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A CHARACTERIZATION OF SKEW b-DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring, α an automorphism of R and b an element of Q, the maximal right ring of quotients of R. The main purpose of this paper is to characterize skew b-derivations in prime rings which satisfy various differential identities. Further, we provide an example to show that the assumed restrictions cannot be relaxed.

Keywords: Automorphism, Derivation, Skew b-derivation, Skew derivation, Prime ring.
2020 MSC: 16N60, 16W25.

1. Introduction

Throughout the discussions, R will represent a nonzero associative ring with center *Z(R)*. A ring R is called *prime* if $xRy = (0)$ (where $x, y \in R$) implies $x = 0$ or $y = 0$. A ring R is said to be *2-torsion free* if for any $x \in R$, $2x = 0$ implies $x = 0$. For any $x, y \in R$, the symbol $[x, y]$ will denote the Lie product $xy - yx$, while the symbol $x \circ y$ will stand for anti-commutator $xy + yx$. The symbol Q means maximal right ring of quotients of R. The center of Q, denoted by C, is called *the extended centroid* of R. It is remarked that R is a prime ring if and only if C is a field, we refer the reader to [4] for these objects. An additive mapping $f : R \to R$ is said to be *a left centralizer* of R if $f(xy) = f(x)y$ holds for all $x, y \in R$. An additive mapping $d : R \to R$ is called *a derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $d : R \to R$ is called *a skew derivation* if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$ with associated automorphism α of R. In the following definition, we extended the notions of skew derivations.

Definition 1.1. Let α be an automorphism of R and $b \in Q$. An additive mapping $d : R \to Q$ is called *a skew b-derivation* with the associated term (b, α) if

(1) $$d(xy) = d(x)y + b\alpha(x)d(y) \text{ for all } x, y \in R.$$

The concept of skew b-derivation with the associated term (b, α) covers the concepts of skew derivation, derivation and left centralizer. For $b = 1$, skew

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b-derivation becomes a skew derivation and for $b = 1$, $\alpha = I_R$, the identity mapping of R, skew b-derivation becomes a derivation. If we take $b = 0$ in (1), then skew b-derivation becomes a left centralizer. Therefore, it is resonant to study about skew b-derivations. The main objective of the present paper is to describe the structure of skew b-derivations on prime rings.

In view of Definition 1.1, we observe that any skew derivation is skew 1-derivation with associated term $(1, \alpha)$ and a derivation is skew 1-derivation with the associated term $(1, I_R)$, where $\alpha : R \to R$ is an automorphism. But the converse need not be true in general. The following example justifies the fact:

Example 1.2. *Let* $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| \, x, y, z \in \mathbb{Z} \right\}$, *where* $\mathbb{Z}$ *is the ring of integers. Define the mappings* $d,\ \alpha : R \to R$ *such that*

$$d \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \text{ and } \alpha \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & -y \\ 0 & z \end{pmatrix}.$$

It can be easily verify that for $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, d *is a skew* b*-derivation of* R *with the associated term* (b, α), *but* d *is neither a skew derivation nor a derivation of* R.

Posner [14] initiated the study of such mappings, and he established a relationship between the commutativity and derivations of prime rings. Later on, many authors studied the action of such types of mappings (like derivations, skew derivations, generalized derivations, module derivations, etc.) on rings and algebras in various directions (see [1, 2, 6–8, 10, 12, 15] and [13], where further references can be found). In [10], Herstein proved the following classical result: if R is a prime ring of characteristic not two admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Further, Daif [8] showed that a 2-torsion free semiprime ring R admits a derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in I$, where I is a nonzero ideal of R and d is nonzero on I, then R contains a nonzero central ideal. In the same paper, he also proved that if a semiprime ring R admits a derivation d which is nonzero on an ideal I of R and satisfying the condition that $d([x, y]) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal. In [9], Daif and Bell showed that a semiprime ring R must be commutative if it admits a derivation d such that either $d([x, y]) - [x, y] = 0$ for all $x, y \in R$ or $d([x, y]) + [x, y] = 0$ for all $x, y \in R$. This result was extended by Rehman and Raza [15] as follows: let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. Then if d is a skew derivation of R with associated automorphism α such that $d([x, y]) = [x, y]_n$ for all $x, y \in I$, then R is commutative.

