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ON THE PROPERTIES OF FORMAL LOCAL COHOMOLOGY MODULES

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Abstract. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, $\mathfrak{a}$ an ideal of R. Let $t \in \mathbb{N}_0$ be an integer and M a finitely generated R-module such that the R-module $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i < t$.We prove that For all minimax submodules N of $\mathfrak{F}^i_{\mathfrak{a}}(M)$, the R-modules

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N) \quad \text{and} \quad \mathrm{Ext}^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N)$$

are minimax. In particular, the set $\mathrm{Ass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M)/N)$ is finite.

Keywords: formal local cohomology, local cohomology, cominimax.
2020 MSC: 13D45, 13E05, 13E99.

1. Introduction

Throughout this paper, $(R, \mathfrak{m})$ is a commutative Noetherian local ring with identity, M is a finitely generated R-module and $\mathfrak{a}$ be an ideal of R. For an R-module M, the i^{th} *local cohomology module* of M with respect to $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(M) := \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

We refer the reader to [8] for more details about local cohomology.

In [18], the author introduced the notion of formal local cohomology modules

$$\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_{n\in\mathbb{N}} H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$$

for each $i \geq 0$. The basic properties of formal local cohomology modules are found in [18] and [4].

In [12], Hartshorne defines an R-module M to be $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}_R M \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \geq 0$. He asks when the local cohomology modules of a finitely generated module are $\mathfrak{a}$-cofinite. In this regard, the best known result is that for a finitely generated R-module M if either $\mathfrak{a}$ is principal or R is local and $\dim R/\mathfrak{a} = 1$, then the modules $H^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite. These results are proved in [14, Theorem 1] and [9, Theorem 1.1], respectively.

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Since for an $\mathfrak{a}$-cofinite module N, we have $\mathrm{Ass}_R N = \mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, N))$, it turns out that $\mathrm{Ass}_R N$ is finite. Huneke [13] raised the following question: If M is a finitely generated R-module, then the set of associated primes of $H^i_{\mathfrak{a}}(M)$ is finite for all ideals $\mathfrak{a}$ of R and all $i \geq 0$. Singh [19] gives a counter-example to this conjecture. However, it is known that this conjecture is true in many situations. For example, Brodmann and Lashgari [7, Theorem 2.2]showed that, if for a finitely generated R-module M and an integer t, the local cohomology modules $H^0_{\mathfrak{a}}(M), H^1_{\mathfrak{a}}(M), \ldots, H^{t-1}_{\mathfrak{a}}(M)$ are all finitely generated, then $\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M))$ is finite.

In [20], Zöschinger introduced the interesting class of *minimax modules*, modules containing some finitely generated submodule such that the quotient module is Artinian. As a generalization of the concept of $\mathfrak{a}$-cofinite modules, the concept of $\mathfrak{a}$-*cominimax modules* was introduced in [5]. An R-module M is said to be $\mathfrak{a}$-*cominimax* if $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^j_R(R/\mathfrak{a}, M)$ is minimax for all j. Since the concept of minimax modules is a natural generalization of the concept of finitely generated modules, many authors studied the minimaxness of local cohomology modules and answered the Hartshorn's question in the class of minimax modules (see [1, 2, 5, 16]).

Finally, the notions of *weakly Laskerian modules* has been introduced by Divaani-Aazar and Mafi in [11]. An R module M is said to be *weakly Laskerian* R-*module* if the set of associated primes of any quotient module of M is finite. Moreover the class of weakly Laskerian R-modules is closed under taking submodules, quotients and extensions, i.e., it is a *Serre subcategory of the category of* R-*modules*([10, lemma 2.3]). They also, as a generalization of *cofinite modules* with respect to an ideal, defined an R-module M to be *weakly cofinite* with respect to ideal $\mathfrak{a}$ of R or $\mathfrak{a}$-weakly cofinite if $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all $i \geq 0$.

The formal local cohomology modules have been studied by several authors (see, for example, [4], [18] and [15]). In [6], was investigated some Artinianness properties of formal local cohomology modules. they proved that, in each of the following cases, $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian iff $\mathfrak{a} \subseteq \sqrt{\mathrm{Ann}_R(\mathfrak{F}^i_{\mathfrak{a}}(M))}$ for all $i \neq t$:

(1) $\mathfrak{a}$ is principal.
(2) $\dim R \leq 2.$,
(3) $\dim R/\mathfrak{a} \leq 1$.

