = 6$, $\beta''(v_1v_{11}) = 2$, $\beta''(v_2v_7) = 6$, $\beta''(v_2v_{10}) = 2$, $\beta''(v_4v_{11}) = 1$, $\beta''(v_4v_{14}) = 1$, $\beta''(v_5v_{10}) = 1$, $\beta''(v_5v_{13}) = 1$, $\beta''(v_7v_{14}) = 2$ and $\beta''(u_8u_{13}) = 2$. So, we define a function $g : V' \longrightarrow V''$ such that $g(u_1) = v_7$, $g(u_2) = v_8$, $g(u_3) = v_3$, $g(u_4) = v_1$, $g(u_5) = v_2$, $g(u_6) = v_6$, $g(u_7) = v_{10}$, $g(u_8) = v_{11}$, $g(u_9) = v_9$, $g(u_{10}) = v_4$, $g(u_{11}) = v_5$, $g(u_{12}) = v_{12}$, $g(u_{13}) = v_{13}$ and $g(u_{14}) = v_{14}$. So, they are isomorphic RL-graphs.*

Theorem 3.22. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be two RL-graphs, while G is a disconnected RL-graph and it has $P_i = (\alpha_{1i}, \beta_{1i})$ on $P_i^* = (V_{1i}, E_{1i})$ partitions. Then $K = (\alpha, \beta)$ on $K^* = (V, E)$ is their kronecker product, where K^* is a disconnected graph,*

$$\bigvee_{q \in V} \alpha(q) = \bigvee_{i} \bigvee_{q' \in V_2} \bigvee_{q \in V_{1i}} \alpha_{1i}(q) \otimes \alpha_2(q')$$

and

$$\bigvee_{qq' \in E} \beta(qq') = \bigvee_{i} \bigvee_{qq' \in E_2} \bigvee_{q_{ii}q_{ij} \in E_{1i}} \beta_{1i}(q_{ii}q_{ij}) \otimes \beta_2(qq').$$

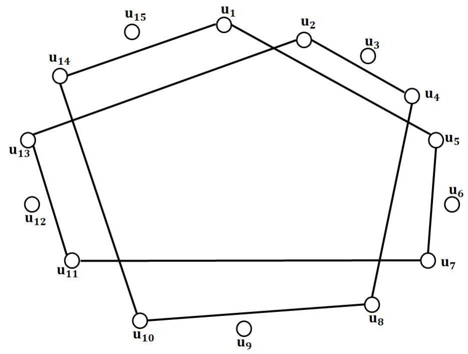

FIGURE 9. The graph $(G_1 \otimes H)^*$

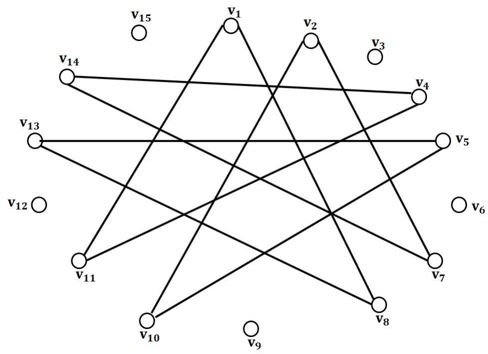

FIGURE 10. The graph $(G_2 \otimes H)^*$

Proof. Since G^* is the disconnected simple graph and H^* is the connected simple graph, by using the definition of the kronecker product of two matrices, K^* is the disconnected graph. The rest of the proofs are straightforward. □

Example 3.23. *Let two RL-graphs G_1 and H in Example 3.21 while H is a disconnected RL-graphs. According to the Example 3.21, we see that $G_1 \otimes H$ on $(G_1 \otimes H)^*$ is the disconnected RL-graphs.*

$$\begin{aligned}\bigvee_{q\in V'} \alpha'(q) &= 8\\ &= (9\otimes 1)\vee(8\otimes 9)\\ &= \bigvee_{q\in V_1}\bigvee_{q'\in V_{21}} \alpha_1(q)\otimes\ \alpha_{21}(q')\vee \bigvee_{q\in V_1}\bigvee_{q'\in V_{22}} \alpha_1(q)\otimes\ \alpha_{22}(q').\end{aligned}$$

Definition 3.24. Let $G = (\alpha, \beta)$ on $G^* = (V, E)$ be an *RL*-graph that G^* is a regular graph. Then G is called the regular *RL*-graph. If α has the same value for all vertices of the regular RL-graph G, then G is α-regular RL-graph. Additionally, if β has the same value for all edges of the regular RL-graph G, then G is β-regular RL-graph. Besides, it is a totally regular RL-graph if G is α-regular and β-regular RL-graph.

