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CHLODOWSKY TYPE (λ, q)-BERNSTEIN STANCU OPERATORS OF PASCAL ROUGH TRIPLE SEQUENCES

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Abstract. The fundamental concept of statistical convergence first was put forward by Steinhaus [20] and at the same time but also by Fast [14] independently both for complex and real sequences. In fact, the convergence in terms of statistical manner can be seen as a generalized form of the common convergence notion that is in the parallel of the theory of usual convergence. Measuring how large a subset of the set of natural number can be possible by means of asymptotic density. It is intuitively known that positive integers are in fact far beyond the fact that they are perfect squares. This is due to the fact that each perfect square is positive and besides at the same time there are many other positive integers. But it is also known that the set consisting of integers which are positive is not larger than that of those which are perfect squares: both of those sets are countable and infinite and therefore can be considered in terms of 1-to-1 correspondence. However, when the natural numbers are scanned for increasing order, then the squares are seen increasingly scarcity. It is at this point that the concept of natural density comes into out help and this intuition becomes more precise. In this study, the above mentioned statistical convergence and asymptotic density concepts are examined in a new space and an attempt is made to fill a gap in the literature as follows. Stancu type extension of the widely known Chlodowsky type (λ, q)-operators is going to be introduced. Moreover, the description of the novel rough statistical convergence having Pascal Fibonacci binomial matrix is going to be presented and several general characteristics of rough statistical convergence are taken into consideration. In the second place, the approximation theory is investigated as the rate of the rough statistical convergence of Chlodowsky type (λ, q)-operators.

Keywords: Chlodowsky type (λ, q)-Bernstein Stancu operators, Rough statistical convergence, Natural density, Triple sequences, Chi sequence, Korovkin type approximation theorems, Pascal Fibonacci matrix, Positive linear operator.
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1. **Introduction**

It is assumed that β is a nonnegative real number and the following formulas are taken into consideration.

$$(x+y+u\beta)^{u}(x+y+v\beta)^{v}(x+y+w\beta)^{w}= \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\binom{u}{m}\binom{v}{n}\binom{w}{k}u^{3}(u+m\beta)^{m-1}(u+n\beta)^{n-1} (u+k\beta)^{k-1}[v+(u-m)(v-n)(w-k)]^{(u-m)+(v-n)+(w-k)} \tag{1}$$

$$(x+y+u\beta)^{u}(x+y+v\beta)^{v}(x+y+w\beta)^{w}= \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\binom{u}{m}\binom{v}{n}\binom{w}{k}(u+m\beta)^{m}(u+n\beta)^{n} (u+k\beta)^{k}v[v+(u-m)(v-n)(w-k)]^{(u-m-1)+(v-n-1)+(w-k-1)} \tag{2}$$

$$(x+y)(x+y+u\beta)^{u-1}(x+y+v\beta)^{v-1}(x+y+w\beta)^{w-1}= \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\binom{u}{m}\binom{v}{n}\binom{w}{k}u^{3}(u+m\beta)^{m-1}(u+n\beta)^{n-1} (u+k\beta)^{k-1}v[v+(u-m)(v-n)(w-k)]^{(u-m-1)+(v-n-1)+(w-k-1)}, \tag{3}$$

where $u,v,w\in\mathbb{R}$ and $u,v,w\geq 1$.

In the present manuscript, Chlodowsky type (λ,q)-Bernstein Stancu operators of triple sequence space defined as follows is going to be constructed

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)= \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\widehat{b}_{rst,mnk}(x;q)\,f\left(\frac{[mnk]_q+\alpha}{[rst]_q+\beta}b_{rst}\right) \tag{4}$$

in which $t,\ s,\ r\in\mathbb{N}$, $0\leq x\leq b_{rst}$, $0<q\leq 1$, and b_{rst} denotes a sequence consisting of positive numbers satisfying the equality $\lim_{r,s,t\to\infty} b_{rst}=\infty$, $\lim_{r,s,t\to\infty}\frac{b_{rst}}{[rst]_q}=0$,

$$\widehat{b}_{rst,mnk}(x;q)=\binom{r}{m}\binom{s}{n}\binom{t}{k}\left(\frac{x}{b_{rst}}\right)^{m+n+k}\left(1-\frac{x}{b_{rst}}\right)^{(r-m)+(s-n)+(t-k)}$$

and $\beta,\ \alpha\in\mathbb{R}$ and $0\leq\alpha\leq\beta$. When $\alpha=\beta=0$ are taken, Chlodowsky type (λ,q)-Bernstein Stancu polynomials are obtained.

The triple Pascal matrix in fact is an infinite matrix composed of the binomial coefficients in its entries. To accomplish this goal, there exist three various ways, namely, using a lower-triangular, upper-triangular, or a symmetric matrix. The 4×4 truncation of those are shown below.

Triple upper triangular

$$U_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 27 & 96 \\ 0 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

Triple lower triangular

$$L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 27 & 1 & 0 \\ 1 & 96 & 500 & 1 \end{pmatrix};$$

Symmetric

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 27 & 500 & 8575 \\ 1 & 96 & 3375 & 87808 \\ 1 & 250 & 15435 & 592704 \end{pmatrix}$$

One can see the pleasing relationship $A_n = L_n U_n$ between those matrices One can also easily observe that all of those three matrices have a value of 1 as their determinants. The entries of the symmetric triple Pascal matrix are composed of binomial coefficients, that is, $A_{ijk} = \binom{r}{m}\binom{s}{n}\binom{t}{k} = \frac{r!}{m!(r-m)!}\frac{s!}{n!(n-s)!}\frac{t!}{k!(k-t)!}$, where $r, s, t = i + j + k$ and $m = i$, $n = j$, $k = t$. That is

$$A_{ijk} =_{i+j+k} C_{ijk} = \frac{(i+j+k)!}{i!j!k!}.$$

Thus the trace of A_n is given by

$$\operatorname{tr}(A_n) = \sum_{m=0}^{r-1}\sum_{n=0}^{s-1}\sum_{k=0}^{t-1} \frac{(2m)!}{(m!)^2}\frac{(2n)!}{(n!)^2}\frac{(2k)!}{(k!)^2}$$

having a few first terms determined by the following sequence $1, 27, 729, 24389, \ldots\cdot$