Now, considering the weakly Laskerianness properties, we develop these concepts for formal local cohomology modules.and continue and investigate cominimaxness properties of formal local cohomology modules.

2. (co)minimaxness properties

We begin with remark which to show that the class of cofinite modules with respect to an ideal is strictly contained in the class of cominimax modules with respect to the same ideal.

Remark 2.1.
(1): ([3, 2.1])Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{p}$ a prime ideal of R such that $\dim R/\mathfrak{p} = 1$. Then R-module $E(R/\mathfrak{p})$ is $\mathfrak{p}$-cominimax but not $\mathfrak{p}$-cofinite; Because for all $i \geq 0$, we have $\mathrm{Ext}^i_R(R/\mathfrak{p}, E(R/\mathfrak{p})) = 0$. On the other hand, $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p})) \cong (0 :_{E(R/\mathfrak{p})} \mathfrak{p}) \cong E_{R/\mathfrak{p}}(R/\mathfrak{p})$ is not finitely generated.
(2): If M is Noetherian or Artinian, then M is $\mathfrak{a}$-cominimax.

The following lemma is used in the sequel.

Lemma 2.2. *Let* $\mathfrak{a}$ *be an ideal of a Noetherian ring* R *and* M *a minimax* R*-module such that* $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$*. Then the* R*-modules*

$$\mathrm{Tor}^R_i(R/\mathfrak{a}, M),\ \mathrm{Ext}^i_R(R/\mathfrak{a}, M)$$

is minimax for all $i \geq 1$*.*

Proof. The assertion follows from [5] [Corollary 2.5]. □

Theorem 2.3. *Let* $\mathfrak{a}$ *be an ideal of a Noetherian ring* R *and* M *a weakly Laskerian* R*-module such that* $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$*. Then the following statements are equivalent:*
(a) M *is* $\mathfrak{a}$*-cominimax.*
(b) *The* R*-modules* $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ *and* $\mathrm{Ext}^1_R(R/\mathfrak{a}, M)$ *are minimax.*

Proof. According to Theorem 2.3,the conclusion (b) follows from (a) is obvious. In order to prove (b) $\Rightarrow$ (a), by definition of weakly Laskerian R-module, there is a finitely generated submodule N of M such that $\dim(M/N) \leq 1$ and $\mathrm{Supp}(M/N) \subseteq V(\mathfrak{a})$. Also, the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0 \quad (\star)$$

induces the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, N) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, M) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, M/N) \longrightarrow$$

$$\mathrm{Ext}^1_R(R/\mathfrak{a}, N) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, M) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, M/N) \longrightarrow \mathrm{Ext}^2_R(R/\mathfrak{a}, N).$$

Hence, because $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, M)$ are minimax it follows that the R-modules $\mathrm{Hom}_R(R/\mathfrak{a}, M/N)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, M/N)$ are finitely generated. Therefore, in view of lemma (2.2), the R-module M/N is $\mathfrak{a}$-cominimax. Now it follows from the exact sequence $(\star)$ that M is $\mathfrak{a}$-cominimax. □

Theorem 2.4. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *be a is nonzero finitely generated* R*-module. Let* $t \in \mathbf{N}_0$*. Suppose that the* R*-module* $\mathfrak{F}^i_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cominimax for all* $i < t$*, and the* R*-modules* $\mathrm{Ext}^t_R(R/\mathfrak{a}, M)$ *and* $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, M)$ *are minimax. Then the* R*-modules* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)$ *and* $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ *are minimax.*

Proof. We use induction on t.The exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow 0 \qquad (\star\star)$$

induces the long exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, M) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$$

$$\longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, M) \longrightarrow \cdots .$$

Since $\mathrm{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) = 0$, so $\mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ are minimax. Assume inductively that $t > 0$ and that we have established the result for nonnegative integers smaller than t. By applying the functor $\mathrm{Hom}_R(R/\mathfrak{a}, -)$ to the exact sequence $(\star\star)$, we can deduce that $\mathrm{Ext}^j_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is minimax for $j = t, t+1$. On the other hand, $\mathfrak{F}^0_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$ and $\mathfrak{F}^j_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \simeq \mathfrak{F}^j_{\mathfrak{a}}(M)$ for all $j > 0$. Therefore we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Let E be an injective hull of M and put $N = E/M$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, E) = 0 = \Gamma_{\mathfrak{a}}(E)$. Hence $\mathrm{Ext}^j_R(R/\mathfrak{a}, N) \simeq \mathrm{Ext}^{j+1}_R(R/\mathfrak{a}, M)$ and $\mathfrak{F}^j_{\mathfrak{a}}(N) \simeq \mathfrak{F}^{j+1}_{\mathfrak{a}}(M)$ for all $j \geq 0$. Now, the induction hypothsis yields that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ are minimax, as required. □

We are now ready to state and prove the main theorem.