This paper aims to study the various differential identities of prime rings and depict the specific form of skew b-derivations. Notably, we examine the results mentioned above for skew b-derivations and prove that if d is a skew b-derivation of a noncommutative prime ring R satisfying one of the following conditions: (i) $[d(x),$
$d(y)] = 0$ (ii) $d([x, y]) = 0$ (iii) $d(x)d(y) \pm d([x, y]) = 0$ for all $x, y \in R$, then $d = 0$. Besides these results, we also substantiate that if R is a noncommutative prime ring and d a nonzero skew b-derivation of R which satisfies either $d([x, y]) \pm [x, y] = 0$ or $d(x \circ y) \pm x \circ y = 0$ for all $x, y \in R$, then $d(x) = \mp x$ for $x \in R$.

2. **The results**

We commence our discussions with the following result:

Theorem 2.1. *Let R be a 2-torsion free prime ring, α an automorphism of R, $b \in Q$ and d a skew b-derivation of R with the associated term (b, α). If R is noncommutative and $[d(x), d(y)] = 0$ for all $x, y \in R$, then $d = 0$.*

Proof. On the contrary, we suppose that $d \neq 0$. By the assumption, we have

$$[d(x), d(y)] = 0 \tag{2}$$

for all $x, y \in R$. Replacing y by yt in (2), we get

$$d(y)[d(x), t] + [d(x), b\alpha(y)]d(t) = 0$$

for all $x, y, t \in R$. Substituting tr in place of t in the last relation, we find that

$$d(y)t[d(x), r] + [d(x), b\alpha(y)]b\alpha(t)d(r) = 0 \tag{3}$$

for all $x, y, t, r \in R$. Replacing t by ts in (3), we get

$$d(y)ts[d(x), r] + [d(x), b\alpha(y)]b\alpha(t)\alpha(s)d(r) = 0 \tag{4}$$

for all $x, y, r, s, t \in R$. Writing rs instead of r in (3), we obtain

$$d(y)ts[d(x), r] + [d(x), b\alpha(y)]b\alpha(t)b\alpha(s)d(r) = 0 \tag{5}$$

for all $x, y, r, s, t \in R$. Combining (4) and (5), we find that

$$[d(x), b\alpha(y)]b\alpha(t)(b\alpha(s) - \alpha(s))d(r) = 0 \tag{6}$$

for all $x, y, r, s, t \in R$. Since α is an automorphism of R, it follows that

$$[d(x), bz]bR(bu - u)d(r) = (0)$$

for all $x, z, r, u \in R$. By the primeness of R and the fact $d \neq 0$ yields

$$[d(x), bz]b = 0 \text{ or } bu - u = 0.$$

Now we consider the following cases:
Case 1: Consider the first case

$$[d(x), bz]b = 0 \tag{7}$$

for all $x, z \in R$. Replacing y by $\alpha^{-1}(y)$ in (3) and using (7), we get

$$d(\alpha^{-1}(y))t[d(x), r] = 0$$

for all $x, y, r, t \in R$. This implies that $d(y)R[d(x), r] = (0)$ for all $x, y, r \in R$. Using the primeness of R and $d \neq 0$, we find that $[d(x), r] = 0$ for all $x, r \in R$. That is,

$$(8) \qquad d(x) \in Z(R)$$

for all $x \in R$. Thus we have

$$\begin{aligned} 0 &= [d(xy), y] \\ &= [d(x)y + b\alpha(x)d(y), y] \\ &= [b\alpha(x), y]d(y). \end{aligned}$$