Let A_n be $n \times n \times n$ matrix of which skew diagonal entries are composed of successive rows (if necessary truncated) of pascals triangle. Generally, $A_n = (a_{ijk})$, in which

$$a_{ijk} = \binom{i+j+k}{i}\binom{i+j+k}{j}\binom{i+j+k}{k} \quad \text{for } i, j, k = 0, 1, 2, \ldots, n-1.$$

The following the factorization procedure

$$A_n = L_n L_n^T, \tag{5}$$

in which L_n^T stands for the transpose of L_n. When $[ijk]^{th}$ section of element is considered for the product, the following equalities hold

$$\begin{aligned}&= \text{coefficient of } x^{ijk} in \left(1+x\right)^i \left(1+x\right)^j \left(1+x\right)^k \\ &= a_{ijk} = \binom{i+j+k}{i}\binom{i+j+k}{j}\binom{i+j+k}{k}\end{aligned}$$

and clearly

$$|L_n| = 1, \tag{6}$$

such that

$$|A_n| = \left|L_n L_n^T\right| = |L_n|^2 = 1$$

it is seen that L_n^{-1} is related to L_n in a simple way. For instance

$$L_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -27 & 1 & 0 \\ 1 & 96 & -500 & 1 \end{pmatrix};$$

and in general

$$L_n^{-1} = (-1)^{i+j-2k} I_{ijk} \tag{7}$$

Besides, 1 is an eigen value of A_n if n is odd and if λ is the eigen value of A_n then thus is λ^{-1}. One can easily verify these conjectures for small values of n. Generally it is assumed that

$$P_n(\lambda) = |\lambda I_n - A_n|,$$

in which I_n is the $n \times n \times n$ identity matrix. Then by (4), (5) and (6)

$$\begin{aligned} P_n(\lambda) &= \left|\lambda\, L_n L_n^{-1} - L_n L_n^T\right| \\ &= |L_n| \left|\lambda\, L_n^{-1} - L_n^T\right| \\ &= \left|\left((-1)^{i+j-2k} \lambda\, l_{ijk} - l_{kji}\right)\right| \\ &= (-\lambda)^n \left|\left(\lambda^{-1} l_{kji} - (-1)^{i+j-2k} l_{ijk}\right)\right|. \end{aligned}$$

If one multiplies odd numbered columns and rows of this matrix by -1 and transpose, one obtains $P_n(\lambda) = (-\lambda)^n \left|\left((-1)^{i+j-2k} \lambda^{-1} l_{ijk} - l_{kji}\right)\right|$

$$P_n(\lambda) = (-\lambda)^n P_n\left(\frac{1}{\lambda}\right) \tag{8}$$

However, eigen values of A_n are at the same time the roots of $P_n(\lambda) = 0$. Therefore, one can conclude from (8) that if λ is an eigen value of A_n then so is λ^{-1}.

2. The triple Pascal matrix of inverse and triple Pascal sequence spaces

It is assumed that P denote the Pascal means defined by the Pascal matrix as is defined by

$$P = \left[P^{rst}_{mnk}\right] = \begin{cases} \binom{r}{m}\binom{s}{n}\binom{t}{k}, & \text{if } (0 \leq m \leq r, n \leq s, k \leq t) \\ 0, & \text{if } \left((m > r, n > s, k > r\right), r, s, t, m, n, k \in \mathbb{N}) \end{cases}$$

and the inverse of Pascal's matrix

$$\begin{aligned} P &= \left[P^{rst}_{mnk}\right]^{-1} \\ &= \begin{cases} (-1)^{(r-m)+(s-n)+(t-k)} \binom{r}{m}\binom{s}{n}\binom{t}{k}, & \text{if } (0 \leq m \leq r, n \leq s, k \leq t) \\ 0, & \text{if } (m > r, n > s, k > r, r, s, t, m, n, k \in \mathbb{N}) \end{cases} \end{aligned} \tag{9}$$

There is some interesting properties of Pascal matrix. For example, we can form three types of matrix; symmetric, lower triangular and upper triangular; for any integer $i, j, k > 0$. The Pascal matrix which is symmetric and having order $n \times n \times n$ is defined by

$$A_{ijk} = a_{ijk} = \binom{i+j+k}{i}\binom{i+j+k}{j}\binom{i+j+k}{k} \quad \text{for } i, j, k = 0, 1, 2, \ldots, n. \tag{10}$$

We can define the lower triangular Pascal matrix having the of order $n \times n \times n$ as follows

$$L_{ijk} = (L_{ijk}) = \frac{1}{(-1)^{i+j-2k} I_{ijk}}; i, j, k = 1, 2, \ldots, n, \tag{11}$$

and the upper triangular Pascal matrix having the order of $n \times n \times n$ can be described as follows

$$U_{ijk} = (U_{ijk}) = \frac{1}{(-1)^{k-(i+j)} I_{ijk}}; i, j, k = 1, 2, \ldots, n. \tag{12}$$

It is well known that $U_{ijk} = (L_{ijk})^T$ for any positive integer i, j, k.

(i) If A_{ijk} is the symmetric Pascal matrix having the order of $n \times n \times n$ described by (1), L_{ijk} is the lower triangular Pascal matrix having the order of $n \times n \times n$ described by (11), then $A_{ijk} = L_{ijk}U_{ijk}$ and $\det(A_{ijk}) = 1$.

(ii) If both A and also B are $n \times n \times n$ matrices. It is said that A is similar to B if there exists an invertible $n \times n \times n$ matrix P satisfying the equality $P^{-1}AP = B$.

(iii) If A_{ijk} is the symmetric Pascal matrix having the order $n \times n \times n$ given by (10), then A_{ijk} is similar to its inverse A^{-1}_{ijk}.

(iv) Let L_{ijk} be the lower triangular Pascal matrix of order $n \times n \times n$ described by (11), then $L^{-1}_{n} = L^{-1}_{ijk} = (-1)^{i+j-2k} I_{ijk}$.