Example 3.25. *Consider L in Example 3.2, and an RL-graph $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$, as Figure 11, where $V_1 = \{q_1, q_2, q_3, q_4\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_1q_4, q_2q_4, q_1q_3\}$, $\alpha_1(q_i) = \{a, b, c\}$, for every $q_i \in V_1$, $\beta_1(q_iq_j) = \{a, b\}$, for every $q_iq_j \in E_1$. Then this RL-graph is totally regular RL-graph.*

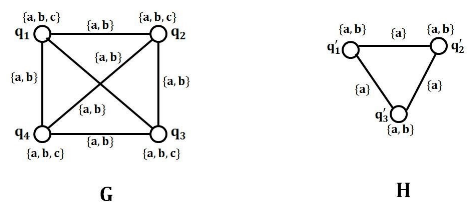

FIGURE 11. The totally regular RL-graphs G and H

Theorem 3.26. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be two RL-graphs. Then*

(i) *If they are regular RL-graphs, then their kronecker product is a regular RL-graph.*
(ii) *If they are α-regular RL-graphs, then their kronecker product is an α-regular RL-graph.*
(iii) *If they are β-regular RL-graphs, then their kronecker product is a β-regular RL-graph.*
(iv) *If they are totally regular RL-graphs, then their kronecker product is a totally regular RL-graph.*

Proof. (i) Consider that G is a k-regular RL-graph, and H is a k'-regular RL-graph. Since every vertices of k-regular RL-graph G connect to k vertices, each row of its adjacency matrix has k rows equal to 1. So, when this matrix is kronecker product by the adjacency matrix H, then each row will have $k \times k'$ rows equal to 1. Besides, their kronecker product is $k \times k'$-regular RL-graph. The proof of (ii), (iii), and (iv) are similar to above by some modifications. □

Example 3.27. *Let L and RL-graph G in Example 3.25, and a totally regular RL-graph $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$, as in Figure 11, $V_2 = \{q'_1, q'_2, q'_3\}$, $E_2 = \{q'_1q'_2, q'_2q'_3, q'_1q'_3\}$, $\alpha_2(q'_i) = \{a, b\}$ for every $q'_i \in V_2$ and $\beta_2(q'_iq'_j) = \{a\}$ for every $q_iq_j \in E_2$. So, these RL-graphs are totally regular RL-graphs. Consider their kronecker product $G \otimes H = (\alpha, \beta)$ on $(G \otimes H)^* = (V, E)$, as in Figure 12, where $V = \{q''_1, q''_2, \dots, q''_{12}\}$, $E = \{q''_1q''_5, q''_1q''_6, q''_1q''_8, q''_1q''_9, q''_1q''_{11}, q''_1q''_{12}, q''_2q''_4, q''_2q''_6,$*

$q''_2q''_7, q''_2q''_9, q''_2q''_{10}, q''_2q''_{12}, q''_3q''_4, q''_3q''_5, q''_3q''_7, q''_3q''_8, q''_3q''_{10}, q''_3q''_{11}, q''_4q''_8, q''_4q''_9, q''_4q''_{11}, q''_4q''_{12}, q''_5q''_7, q''_5q''_9, q''_5q''_{10}, q''_5q''_{12}, q''_6q''_7, q''_6q''_8, q''_6q''_{10}, q''_6q''_{11}, q''_7q''_{11}, q''_7q''_{12}, q''_8q''_{10}, q''_8q''_{12}, q''_9q''_{10}, q''_9q''_{11}\}$, $\alpha(q''_i) = \{a, b\}$ *for every* $q''_i \in V$ *and* $\beta(q''_iq''_j) = \{a\}$ *for every* $q''_iq''_j \in E$. *So, it is totally regular RL-graph.*