This further gives that

$$[bz, y]Rd(y) = (0) \text{ for all } y, z \in R.$$

The primeness of R infers that $d(y) = 0$ or $[bz, y] = 0$. Set $A := \{y \in R \mid d(y) = 0\}$ and $B := \{y \in R \mid [bz, y] = 0 \text{ for all } z \in R\}$. Clearly, A and B are additive subgroups of R such that $A \cup B = R$. But, a group cannot be written as a union of its two proper subgroups, consequently $A = R$ or $B = R$. The first case contradicts our supposition that $d \neq 0$. Thus, we have $[bz, y] = 0$ for all $y, z \in R$. Replacing z by zr in the last relation, we obtain $bz[r, y] = 0$ for all $y, z, r \in R$. Since R is a noncommutative prime ring, it follows that $b = 0$. Therefore, $d(xy) = d(x)y$ for all $x, y \in R$. By [3, Lemma 2.3], we get $d(x) = qx$ for all $x \in R$ for some $q \in Q$. Thus (2) becomes

$$[qx, qy] = 0 \text{ for all } x, y \in R.$$

Substituting yz for y in the above relation, we obtain

$$qy[qx, z] = 0 \text{ for all } x, y \in R.$$

In view of primeness of R, we conclude that $q = 0$ or $[qx, z] = 0$. In the either case, we can easily conclude that $q = 0$, which gives $d = 0$, a contradiction.

Case 2: If $bs - s = 0$ for all $s \in R$, then (3) reduces to

$$(9) \qquad d(y)t[d(x), r] + [d(x), \alpha(y)]\alpha(t)d(r) = 0$$

for all $x, y, t, r \in R$. In particular, for $r = d(r)$, we have

$$[d(x), \alpha(y)]\alpha(t)d^2(r) = 0$$

for all $x, y, t, r \in R$. This gives

$$[d(x), y]Rd^2(r) = (0)$$

for all $x, y, r \in R$. Since R is prime, either $[d(x), y] = 0$ or $d^2(r) = 0$. First let $d^2(r) = 0$. Now replace r by rs for all $r, s \in R$, we get $2d(r)d(s) = 0$ for all $r, s \in R$. This implies that $d(r)Rd(s) = 0$ for all $r, s \in R$ and hence $d(r) = 0$ for all $r \in R$, which is a contradiction. Consequently, we conclude that $[d(x), y] = 0$ for all $x, y \in R$ and hence $d(x) \in Z(R)$. Using the same arguments as we have used in Case 1 after (8), we get the required result. Therefore the proof is completed. □

Following is immediate corollary of Theorem 2.1.

Corollary 2.2. *Let R be a 2-torsion free prime ring, α an automorphism of R and d a skew derivation of R with the associated automorphism α. If R is noncommutative and $[d(x), d(y)] = 0$ for all $x, y \in R$, then $d = 0$.*

Theorem 2.3. *Let R be a prime ring, α an automorphism of R, $b \in Q$ and d a nonzero skew b-derivation of R with the associated term (b, α). If R is noncommutative and $d([x, y]) - [x, y] = 0$ for all $x, y \in R$, then $b = 0$ and $d(x) = x$ for all $x \in R$.*

Proof. By the assumption, we have

$$d([x, y]) - [x, y] = 0 \tag{10}$$

for all $x, y \in R$. Replacing y by yt in (10), we get

$$b\alpha([x, y])d(t) + d(y)[x, t] + b\alpha(y)d([x, t]) - y[x, t] = 0 \tag{11}$$

for all $x, y, t \in R$. By using (10) in (11), we get

$$b\alpha([x, y])d(t) + d(y)[x, t] + b\alpha(y)[x, t] - y[x, t] = 0,$$

which implies

$$b\alpha([x, y])d(t) + (d(y) + b\alpha(y) - y)[x, t] = 0 \tag{12}$$

for all $x, y, t \in R$. Replacing t by tr in (12), we get

$$b\alpha([x, y])b\alpha(t)d(r) + (d(y) + b\alpha(y) - y)t[x, r] = 0 \tag{13}$$

for all $x, y, t, r \in R$. Putting $r = x$ in (13), we get $b\alpha([x, y])b\alpha(t)d(x) = 0$. This implies