2.1. **Densities and rough statistical convergence.** When the theory of numbers is searched, one can encounter various definitions about the concept of density. But it is a widely known fact that among those the most popular one is asymptotic density. But, this widely known asymptotic density is not available for all sequences. Now in order to fill the gaps and also to help various aims, densities are proposed.

Measuring how large a subset of the set of natural number is also made possible by the concept of asymptotic density. It is intuitively known that the number of positive integers is larger than that of perfect squares. It is a fact that every perfect square is also at the same time positive, while there are many other positive integers. But it is also a known fact that the set composed of positive integers is not larger than that of perfect squares: both of the sets are countable infinite and thus can be placed in one-to-one correspondence. Nevertheless when one continues counting through the natural numbers in increasing order, the number of squares become scarce in number. It is this point that natural density comes to in our help makes this intuition invaluable.

It is assumed that α denotes a subset of positive integer. When the solution domain $[1, n]$ is considered, and an integer in this solution domain is chosen randomly, then it is probable that the ratio of the of elements of $\alpha \in [1, n]$ to the total of elements in $[1, n]$ belongs to α. When $n \to \infty$ is taken, and this probability occurs, in other words if the probability tends to a limit, under these conditions this very limit is utilized as the asymptotic density of the given set α. This fact tells us that the given asymptotic density can be considered as a type of probability of selecting a number randomly from the given set α.

$\mathbb{Z}^+$ is going to denote the set of those positive integers and it is assumed that α and β are the subsets of $\mathbb{Z}^+$. When the symmetric difference given by $\alpha\Delta\beta$ is finite, under this condition it can be said that α is asymptotically equal to β and this equality is denoted by $\alpha \approx \beta$. The very idea of the lower asymptotic density was first introduced by Freedman and Sember and they also described the concept of convergence in density.

Definition 2.1. If f is a given function which is described for all sets of natural numbers and also it is assumed that the values in the range of $[0, 1]$ are taken. It is said that the function f is a lower asymptotic density, when the following conditions are satisfied:

(i) $f(\alpha) = f(\beta)$, when $\alpha \approx \beta$,
(ii)$f(\alpha) + f(\beta) \leq f\left(\alpha \bigcup \beta\right)$, when $\alpha \bigcap \beta = \phi$,
(iii)$f(\alpha) + f(\beta) \leq 1 + f\left(\alpha \bigcap \beta\right)$, for all α,
(iv) $f(\mathbb{Z}^+) = 1$.

Using the given definition for lower density, the upper density can also be defined as follows:

It is assumed that f is any given density. Under this assumption, for any given set of natural numbers α, it is said that $\bar{f}$ is upper density associated with f, when the condition $\bar{f}(\alpha) = 1 - f(\mathbb{Z}^+ \setminus \alpha)$ is satisfied.

When the set $\alpha \subset \mathbb{Z}^{+}$is taken into consideration and the condition $f(\alpha) = \bar{f}(\alpha)$ is satisfied, then one can conclude that the set denoted by α has got a natural density w.r.t. α. In fact, we usually use the term asymptotic density for the following function

$$d(\alpha) = \liminf_{u,v,w\to\infty} \frac{\alpha(u,v,w)}{uvw},$$

in which $\alpha \subset \mathbb{N}$ and $\alpha(u,v,w) = \sum_{(a,b,c)\leq(u,v,w),(a,b,c)\in\alpha} 1$.

At the same time, the following notation is used to denote the natural density of α

$$d(\alpha) = \lim_{u,v,w} \frac{1}{uvw} |\alpha(u,v,w)|$$

in which $|\alpha(u,v,w)|$ stands for the number of elements in $\alpha(u,v,w)$.

The spectra and the fine spectra of the operator Δ^{uv} over the sequence space c_0 has been studied by Fathi and Lashkaripour [13]. The fundamental concept of statistical convergence was first put forward by Steinhaus [20] and at the same time but in an independent way by Fast [14] for both the real or complex sequences. In fact, statistical convergence is known as the generalized form of the usual notion of convergence, which is also in the parallel of the theory of widely known ordinary convergence.

One can define a triple sequence (either real or complex) as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, in which $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{R}$ stand for the set of complex, natural and real numbers respectively. Several various kinds of notions for triple sequence have been put forward and scrutinized initially by Hazarika et al. [15], Sahiner et al. [18, 19], Esi et al. [4–11], Dutta et al. [3], Subramanian et al. [12, 21–24], Debnath et al. [2] and many others.

It is assumed that K is a subset of the set $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, and also show the set $\{(m,n,k) \in K : m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Under these assumptions, the natural density of K is presented by $\delta(K) = \lim_{uvw\to\infty} \frac{|K_{uvw}|}{uvw}$, in which $|K_{uvw}|$ shows the number of existing elements in K_{uvw}. It is obvious that a finite subset has natural density zero, and one has $\delta(K^c) = 1 - \delta(K)$ in which $K^c = \mathbb{N}\backslash K$ is the given complement of K. When $K_1 \subseteq K_2$, one has the condition $\delta(K_1) \leq \delta(K_2)$.

Let us take a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}$, $m, n, k \in \mathbb{N}$ into consideration.

One says that a triple sequence $x = (x_{mnk})$ is statistically convergent to the real number $0 \in \mathbb{R}$, and writes as $st - \lim\ x = 0$, under the condition that the set satisfy the inequality given below

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - 0| \geq \epsilon\right\}$$

has got natural density zero for every given $\epsilon > 0$. Under this condition, 0 is known as the statistical limit of the triple sequence x.

When a given triple sequence satisfies the statistically convergent condition, then for each given $\epsilon > 0$, infinitely many terms of this sequence might lie outside the ϵ-neighbourhood of this given statistical limit, but for this to happen, the natural density of the set composed of the indices of those terms should be zero. This is an outstanding characteristics distinguishing statistical convergence from ordinary one. Since the natural density of a finite set is zero, one can conclude that every ordinary convergent sequence is also statistically convergent.

When a given triple sequence $x = (x_{mnk})$ meets some properties P for all m, n, k except for a set of natural density zero, then one can say that the triple sequence x satisfies P for "almost all (m, n, k)"and this condition is abbreviated by "a.a. (m, n, k)".