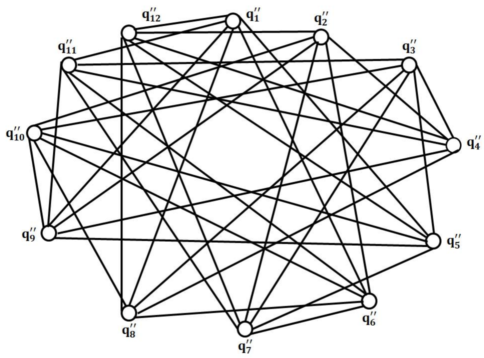

FIGURE 12. The graph $(G \otimes H)^*$

4. Applications of the kronecker product of two RL-graphs

The kronecker product of two RL-graphs has some applications. In this section, two applications of this operation are stated that one of them is determined the maximum efficiency work among its members and anther one is estimated the maximum interact among its members. The issue is clarified by an example.

Application 4.1. **a:** Let two construction companies. We tend to calculate their work efficiency when these two companies work together. Accordingly, we model these two companies by RL-graphs and we calculate their work efficiency when these two companies work together by the kronecker product.

If $L = (\{1, 2, \ldots, 100\}, \vee, \wedge, \otimes, \rightarrow, 1, 10)$, where $a \otimes b = a \wedge b$, $a \rightarrow b = \begin{cases} 100 & if\ a \leq b, \\ b & if\ b < a, \end{cases}$ then the first company is modeled the RL-graph $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$, where

(i) Each member of this company is labeled with a_i for every $1 \leq i \leq n$. So, $V_1 = \{a_i | \ 1 \leq i \leq n\}$.
(ii) We put an edge between the two members a_i and a_j for every a_i and a_j. So, $E_1 = \{a_ia_j | \ 1 \leq i \neq j \leq n\}$.
(iii) $\alpha_1(a_i)$ equals the amount of work efficiency.
(iv) $\beta_1(a_ia_j) = \alpha(a_i) \otimes \alpha(a_j)$ for every two members a_i and a_j.

By considering a suitable change, the second company, like the first company, is represented by an RL-graph $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$. So, their kronecker product $K = (\alpha, \beta)$ on $K^* = (V, E)$ can now be used to determine their work efficiency. Thus, the maximum of β is the maximum of their work efficiency.

b: Consider two separate groups working in a specific field. The first group has n members but not all of these people have the same social interaction with each other. Hence, $L = (\{1, 2, \ldots, 10\}, \vee, \wedge, \otimes, \rightarrow, 1, 10)$, where $a \otimes b = \begin{cases} (a+b-10) & if\ a+b > 10, \\ 1 & if\ a+b \leq 10, \end{cases}$ $a \rightarrow b = \begin{cases} 10 & if\ b-a \geq 0, \\ (10-a+b) & if\ b-a < 0, \end{cases}$ and this group is modeled by RL-graph $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$, where

(i) each of these people in this group is labeled with a_i for every $1 \leq i \leq n$ and $V_1 = \{a_i | 1 \leq i \leq n\}$,

(ii) if two people from this group have worked together so far, they will be connected to each other by one edge, which is shown with two vertices,

(iii) $\alpha_1(a_i)$ equals the amount of interaction people have,

(iv) the interaction of two people with $\beta_1(a_i a_j)$ is shown.

By considering a suitable change, the second group, like the first group, is represented by an RL-graph $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$. So, their kronecker products can now be used to estimate compatibility and interaction, and determine the four people who will interact the most. Thus, the maximum of β of the kronecker products is the maximum interact groups that have four members.