$$b\alpha([x, y])bRd(x) = (0) \tag{14}$$

for all $x, y \in R$. Using the primeness of R, we have $b\alpha([x, y])b = 0$ or $d(x) = 0$. Set $A = \{x \in R \mid d(x) = 0\}$ and $B = \{x \in R \mid b\alpha([x, y])b = 0 \text{ for all } y \in R\}$. Clearly, A and B are additive subgroups of R such that $A \cup B = R$. But, a group cannot be written as a union of its two proper subgroups, consequently $A = R$ or $B = R$. The first case contradicts our supposition that $d \neq 0$. Thus, we are left with the case

$$b\alpha([x, y])b = 0 \tag{15}$$

for all $x, y \in R$. By using (15) in (13), we have

$$(d(y) + b\alpha(y) - y)t[x, r] = 0$$

for all $x, y, t, r \in R$. Since R is a noncommutative prime ring, so the above expression gives that

$$d(y) + b\alpha(y) - y = 0 \tag{16}$$

for all $y \in R$. Replacing y by yt in (16), we obtain

$$d(y)t + b\alpha(y)d(t) + b\alpha(y)\alpha(t) - yt = 0$$

for all $y, t \in R$. By using (16) in the last relation, we have

$$-b\alpha(y)t + b\alpha(y)d(t) + b\alpha(y)\alpha(t) = 0$$

for all $y, t \in R$. Replacing y by $\alpha^{-1}(y)$ in the above expression, we obtain

$$by(-t + d(t) + \alpha(t)) = 0$$

for all $y, t \in R$. This implies $b = 0$ or $-t + d(t) + \alpha(t) = 0$. If $b = 0$, then from (16), we get $d(y) = y$ for all $y \in R$, which is the required result. On the other hand, if

$$-t + d(t) + \alpha(t) = 0$$

for all $t \in R$. This can be written as

$$d(t) = t - \alpha(t) \tag{17}$$

for all $t \in R$. Replacing t by $[x, y]$ in (17), we get

$$d([x, y]) = [x, y] - \alpha([x, y]) \tag{18}$$

for all $x, y \in R$. By using (10), we have $\alpha([x, y]) = 0$ for all $x, y \in R$, which yields $[x, y] = 0$ for all $x, y \in R$. This contradicts the fact that R is noncommutative. This completes the proof. □

Corollary 2.4. *Let R be a prime ring, α an automorphism of R, $b \in Q$ and d a nonzero skew b-derivation of R with the associated term (b, α). If R is noncommutative and $d(xy) - xy = 0$ for all $x, y \in R$, then $b = 0$ and $d(x) = x$ for all $x \in R$.*

Proof. We have

$$d(xy) - xy = 0 \text{ for all } x, y \in R. \tag{19}$$

Interchanging the role of x and y, we get

$$d(yx) - yx = 0 \text{ for all } x, y \in R. \tag{20}$$

Subtracting (19) from (20), we obtain

$$d([x, y]) - [x, y] = 0 \text{ for all } x, y \in R.$$

Application of Theorem 2.3 gives the required result. □

Corollary 2.5. *Let R be a noncommutative prime ring and f a nonzero left centralizer of R. If $f(xy) - xy = 0$ for all $x, y \in R$, then $f(x) = x$ for all $x \in R$.*

Using a similar approach with necessary variations, we can prove the following:

Theorem 2.6. *Let R be a prime ring, α an automorphism of R, $b \in Q$ and d a nonzero skew b-derivation of R with the associated term (b, α). If R is noncommutative and $d([x, y]) + [x, y] = 0$ for all $x, y \in R$, then $b = 0$ and $d(x) = -x$ for all $x \in R$.*