It is assumed that $\left(x_{m_i n_j k_\ell}\right)$ is a given sub sequence of $x = (x_{mnk})$. When it is known that the natural density of the set $K = \left\{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i, j, \ell) \in \mathbb{N}^3\right\}$ is not zero, one can call $\left(x_{m_i n_j k_\ell}\right)$ as a non thin sub sequence of a triple sequence x.

The constant $c \in \mathbb{R}$ is known as the statistical cluster point of a given triple sequence $x = (x_{mnk})$ only when the natural density of the following set

$$\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\right\}$$

is not zero for each given $\epsilon > 0$. The notation Γ_x denotes the set composed of all statistical cluster points of the given sequence x.

If there is any positive number M satisfying the following inequality, then the triple sequence $x = (x_{mnk})$ is known to be statistically analytic

$$\delta\left(\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0.$$

One can count the areas of summability theory, trigonometric series, turnpike theory, measure theory, , fuzzy set theory, approximation theory and so on as areas in which the theory of statistical convergence is discussed.

In the present article, we are going to give the definition of the Pascal Fibonacci binomial matrix $F = \left(f_{ij\ell}^{mnk}\right)_{m,n,k=1}^{\infty}$, which is different from the preexisting Pascal Fibonacci binomial matrix in terms of using Fibonacci numbers $f_{ij\ell}$ and at the same time present several novel triple sequence spaces. It is time to describe the Pascal Fibonacci binomial matrix $Ab^{rs} = Ab^{rs}_{uvw,mnk}$, where

$$\begin{aligned} b^{rs} &= b^{rs}_{uvw,mnk} \\ &= \begin{cases} \frac{f_{sr}}{f_{(s+r)^{u+v+w}}} \binom{u}{m} \binom{v}{n} \binom{w}{k} s^{(u-m)+(v-n)+(w-k)}, & \text{if } m \leq u, n \leq v, k \leq w \\ 0, & \text{otherwise } m > u, n > v, k > w \end{cases} \end{aligned}$$

The fundamental concept of rough convergence was first put forward by Phu [17]. He had at the same time put forward the concepts of both roughness degree and rough limit points. One can naturally encounter the idea of rough convergence in the field of numerical analysis and the idea has got very interesting applications. Aytar [1] has extended the concept of rough convergence to the concept of rough statistical convergence by means of the notion of natural density in the same manner as usual convergence has been extended to statistical convergence. Pal et al. [16] extended the concept of rough convergence by means of the concept of ideals that automatically extends the previous concepts of both rough statistical convergence and rough convergence.

In the present article, the concept of rough statistical convergence of triple sequences is going to be introduced. Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix criteria associated with the set of rough statistical convergence are going to be obtained. From now on, r is going to denote a nonnegative real number.

Definition 2.2. A Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ is known as rough convergent and illustrated by ($r - convergent$) to l (Pringsheim's sense), illustrated as $\mu_{mnk} \to^r l$, when the following conditions are satisfied

$$\forall \epsilon > 0, \exists\, i_\epsilon \in \mathbb{N} : m,n,k \geq i_\epsilon \Rightarrow \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - l \right| < r + \epsilon, \tag{13}$$

or in an equivalent manner, if

$$\limsup \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - l \right| \leq r, \tag{14}$$

in which r is known as the roughness of degree. If $r = 0$, the ordinary convergence of a Chlodowsky type (λ, q)-Bernstein operators of Pascal triple sequence is obtained.

Definition 2.3. The fact that the r-limit set of a Chlodowsky type (λ, q)-Bernstein operators of Pascal triple sequence is not unique is obvious. One can define the r-limit set of the Cheney and Sharma operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ as $\mathrm{LIM}^r := \left\{ l \in \mathbb{R} : B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) \to^r l \right\}$.

Definition 2.4. A Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ is known to be r-convergent if $\mathrm{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) \neq \phi$. Under these conditions, r is known as the convergence degree of the Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$. When $r = 0$ is taken, one obtains the ordinary convergence.

Definition 2.5. A Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ is known as r-statistically convergent to l, denoted by $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) \to^{rst} l$, provided that the set

$$\left\{ (m,n,k) \in \mathbb{N}^3 : \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - l \right| \geq r + \epsilon \right\}$$

has natural density zero for every $\epsilon > 0$, or equivalently, if the condition

$$st - \limsup \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) - l \right| \leq r$$

is satisfied.

Additionally, one can state $B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) \to^{rst} l$ iff the inequality

$$\left| B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) - l \right| < r + \epsilon$$

is valid for every $\epsilon > 0$ and almost all (m,n,k). Here r is known as the roughness of degree. When $r = 0$ is taken, one obtains the statistical convergence of Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequences.

Definition 2.6. A Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x)$ is said to be rough statistically Cauchy sequence if for every $\epsilon > 0$ and r be a positive number there is positive integer $N = N(r+\epsilon)$ such that

$$d\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) - B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x)_{N(r+\epsilon)} \right| \geq r + \epsilon \right\}\right) = 0.$$

In a similar way to the concept of classical rough convergence, the concept of rough statistical convergence of a Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence may have the following interpretation:

When it is assumed that a Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $F^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x)$ is statistically convergent and at the same time can not be measured or calculated in an exact manner; we have to do with an approximated (or statistically approximated) triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x)$ satisfying $\left| B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) - F^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) \right| \leq r$ for all m, n, k (or for almost all (r,s,t), i.e.,

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) - F^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) \right| > r \right\}\right) = 0.$$

Then the Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x)$ is not statistically convergent any more, but as the inclusion

$$\begin{aligned} &\left\{(m,n,k) \in \mathbb{N}^3 : \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) - l \right| \geq \epsilon \right\} \\ &\qquad \supseteq \left\{(m,n,k) \in \mathbb{N}^3 : \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) - l \right| \geq r + \epsilon \right\} \end{aligned} \tag{15}$$

holds and we have

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) - l \right| \geq \epsilon \right\}\right) = 0,$$

i.e., we get

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left| B^{\alpha,\beta}_{(r,s,t),\lambda,q} (f;x) - l \right| \geq r + \epsilon \right\}\right) = 0,$$

i.e., the Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence spaces μ is r-statistically convergent.