Example 4.2. **a:** *Let A and B be two construction companies. Also, the company A has* 4 *members that are labeled by* a_1, a_2, a_3 *and* a_4 *such that the efficiency of* a_1 *work equals* %80*, the efficiency of* a_2 *work equals* %70*,* a_3 *efficiency work equals* %20 *and* a_4 *efficiency work equals* %50*. So, the company A is modeled by the RL-graph* $G = (\alpha_1, \beta_1)$ *on* $G^* = (V_1, E_1)$*, as in Figure 13, where* $V_1 = \{a_1, a_2, a_3, a_4\}$, $E_1 = \{a_1a_2, a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_3a_4\}$, $\alpha_1(a_1) = 80$, $\alpha_1(a_2) = 70$, $\alpha_1(a_3) = 20$, $\alpha_1(a_4) = 50$, $\beta_1(a_1a_2) = 70$, $\beta_1(a_1a_3) = 20$, $\beta_1(a_1a_4) = 50$, $\beta_1(a_2a_3) = 20$, $\beta_1(a_2a_4) = 50$ *and* $\beta_1(a_3a_4) = 20$*. On the other hands, the company B has* 3 *members that are labeled by* b_1, b_2 *and* b_3 *such that the efficiency of* b_1 *work equals* %40*, the efficiency of* b_2 *work equals* %50 *and* b_3 *efficiency work equals* %90*. So, the company B is modeled by the RL-graph* $H = (\alpha_2, \beta_2)$ *on* $H^* = (V_2, E_2)$*, as in Figure 13, where* $V_2 = \{b_1, b_2, b_3\}$, $E_2 = \{b_1b_2, b_1b_3, b_2b_3\}$, $\alpha_2(b_1) = 40$, $\alpha_2(b_2) = 50$, $\alpha_2(b_3) = 90$, $\beta_2(b_1b_2) = 40$, $\beta_2(b_1b_3) = 40$ *and* $\beta_2(b_2b_3) = 50$*. Then their kronecker product is* $K = (\alpha, \beta)$ *on* $K^* = (V, E)$*, as in Figure 13, where* $V = \{c_1, c_2, \ldots, c_{12}\}$, $E = \{c_1c_5, c_1c_6, c_1c_8, c_1c_9, c_1c_{11}, c_1c_{12}, c_2c_4,$

$c_2c_6, c_2c_7, c_2c_9, c_2c_{10}, c_2c_{12}, c_3c_4, c_3c_5, c_3c_7, c_3c_8, c_3c_{10}, c_3c_{11}, c_4c_8, c_4c_9, c_4c_{11}, c_4c_{12}, c_5c_7, c_5c_9, c_5c_{10}, c_5c_{12}, c_6c_7, c_6c_8, c_6c_{10}, c_6c_{11}, c_7c_{11}, c_7c_{12}, c_8c_{10}, c_8c_{12}, c_9c_{10}, c_9c_{11}\}$, $\alpha(c_1) = 40$, $\alpha(c_2) = 50$, $\alpha(c_3) = 80$, $\alpha(c_4) = 40$, $\alpha(c_5) = 50$, $\alpha(c_6) = 70$, $\alpha(c_7) = 20$, $\alpha(c_8) = 20$, $\alpha(c_9) = 20$, $\alpha(c_{10}) = 40$, $\alpha(c_{11}) = 50$, $\alpha(c_{12}) = 50$, $\beta(c_1c_5) = 40$, $\beta(c_1c_6) = 40$, $\beta(c_1c_8) = 20$, $\beta(c_1c_9) = 20$, $\beta(c_1c_{11}) = 40$, $\beta(c_1c_{12}) = 40$, $\beta(c_2c_4) = 40$, $\beta(c_2c_6) = 50$, $\beta(c_2c_7) = 20$, $\beta(c_2c_9) = 20$, $\beta(c_2c_{10}) = 40$, $\beta(c_2c_{12}) = 50$, $\beta(c_3c_4) = 40$, $\beta(c_3c_5) = 50$, $\beta(c_3c_7) = 20$, $\beta(c_3c_8) = 20$, $\beta(c_3c_{10}) = 40$, $\beta(c_3c_{11}) = 50$, $\beta(c_4c_8) = 20$, $\beta(c_4c_9) = 20$, $\beta(c_4c_{11}) = 40$, $\beta(c_4c_{12}) = 40$, $\beta(c_5c_7) = 20$, $\beta(c_5c_9) = 20$, $\beta(c_5c_{10}) = 40$, $\beta(c_5c_{12}) = 50$, $\beta(c_6c_7) = 20$, $\beta(c_6c_8) = 20$, $\beta(c_6c_{10}) = 40$, $\beta(c_6c_{11}) = 50$, $\beta(c_7c_{11}) = 20$, $\beta(c_7c_{12}) = 20$, $\beta(c_8c_{10}) = 20$, $\beta(c_8c_{12}) = 20$, $\beta(c_9c_{10}) = 20$ and $\beta(c_9c_{11}) = 20$. Since