Corollary 2.7. *Let R be a prime ring, α an automorphism of R, $b \in Q$ and d a nonzero skew b-derivation of R with the associated term (b, α). If R is noncommutative and $d(xy)+xy = 0$ for all $x, y \in R$, then $b = 0$ and $d(x) = -x$ for all $x \in R$.*

Corollary 2.8. *Let R be a noncommutative prime ring and f a nonzero left centralizer of R. If $f([x, y]) + [x, y] = 0$ for all $x, y \in R$, then $f(x) = -x$ for all $x \in R$.*

Theorem 2.9. *Let R be a prime ring, α an automorphism of R, $b \in Q$ and d a nonzero skew b-derivation of R with the associated term (b, α). If R is noncommutative and $d(x \circ y) - x \circ y = 0$ for all $x, y \in R$, then $b = 0$ and $d(x) = x$ for all $x \in R$.*

Proof. By the assumption, we have

$$d(x \circ y) - x \circ y = 0 \tag{21}$$

for all $x, y \in R$. Replacing y by yt in (21), we get

$$b\alpha(x \circ y)d(t) - d(y)[x, t] - b\alpha(y)d([x, t]) + y[x, t] = 0 \tag{22}$$

for all $x, y, t \in R$. For $t = x$, the above expression becomes

$$b\alpha(x \circ y)d(x) = 0$$

for all $x, y \in R$. This can be written as

$$b\alpha(x)zd(x) + bz\alpha(x)d(x) = 0$$

for all $x, z \in R$. In view of [11, Lemma 1.3.2] and our supposition $d \neq 0$, we conclude that $b = 0$ or $b\alpha(x)$ and b are linearly independent over C that is, $b = \delta b\alpha(x)$ for some $\delta \in C$. Taking $x = 0$ in the later case, we obtain $b = 0$. Thus in either cases, we have $b = 0$. Therefore (22) reduces to

$$d(y)[x, t] - y[x, t] = 0$$

for all $x, y, t \in R$. This further gives that

$$(d(y) - y)R[x, t] = (0)$$

for all $x, y, t \in R$. Since R is a noncommutative prime ring, we deduce that $d(y) = y$ for all $y \in R$. This proves the theorem. □

Application of Theorem 2.9 gives the following analogy.

Corollary 2.10. *Let R be a noncommutative prime ring and f a nonzero left centralizer of R. If $f(x \circ y) - x \circ y = 0$ for all $x, y \in R$, then $f(x) = x$ for all $x \in R$.*

Correspondingly, we can establish the following results.

Theorem 2.11. *Let R be a prime ring, α an automorphism of R, $b \in Q$ and d a nonzero skew b-derivation of R with the associated term (b, α). If R is noncommutative and $d(x \circ y) + x \circ y = 0$ for all $x, y \in R$, then $b = 0$ and $d(x) = -x$ for all $x \in R$.*

Corollary 2.12. *Let R be a noncommutative prime ring and f a nonzero left centralizer of R. If $f(x \circ y) + x \circ y$ for all $x, y \in R$, then $f(x) = -x$ for all $x \in R$.*

Theorem 2.13. *Let R be a noncommutative prime ring, α an automorphism of R, $b \in Q$ and d a skew b-derivation of R with the associated term (b, α). There is no nonzero skew b-derivation d of R which satisfies the condition $d([x, y]) = 0$ for all $x, y \in R$.*