2.2. **Approximation theory.** It is known that Korovkin kind approximation theorems have been practical tools to control whether a given Pascal triple sequence $(\alpha_{mnk})_{mnk\geq 1}$ of positive linear operators on $C[a,b]$ of all continuous functions on the real interval $[a,b]$ is an approximation process. In other words, those theorems put forward a variety of test functions providing that the approximation property is valid on the whole space if it holds for them such a property was determined by Korovkin in the year of 1953 for the functions 1, x, x^2 in the space $C[a,b]$ as well as for the functions 1, $\cos x$ and $\sin x$ in the space of all continuous 2π periodic functions on the real line.

3. A Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci Binomial of rough statistical convergence

A Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ is said to be triple analytic if

$$\sup_{m,n,k}\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|^{\frac{1}{m+n+k}}<\infty.$$

The vector space of all Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple analytic sequences are usually denoted by P_{Λ^3}. The Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence space P_{Λ^3} is at the same time a metric space having the following metric

(16)

$$d(x,y)=\sup_{m,n,k}\left\{\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)-F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|^{\frac{1}{m+n+k}}:m,n,k=1,2,3,...\right\},$$

for all $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ and $F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ in P_{Λ^3}. Then,

$$P_{\chi^3}\left(Ab^{rs}_{uvw,mnk}\right)=\left\{B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\in w:\left(Ab^{rs}_{uvw,mnk}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\in P_{\chi^3}\right\}.$$

One can obviously see that if P_{χ^3} is a linear space then $P_{\chi^3}\left(Ab^{rs}_{uvw,mnk}\right)$ is also a linear space.

If P_{χ^3} is a complete metric space, then $P_{\chi^3}\left(Ab^{rs}_{uvw,mnk}\right)$ is also a given complete metric space having the following metric

$$d(x,y)=\sup\left\{\left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)-Ab^{rs}F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|:m,n,k=1,2,3,\ldots\right\}_{P_{\chi^3}}.$$

Lemma 3.1. *If* $P_{\chi^3_{B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)}}\subset P_{\chi^3_{F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)}}$ *then*

$$P_{\chi^3}\left(Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\subset P_{\chi^3}\left(Ab^{rs}F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right).$$

Proof. It is trivial. $\square$

Theorem 3.2. *Consider that P_{χ^3} is a complete metric space and α is closed subset of P_{χ^3}. Then $\alpha\left(Ab^{rs}\right)$ is at the same time closed in $P_{\chi^3(Ab^{rs})}$.*

Proof. Because of the fact that α is a given closed subset of P_{χ^3} using the Lemma 3.1, then one may write

$$\alpha\left(Ab^{rs}\right) \subset P_{\chi^3}\left(Ab^{rs}\right).$$

$\overline{\alpha\left(Ab^{rs}\right)}$, $\bar{\alpha}$ denote the closure of $\alpha\left(Ab^{rs}\right)$ and α, respectively. Proving the equality $\overline{\alpha\left(Ab^{rs}\right)} = \bar{\alpha}\left(Ab^{rs}\right)$ is going to be enough. First of all, one consider $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) \in \overline{\alpha(Ab^{rs})}$, there exists a sequence $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)^{uvw}\right) \in \alpha(Ab^{rs})$ such that $\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right)^{uvw} - x\right|_{Ab^{rs}} \to 0$ in $\alpha\left(Ab^{rs}\right)$ for $u,v,w \to \infty$. Thus, $\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right)^{uvw} - B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right)\right|_{Ab^{rs}} \to 0$ as $u,v,w \to \infty$ in $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right) \in \alpha\left(Ab^{rs}\right)$ so that

$$\sum_{r=1}^{i}\sum_{s=1}^{j}\sum_{t=1}^{\ell}\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right)^{uvw} - B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right)\right| + \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right)^{uvw} - Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right)\right| \to 0$$
$$\text{for } (u,v,w) \to \infty, \text{ in } \alpha.$$

Therefore, $Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right) \in \bar{\alpha}$ and so $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right) \in \bar{\alpha}\left(Ab^{rs}\right)$. Conversely, if we take $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right) \in \overline{\alpha\left(Ab^{rs}\right)}$, then $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right) \in \alpha\left(Ab^{rs}\right)$. Since α is closed, we have $\overline{\alpha\left(Ab^{rs}\right)} = \bar{\alpha}\left(Ab^{rs}\right)$. Hence $\alpha\left(Ab^{rs}\right)$ is a closed subset of $P_{\chi^3}\left(Ab^{rs}\right)$. $\square$

Corollary 3.3. *When P_{χ^3} is a separable space, thus $P_{\chi^3}\left(Ab^{rs}\right)$ is at the same time a separable one.*

Definition 3.4. A Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ is said to be Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix on rough statistically convergence if there exists a scalar l in such a way that for each $\epsilon > 0$ and r is a positive scalar, the following set

$$K_{r+\epsilon}\left(Ab^{rs}\right) := \left\{m \leq u, n \leq v, k \leq w : \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right) - l\right| \geq r+\epsilon\right\}$$

has got natural density of zero, that is; $d\left(K_{r+\epsilon}\left(Ab^{rs}\right)\right) = 0$. In other words

$$\lim_{u,v,w\to\infty}\frac{1}{uvw}\left|\left\{m \leq u, n \leq v, k \leq w : \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x\right) - l\right| \geq r+\epsilon\right\}\right| = 0.$$

Under these conditions, one has the right to define $d(Ab^{rs})-\lim B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) = l$ or $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) \to l(rs(Ab^{rs}))$. The set of Ab^{rs}-rough statistically convergent of Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence space are going to be stated as $rs\,(Ab^{rs})$. Under these conditions $l = 0$, one can write $rs_0\,(Ab^{rs})$.