$$\begin{aligned}\bigvee_{q_iq_j \in V} \beta(q_iq_j) &= 50 \\ &= \beta(c_2c_6) = \beta(c_2c_{12}) = \beta(c_3c_{11}) = \beta(c_3c_5) \\ &= \beta(c_5c_{12}) = \beta(c_6c_{11}),\end{aligned}$$

the group that includes people a_1, a_2, b_2 and b_3 or a_1, a_4, b_2 and b_3 or a_2, a_4, b_2 and b_4 or a_2, a_4, b_2 and b_3 has the maximum work efficiency.

b: *Suppose two groups that the first group has three members that are labeled by a_1, a_2 and a_3, where the interact of a_1 equals 9, the interact of a_2 equals 6, the interact of a_3 equals 4, the interact of a_1a_2 equals 5, the interact of a_1a_3 equals 3 and the interact of a_2a_3 equals 1. The second group has four members that are labeled by b_1, b_2, b_3 and b_4, where the interact of b_1 equals 10, the interact of b_2 equals 9, the interact of b_3 equals 5, the interact of b_4 equals 3, the interact of b_1b_2 equals 9, the interact of b_2b_3 equals 4, the interact of b_3b_4 equals 1 and the interact of b_1b_4 equals 3. Also, $L = (\{1, 2, \dots, 10\}, \vee, \wedge, \otimes, \rightarrow, 1, 10)$, where*

$$a \otimes b = \begin{cases} (a + b - 10) & if\ a + b > 10, \\ 1 & if\ a + b \leq 10, \end{cases}$$

$$a \rightarrow b = \begin{cases} 10 & if\ b - a \geq 0, \\ (10 - a + b) & if\ b - a < 0. \end{cases}$$

Then, their models are two RL-graphs $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ as in Figure 14, where $V_1 = \{a_1, a_2, a_3\}$, $E_1 = \{a_1a_2, a_1a_3, a_2a_3\}$, $V_2 = \{b_1, b_2, b_3, b_4\}$, $E_2 = \{b_1b2, b_2b_3, b_3b_4, b_1b_4\}$, $\alpha_1(a_1) = 9$, $\alpha_1(a_2) = 6$, $\alpha_1(a_3) = 4$, $\beta_1(a_1a_2) = 5$, $\beta_1(a_1a_3) = 3$, $\beta_1(a_2a_3) = 1$, $\alpha_2(b_1) = 10$, $\alpha_2(b_2) = 9$, $\alpha_2(b_3) = 5$, $\alpha_2(b_4) = 3$, $\beta_2(b_1b_2) = 9$, $\beta_2(b_2b_3) = 4$, $\beta_2(b_3b_4) = 1$, and $\beta_2(b_1b_4) = 3$. So, their kronecker product RL-graph is $K = (\alpha, \beta)$ on $K^* = (V, E)$, as in Figure 14, where $V = \{c_1, c_2, \dots c_{12}\}$, $E = \{c_1c_6, c_1c_8, c_1c_{10}, c_1c_{12}, c_2c_5, c_2c_7, c_2c_9, c_2c_{11}, c_3c_6, c_3c_8, c_3c_{10}, c_3c_{12}, c_4c_5, c_4c_7, c_4c_9, c_4c_{11}, c_5c_{10}, c_5c_{12}, c_6c_9, c_6c_{11}, c_7c_{10}, c_7c_{12}, c_8c_9, c_8c_{11}\}$, $\alpha(c_1) = 9$, $\alpha(c_2) = 3$, $\alpha(c_3) = 4$, $\alpha(c_4) = 2$, $\alpha(c_5) = 6$, $\alpha(c_6) = 5$, $\alpha(c_7) = 1$, $\alpha(c_8) = 1$, $\alpha(c_9) = 4$, $\alpha(c_{10}) = 3$,