Proof. By the given condition, we have

$$d([x, y]) = 0 \tag{23}$$

for all $x, y \in R$. Replacing y by yt in (23), we get

$$b\alpha([x, y])d(t) + d(y)[x, t] = 0 \tag{24}$$

for all $x, y, t \in R$. Substituting tr in place of t in (24), we obtain

$$b\alpha([x, y])b\alpha(t)d(r) + d(y)t[x, r] = 0 \tag{25}$$

for all $x, y, t, r \in R$. In particular, for $r = x$ we have

$$b\alpha([x, y])b\alpha(t)d(x) = 0$$

for all $x, y, t \in R$. Using the same arguments, as we have used in Theorem 2.3 after (14), we find that $d(x) = 0$ for all $x \in R$ or $b\alpha([x, y])b = 0$ for all $x, y \in R$. If $d(x) = 0$ for all $x \in R$, then proof is done. On the other hand, if $b\alpha([x, y])b = 0$ for all $x, y \in R$, then (25) reduces to

$$d(y)t[x, r] = 0$$

for all $x, y, t, r \in R$. This gives again $d = 0$, which is the required result. □

Corollary 2.14. *Let R be a noncommutative prime ring, α an automorphism of R and d a skew derivation of R with the associated automorphism α. There is no nonzero skew derivation d of R which satisfies the condition $d([x, y]) = 0$ for all $x, y \in R$.*

Corollary 2.15. *Let R be a noncommutative prime ring and d a derivation of R. There is no nonzero derivation d of R which satisfies the condition $d([x, y]) = 0$ for all $x, y \in R$.*

Corollary 2.16. *Let R be a noncommutative prime ring and f a left centralizer of R. If $f([x, y]) = 0$ for all $x, y \in R$, then $f = 0$.*

Theorem 2.17. *Let R be a noncommutative prime ring, α an automorphism of R, $b \in Q$ and d a nonzero skew b-derivation of R with the associated term (b, α). If $d(x)d(y) - xy = 0$ for all $x, y \in R$, then $b = 0$ and $d(x) = qx$ for all $x \in R$ and for some $q \in C$ such that $q^2 = 1$.*

Proof. By the hypothesis, we have

$$d(x)d(y) - xy = 0 \tag{26}$$

for all $x, y \in R$. Replacing y by yt in (26), we find that

$$d(x)b\alpha(y)d(t) = 0 \tag{27}$$

for all $x, y, t \in R$. This gives

$$d(x)bRd(x)b = (0)$$

for all $x \in R$. Using the primeness of R and the fact that $d \neq 0$, we conclude that $b = 0$. Thus $d(xy) = d(x)y$ and hence by [3, Lemma 2.3], there is, $q \in Q$ such that $d(x) = qx$. Now, from (26), we have

$$qxqy - xy = 0$$

for all $x, y \in R$. This implies

$$qxq = x$$

for all $x \in R$. Since R and Q satisfies the same polynomial identities [4, Theorem 6.4.4], so

$$qxq = x \tag{28}$$

for all $x \in Q$. In particular, for $x = 1$ we have $q^2 = 1$. Using the last relation in (28), we easily conclude that $xq = qx$ for all $x \in Q$, that is, $q \in C$. This proves the theorem. □

Corollary 2.18. *Let R be a noncommutative prime ring and f a nonzero left centralizer of R. If $f(x)f(y) - xy = 0$ for all $x, y \in R$, then $f(x) = qx$ for all $x \in R$ and for some $q \in C$ such that $q^2 = 1$.*

Theorem 2.19. *Let R be a noncommutative prime ring, α an automorphism of R, $b \in Q$ and d a skew b-derivation of R with the associated term (b, α). There is no nonzero skew b-derivation d of R which satisfies the condition $d(x)d(y) - d([x, y]) = 0$ for all $x, y \in R$.*