Definition 3.5. A Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x)$ is said to be Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix on rough statistically Cauchy when there is a number $N = N\,(r+\epsilon)$ such that for every $\epsilon > 0$ and r be a positive number the set

$$\lim_{u,v,w\to\infty} \frac{1}{uvw} \left|\left\{m \le u, n \le v, k \le w : \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)_N\right| \ge r+\epsilon\right\}\right| = 0.$$

Theorem 3.6. *If a Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence space $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x)$ is a Chlodowsky type (λ, q)-operators of Pascal Fibonacci binomial matrix on rough statistically convergent sequence then $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x)$ is a Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix on rough statistically Cauchy sequence.*

Proof. If $\epsilon > 0$, r is a scalar such that taken as positive real number. It is assumed that $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x) \to l\,(rs\,(Ab^{rs}))$. Then

$$\left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x) - l\right| < \frac{r+\epsilon}{2} \text{ for almost all } m, n, k.$$

When one chooses N as follows

$$\left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x)_N - l\right| < \frac{r+\epsilon}{2}$$

then we have

$$\begin{aligned}
&\left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x) - Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x)_N\right| \\
&< \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x) - l\right| + \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x)_N - l\right| \\
&< \left(\frac{r+\epsilon}{2}\right) + \left(\frac{r+\epsilon}{2}\right) \\
&= r+\epsilon
\end{aligned}$$

for almost all m,n,k. Hence, $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ is Chlodowsky type (λ,q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix on rough statistically Cauchy sequence. □

Theorem 3.7. *If $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\,(f;x)$ is Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence for which there is a Cheney and Sharma* *operators of Pascal Fibonacci binomial matrix on rough statistically convergent sequence* $F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ *such that* $Ab^{rs}\mu_{mnk} = Ab^{rs}\gamma_{mnk}$ *for almost all* m,n,k*, then* $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ *is Chlodowsky type* (λ,q)*-Bernstein Stancu operators of Pascal Fibonacci binomial matrix on rough statistically convergent sequence.*

Proof. Suppose that $Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) = Ab^{rs}F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ for almost all m,n,k, and $\left(F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) \to l\left(rs\left(Ab^{rs}\right)\right)$. Then, $\epsilon > 0$ and r be a positive real number and for each u,v,w,

$$\left\{(m,n,k)\leq(u,v,w): \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)-l\right|\geq r+\epsilon\right\} \subseteq \left\{(m,n,k)\leq(u,v,w): \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\neq Ab^{rs}F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|\geq r+\epsilon\right\} \bigcup\left\{(m,n,k)\leq(u,v,w): \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)-l\right|\leq r+\epsilon\right\}.$$

Since $\left(F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) \to l\left(rs\left(Ab^{rs}\right)\right)$, the latter set contains a predefined number of integers, say $g = g(r+\epsilon)$. Thus

$$Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) = Ab^{rs}F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$$

for almost all m,n,k,

$$\lim_{uvw\to\infty}\frac{1}{uvw}\left|\left\{(m,n,k)\leq(u,v,w): \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)-l\right|\geq r+\epsilon\right.\right. \leq \lim_{uvw\to\infty}\frac{1}{uvw}\left|\left\{(m,n,k)\leq(u,v,w): \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\neq Ab^{rs}F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|\right\}\right. + \lim_{uvw}\frac{g(r+\epsilon)}{uvw} = 0.$$

Hence $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) \to l\left(rs\left(Ab^{rs}\right)\right)$. □

Definition 3.8. A Chlodowsky type (λ,q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ is said to be rough statistically analytic if there exists some $l \geq 0$ such that

$$d\left(\left\{(m,n,k)\in\mathbb{N}^3: \left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|^{1/m+n+k} > l\right\}\right) = 0,$$

i.e., $\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|^{1/m+n+k} \leq l$ *a.a.k.* Analytic sequences are obviously rough statistically analytic as the empty set has zero natural density. But its converse is not true.

For instance, one may take into consideration the Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence

$$\begin{aligned} B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) &= \left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)_{uvw}\right) \\ &= \begin{cases} (uvw)^{u+v+w}, & \text{if } m,n,k \text{ are squares} \\ 0, & \text{if } m,n,k \text{ are not squares} \end{cases} \end{aligned}$$

clearly the Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ is not a analytic sequence. But,

$$d\left(\left\{(m,n,k)\in\mathbb{N}^3 : \left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|^{1/m+n+k} > \frac{1}{6}\right\}\right) = 0,$$

as the of squares has zero natural density and hence the Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal triple sequence $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ is rough statistically analytic.

Corollary 3.9. *Every convergent sequence is rough statistically triple Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal analytic.*

Corollary 3.10. *Every rough statistical convergent sequence is rough statistically triple Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal analytic.*

Corollary 3.11. *Every Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix of rough statistical convergent sequence is Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix of rough statistically triple Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal analytic.*

4. Rate of Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix on rough statistical convergence

Let us assume that $F(\mathbb{R})$ denotes the linear space of real value function on $\mathbb{R}$ and $C(\mathbb{R})$ is the space of all real-valued and continuous functions f defined on $\mathbb{R}$. $C(\mathbb{R})$ having the metric described as:

$$d\left(\left(f, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right), \left(f, F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right) = \sup_{B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\in\mathbb{R}} \left|\left(f, vB^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - \left(f, F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right|^{1/m+n+k}, f \in C(\mathbb{R})$$

and we denote $C_{2\pi}(\mathbb{R})$ the space of all 2π periodic functions $f \in C(\mathbb{R})$ with the metric is given by

$$d\left(\left(f, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right), \left(f, F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right)_{2\pi} = \sup_{t\in\mathbb{R}}\left|\left(f,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)(t)\right) - \left(f,F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;vx)(t)\right)v\right|^{1/m+n+k}, f \in C(\mathbb{R}).$$

It is estimated that the rate of Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix on rough statistical convergence of a triple Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal sequence of positive linear operators defined $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

Definition 4.1. Let (a_{uvw}) be a positive non-increasing sequence. The triple Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal sequence $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ is rate of Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal Fibonacci binomial matrix on rough statistical convergence to l having the rate $o(a_{uvw})$ if for every $\epsilon > 0$ and r be a real number in such a way that

$$\lim_{u,v,w\to\infty} \frac{1}{h_{uvw}}\left|\left\{m \le u, n \le v, k \le w : \left|Ab^{rs}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - l\right| \ge r+\epsilon\right\}\right| = 0.$$

Then it can be written as $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - l = d(Ab^{rs}) - o(a_{uvw})$.