Theorem 2.5. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module, and $t \in \mathbf{N}_0$ such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ are minimax for all $i \leq t+1$. Let the R-modules $\mathfrak{F}^i_{\mathfrak{a}}(M)$ be weakly Laskerian R-modules for all i such that $i < t$. Then the following assertions hold:*

(a) *The R-modules $\mathfrak{F}^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cominimax for all i such that $i < t$.*

(b) *For all minimax submodules N of $\mathfrak{F}^i_{\mathfrak{a}}(M)$, the R-modules*

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N) \quad \text{and} \quad \mathrm{Ext}^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N)$$

are minimax. In particular, the set $\mathrm{Ass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M)/N)$ is finite.

Proof.

(a) We proceed by induction on t. In the case $t = 0$ there is nothing to prove. So, let $t > 0$ and suppose the result has been proved for smaller values of t. By the inductive assumption, $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for $i = 0, 1, \cdots, t-2$. Hence by theorem 2.4 and the assumption, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^{t-1}_{\mathfrak{a}}(M))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathfrak{F}^{t-1}_{\mathfrak{a}}(M))$ are minimax. Therefore by Theorem 2.3, $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i < t$. This completes the inductive step.

(b) In view of (a) and theorem 2.4, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ are minimax. On the other hand, according to Lemma 2.2, N is $\mathfrak{a}$-cominimax. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M)/N \longrightarrow 0.$$

induces the exact sequence:

$$\longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, N)$$

$$\longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N) \longrightarrow \mathrm{Ext}^2_R(R/\mathfrak{a}, N)$$

Consequently,

$$\mathrm{Hom}_{\mathrm{R}}(\mathrm{R}/\mathfrak{a}, \mathfrak{F}^{\mathrm{t}}_{\mathfrak{a}}(\mathrm{M})/\mathrm{N}) \quad \text{and} \quad \mathrm{Ext}^{1}_{\mathrm{R}}(\mathrm{R}/\mathfrak{a}, \mathfrak{F}^{\mathrm{t}}_{\mathfrak{a}}(\mathrm{M})/\mathrm{N}).$$

are minimax, as required. □

Corollary 2.6. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(\mathrm{R}, \mathfrak{m})$ *and* M *a finitely generated* R*-module such that* $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{R}/\mathfrak{a}, \mathrm{M})$ *are minimax for all* i *and the* R*-modules* $\mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M})$ *are weakly Laskerian* R*-modules for all* i*. Then:*

(a) *The* R*-modules* $\mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M})$ *are* $\mathfrak{a}$*-cominimax for all* i.

(b) *For any* $\mathrm{i} \geq 0$ *and for any minimax submodule* N *of* $\mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M})$*, the* R*-module* $\mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M})/\mathrm{N}$ *is* $\mathfrak{a}$*-cominimax.*

Proof. (a) obvious. (b) In view of (a) the R-module $\mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M})$ is $\mathfrak{a}$-cominimax for all i. Hence the R-module $\mathrm{Hom}_{\mathrm{R}}(\mathrm{R}/\mathfrak{a}, \mathrm{N})$ is minimax, and so it follows from Lemma 2.2 that N is $\mathfrak{a}$-cominimax. Now, the exact sequence

$$0 \longrightarrow \mathrm{N} \longrightarrow \mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M}) \longrightarrow \mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M})/\mathrm{N} \longrightarrow 0$$

implies that the R-module $\mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M})/\mathrm{N}$ is $\mathfrak{a}$-cominimax. □

Corollary 2.7. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(\mathrm{R}, \mathfrak{m})$ *and* M *a finitely generated* R*-module such that the* R*-modules* $\mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M})$ *are weakly Laskerian* R*-modules for all* i*. Then the following conditions are equivalent:*

(a) *The* R*-modules* $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{R}/\mathfrak{a}, \mathrm{M})$ *are minimax for all* i.

(b) *The* R*-modules* $\mathfrak{F}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M})$ *are* $\mathfrak{a}$*-cominimax for all* i.

Proof. (a) ⇒ (b) follows by Corollary 2.6.
(a) ⇒ (b) follows by [17], Proposition 3.9. □

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