Proof. By the given condition, we have

$$d(x)d(y) - d([x, y]) = 0 \tag{29}$$

for all $x, y \in R$. Replacing y by yt in (29), we obtain

$$d(x)d(y)t + d(x)b\alpha(y)d(t) - d([x, y])t - b\alpha([x, y])d(t) - d(y)[x, t] - b\alpha(y)d([x, t]) = 0$$

for all $x, y, t \in R$. Using (29) in the above expression, we find that

$$d(x)b\alpha(y)d(t) - b\alpha([x, y])d(t) - d(y)[x, t] - b\alpha(y)d(x)d(t) = 0 \tag{30}$$

for all $x, y, t \in R$. Writing tr instead of t in (30), we get

(31)
$$d(x)b\alpha(y)b\alpha(t)d(r) - b\alpha([x, y])b\alpha(t)d(r) - d(y)t[x, r] - b\alpha(y)d(x)b\alpha(t)d(r) = 0$$

for all $x, y, t, r \in R$. In particular, for $r = x$ we have

$$d(x)b\alpha(y)b\alpha(t)d(x) - b\alpha([x, y])b\alpha(t)d(x) - b\alpha(y)d(x)b\alpha(t)d(x) = 0$$

for all $x, y, t \in R$. This further gives that

$$(d(x)b\alpha(y)b - b\alpha([x, y])b - bd(y)d(x)b)Rd(x) = (0)$$

for all $x, y \in R$. The primeness of R yields $d(x)b\alpha(y)b \pm b\alpha([x, y])b - bd(y)d(x)b = 0$ or $d(x) = 0$. The later case give the required result. In the first case, if

$$d(x)b\alpha(y)b \pm b\alpha([x, y])b - bd(y)d(x)b = 0 \quad (32)$$

for all $x, y \in R$, then (31) reduces to $d(y)t[x, r] = 0$ for all $x, y, t, r \in R$. Since R is a noncommutative prime ring, so the last relation yields $d = 0$. Thereby the proof is completed. □

We obtain the following results with immediate consequences of the above theorem.

Corollary 2.20. *Let R be a noncommutative prime ring, α an automorphism of R and d a skew derivation of R with associated automorphism α. There is no nonzero skew derivation d of R which satisfies the condition $d(x)d(y) - d([x, y]) = 0$ for all $x, y \in R$.*

Corollary 2.21. *Let R be a noncommutative prime ring and d a derivation of R. There is no nonzero derivation d of R which satisfies the condition $d(x)d(y) - d([x, y]) = 0$ for all $x, y \in R$.*

Corollary 2.22. *Let R be a noncommutative prime ring and f a left centralizer of R. If $f(x)f(y) - f([x, y]) = 0$ for all $x, y \in R$, then $f = 0$.*

Theorem 2.23. *Let R be a prime ring, α an automorphism of R, $b \in Q$ and d a skew b-derivation of R with the associated term (b, α). There is no nonzero skew b-derivation d of R which satisfies the condition $d(x)d(y) + d([x, y]) = 0$ for all $x, y \in R$.*

Corollary 2.24. *Let R be a noncommutative prime ring, α an automorphism of R and d a skew derivation of R with associated automorphism α. There is no nonzero skew derivation d of R which satisfies the condition $d(x)d(y) + d([x, y]) = 0$ for all $x, y \in R$.*

Corollary 2.25. *Let R be a noncommutative prime ring and d a derivation of R. There is no nonzero derivation d of R which satisfies the condition $d(x)d(y) + d([x, y]) = 0$ for all $x, y \in R$.*

Corollary 2.26. *Let R be a noncommutative prime ring and f a left centralizer of R. If $f(x)f(y) + f([x, y]) = 0$ for all $x, y \in R$, then $f = 0$.*

The following example shows the necessity of primeness in Theorems 2.1, 2.13, 2.19 and 2.23.

Example 2.27. *Let* $R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \middle| \ x, y, z \in \mathbb{Z} \right\}$*, where* $\mathbb{Z}$ *is the ring of integers. It is straightforward to check that it forms a ring with respect to matrix addition and matrix multiplication. But it is not a prime ring. Define the mappings* $d,\ \alpha : R \to R$ *such that*

$$d \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \alpha \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x & y \\ 0 & 0 & -z \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily verify that for $b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*,* d *is a skew* b*-derivation with the associated term* (b, α) *and satisfy the conditions of Theorems 2.1, 2.13, 2.19 and 2.23. However,* d *is nonzero on* R*.*

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