Lemma 4.2. *Let (a_{uvw}) and (b_{uvw}) be two positive non-increasing sequences. Let $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ and $F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$ be two triple Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal sequences such that $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - l_1 = d(Ab^{rs}) - o(a_{uvw})$ and $\left(F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - l_2 = d(Ab^{rs}) - o(b_{uvw})$. Then one has*

(i) $\alpha\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - l_1\right) = d(Ab^{rs}) - o(a_{uvw})$ *for any scalar* α,

(ii) $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - l_1\right) \pm \left(F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - l_2\right) = d(Ab^{rs}) - o(c_{uvw})$,

(iii) $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - l_1\right) \cdot \left(F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - l_2\right) = d(Ab^{rs}) - o(a_{uvw}b_{uvw})$,

where $c_{uvw} = \max\{a_{uvw}, b_{uvw}\}$.

For any $\delta > 0$, the modulus of continuity of f, $w(f,\delta)$ is given as

$$w(f,\delta) = \sup_{\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right| < \delta} \left|\left(f,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - \left(f,F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right|.$$

A function $f \in C[a,b], \lim_{uvw\to 0^+} w(f,\delta) = 0$. For any $\delta > 0$

$$\left|\left(f, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - \left(f, F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right| \leq$$

$$w(f,\delta)\left(\frac{\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) - F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|}{\delta} + 1\right). \quad (17)$$

Theorem 4.3. *Let $\left(u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ be triple Chlodowsky type (λ, q)-Bernstein Stancu operators of Pascal sequence of positive linear operator from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. It is assumed that*

(i)

$$d\left(u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\left(1, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right), 0\right)_{2\pi} = d(Ab^{rs}) - o(h_{uvw})$$

(ii) $w(f, \theta_{mnk}) = d(Ab^{rs}) - o(g_{uvw})$, *where*

$$\theta_{mnk} = \sqrt{U^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left[sin^2\left(\frac{t - B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)}{2}\right), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right]}.$$

Then for all $f \in C_{2\pi}(\mathbb{R})$, we get

$$d\left(g^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\left(f, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right), 0\right)_{2\pi} = d(Ab^{rs}) - o(e_{uvw}),$$

where $e_{uvw} = \max\{h_{uvw}, g_{uvw}\}$.

Proof. Let $f \in C_{2\pi}(\mathbb{R})$ and $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) \in [-\pi,\pi]$, we can write

$$\left|g^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\left(f,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)-f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right)\right|$$
$$\leq g^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$$
$$\left((f,t)-f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right),B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)+$$
$$\left|f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right|$$
$$\left|g^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\left(1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)-f(1)\right)\right|$$
$$\leq g^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\frac{\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)-F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right|}{\delta}\right.$$
$$\left.+1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$$
$$w(f,\delta)+\left|f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right|g^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$$
$$\left(\left(1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)-f(1)\right)$$
$$\leq g^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\frac{\pi^2}{\delta^2}sin^2\left(\frac{F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)-B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)}{2}\right)\right.$$
$$\left.+1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$$
$$w(f,\delta)+\left|f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right|g^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)$$
$$\left(\left(1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)-f(1)\right)$$
$$\leq\left\{g^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)+\right.$$
$$\left.\frac{\pi^2}{\delta^2}l_{mnk}\left(sin^2\left(\frac{F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)-B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)}{2}\right),x\right)\right\}w(f,\delta)+$$
$$\left|f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right|u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\left(1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)-f(1)\right)$$
$$=u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)+$$
$$\frac{\pi^2}{\delta^2}u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\sin^2\left(\frac{F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)-B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)}{2}\right),\right.$$
$$\left.B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)w(f,\delta)+\left|f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right|$$
$$u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\left(1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)-f(1)\right).$$

By choosing $\sqrt{\theta_{mnk}} = \delta$, we get

$$\begin{aligned}
&d\left(u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\left(f, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - \right.\right.\\
&\left.\left. f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right), 0\right)_{2\pi}\\
\leq & d\left(\left(f, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right), \left(f, F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right)_{2\pi}\\
&d\left(u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\left(f, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - \right.\right.\\
&\left.\left. f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right) + 2w(f, \theta_{mnk}) + w(f, \theta_{mnk})\right)\\
&d\left(u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\left(\left(1, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - \right.\right.\\
&\left.\left. f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right), 0\right)_{2\pi} \leq K\left\{d\left(u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right.\right.\\
&\left.\left(\left(1, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right), 0\right)_{2\pi}\\
&+ w(f,\theta_{mnk}) + w(f,\theta_{mnk})u^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\\
&\left.\left(\left(1, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right) - f\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right)_{2\pi}\right],
\end{aligned}$$

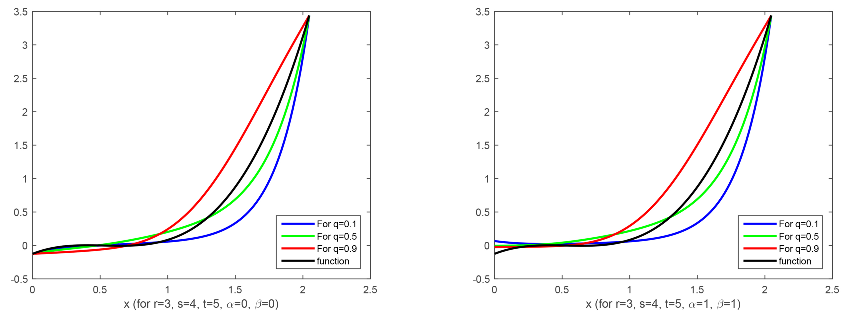
where

$$K = \max\left\{2, d\left(\left(f, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right), \left(f, F^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)\right)_{2\pi}\right\}.$$

□

Example 4.4. *By using the symbolic programming language Matlab, we illustrate comparisons and several explanatory graphics for the convergence of operators (6) to the function* $f(x) = \left(x - \frac{1}{3}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{3}{4}\right)$ *by using various parameters.*