$\alpha(c_{11}) = 1$, $\alpha(c_{12}) = 1$, $\beta(c_1c_6) = 4$, $\beta(c_1c_8) = 1$, $\beta(c_1c_{10}) = 2$, $\beta(c_1c_{12}) = 1$, $\beta(c_2c_5) = 4$, $\beta(c_2c_7) = 1$, $\beta(c_2c_9) = 2$, $\beta(c_2c_{11}) = 1$, $\beta(c_3c_6) = 1$, $\beta(c_3c_8) = 1$, $\beta(c_3c_{10}) = 1$, $\beta(c_3c_{12}) = 1$, $\beta(c_4c_5) = 1$, $\beta(c_4c_7) = 1$, $\beta(c_4c_9) = 1$, $\beta(c_4c_{11}) = 1$, $\beta(c_5c_{10}) = 1$, $\beta(c_5c_{12}) = 1$, $\beta(c_6c_9) = 1$, $\beta(c_6c_{11}) = 1$, $\beta(c_7c_{10}) = 1$, $\beta(c_7c_{12}) = 1$, $\beta(c_8c_9) = 1$ *and* $\beta(c_8c_{11}) = 1$. *So, its maximum* β *are* $\beta(c_2c_5)$ *and* $\beta(c_1c_6)$. *In fact, the group with* a_1, a_2, b_1, *and* b_2, *has the maximum interact among its members.*

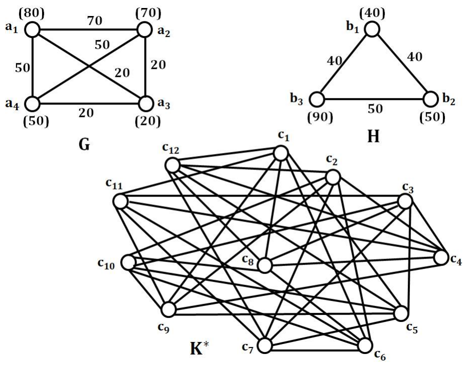

FIGURE 13. The RL-graphs G and H and the graph K^*

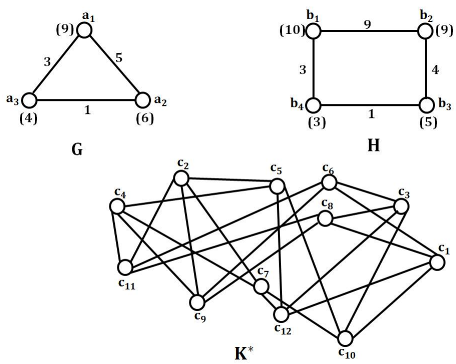

FIGURE 14. The RL-graph G, the RL-graph H and the graph K^*

5. Conclusion

In this study, using kronecker product graphs, the notion of kronecker product RL-graphs has been established from two RL-graphs. In order to identify the close relationship between two RL-graphs and their kronecker product, some theorems and examples have also been presented. The material presented in the mathematical sciences has always helped improve human life, so they have always used these concepts to solve their problems. So we can say that different notions can use as utilities that may apply in many fields. Accordingly, using this kronecker product of two RL-graphs, we can relate two groups unrelated to each other and predict how much their work efficiency will change if these two groups merge. By obtaining this information, more accurate decisions can make. We are willing to investigate this topic in more detail in our future work, gain more insights into these structures, and measure their complexity. We also decided to compare this modeling method with other modeling and show which modeling method is the best. Furthermore, we intend to create a deep relationship between graphs and automata by kronecker product and to study and identify these relationships in detail. In addition, we search for more associations between these structures for application in the computer network.

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