One can observe from Figure 1(a) that as the value the q approaches towards 1 *provided that* $0 < q \leq 1$*, Chlodowsky type* (λ, q)*-Bernstein-Stancu operators given by (6) converges towards the function* $f(x) = \left(x - \frac{1}{3}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{3}{4}\right)$*. One can see in the Figure 1(a) that it can be observed that for* $\alpha = \beta = 0$*, as the value the* (r, s, t) *increases, Chlodowsky type* (λ, q)*-Bernstein-Stancu operators given by (6) converges towards the function. Similarly from Figure 1(b), one can observe that for* $\alpha = \beta = 1$*, as the value the q approaches towards* 1 *or some thing else provided* $0 < q \leq 1$*, Chlodowsky type* (λ, q)*-Bernstein-Stancu operators given by (6) converges towards the function. From Figure 1(b), one can observe that as the value the* $[r, s, t]$ *increases, Chlodowsky type* (λ, q)*-Bernstein-Stancu operators given by* $f(x) = \left(x - \frac{1}{3}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{3}{4}\right)$ *converges towards the function.*



(A) (B)

FIGURE 1. Chlodowsky type (λ, q)-Bernstein-Stancu operators

5. Conclusion

We have put forward Stancu type extension of the well known Chlodowsky type (λ, q)-operators and also are reserved to the description of a novel rough statistical convergence having Pascal Fibonacci binomial matrix has been presented and some general properties of rough statistical convergence are investigated and at the same time approximation theory worked as a rate of the rough statistical convergence of Chlodowsky type (λ, q)-operators. For the reference sections, take into consideration the following introduction given in the main results which are motivating the research. Thus it will be possible to benefit from this article for new studies.

Competing Interests

We, authors, declare that there has not been any conflict of interests with respect to the publication of the present article.

References

[1] S. Aytar, *Rough statistical convergence*, Numerical Functional Analysis and Optimization. An International Journal 29(3-4), 291–303 (2008).

[2] S. Debnath, B. Sarma, B. C. Das, *Some generalized triple sequence spaces of real numbers*, Journal of nonlinear analysis and optimization. Theory and Applications 6(1), 71–78 (2015).

[3] A. J. Dutta, A. Esi, B. C. Tripathy, *Statistically convergent triple sequence spaces defined by Orlicz function*, Journal of Mathematical Analysis 4(2), 16–22 (2013).

[4] A. Esi, *On some triple almost lacunary sequence spaces defined by Orlicz functions*, Research and Reviews: Discrete Mathematical Structures 1(2), 16–25 (2014).

[5] A. Esi, S. Araci, M. Acikgoz, *Statistical convergence of Bernstein operators*, Applied Mathematics & Information Sciences 10(6), 2083–2086 (2016).

[6] A. Esi, S. Araci, A. Esi, *λ-statistical convergence of Bernstein polynomial sequences*, Advances and Applications in Mathematical Sciences 16(3), 113–119 (2017).

[7] A. Esi, M. N. Catalbas, *Almost convergence of triple sequences*, Global Journal of Mathematical Analysis 2(1), 6–10 (2014).

[8] A. Esi, E. Savas, *On lacunary statistically convergent triple sequences in probabilistic normed space*, Applied Mathematics & Information Sciences 9(5), 2529–2534 (2015).

[9] A. Esi, N. Subramanian, *Generalized rough Cesaro and lacunary statistical Triple difference sequence spaces inprobability of fractional order defined by Musielak Orlicz function*, International Journal of Analysis and Applications 16(1), 16–24 (2018).

[10] A. Esi, N. Subramanian, *On triple sequence spaces of Bernstein operator of χ^3 of rough $\lambda-$ statistical convergence in probability of random variables defined by Musielak-Orlicz function*, International Journal of Open Problems in Computer Science and Mathematics 11(2), 62–70 (2018).

[11] A. Esi, N. Subramanian, A. Esi, *On triple sequence space of Bernstein operator of rough $I-$ convergence Pre-Cauchy sequences*, Proyecciones Journal of Mathematics 36(4) 567–587 (2017).

[12] A. Esi, N. Subramanian, V. A. Khan, *The rough intuitionistic fuzzy Zweier lacunary ideal convergence of triple sequence spaces*, Journal of Mathematics and Statistics 14(1), 72–78 (2018).

[13] J. Fathi, R. Lashkaripour, *On the fine spectra of the generalized difference operator Δ^{uv} over the sequence space c_0*, Journal of Mahani Mathematical Research 1(1), 1–12 (2012).

[14] H. Fast, *Sur la convergence statistique*, Colloquium Mathematicum 2, 241–244 (1951).

[15] B. Hazarika, N. Subramanian, A. Esi, *On rough weighted ideal convergence of triple sequence of Bernstein polynomials*, Proceedings of the Jangjeon Mathematical Society. Memoirs of the Jangjeon Mathematical Society 21(3), 497–506 (2018).

[16] S. K. Pal, D. Chandra, S. Dutta, *Rough ideal convergence*, Hacettepe Journal of Mathematics and Statistics, 42(6), 633–640 (2013).

[17] H. X. Phu, *Rough convergence in normed linear spaces*, Numerical Functional Analysis and Optimization. An International Journal 22(1-2), 199–222 (2001).

[18] A. Sahiner, M. Gurdal, F. K. Duden, *Triple sequences and their statistical convergence*, Selcuk Journal of Applied Mathematics 8(2), 49–55 (2007).

[19] A. Sahiner, B. C. Tripathy, *Some I-related properties of triple sequences*, Selcuk Journal of Applied Mathematics 9(2), 9–18 (2008).

[20] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloquium Mathematicum 2, 73–74 (1951).

[21] N. Subramanian, A. Esi, *The generalized tripled difference of χ^3 sequence spaces*, Global Journal of Mathematical Analysis 3(2), 54–60 (2015).

[22] N. Subramanian, A. Esi, *On triple sequence space of Bernstein operator of χ^3 of rough λ-statistical convergence in probability definited by Musielak-Orlicz function p-metric*, Electronic Journal of Mathematical Analysis and Applications 6(1), 198–203 (2018).

[23] N. Subramanian, A. Esi, M. K. Ozdemir, *Rough statistical convergence on triple sequence of Bernstein operator of random variables in probability*, Songklanakarin Journal of Science and Technology 41(3), 567–579 (2018).

[24] S. Velmurugan, N. Subramanian, *Bernstein operator of rough λ-statistically and ρ Cauchy sequences convergence on triple sequence spaces*, Journal of Indian Mathematical Society 85(1–2), 257-265 (2018).

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