Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

SOME RESULTS ON UNCORRELATED DEPENDENT RANDOM VARIABLES

A. Dolati ✉, M. Amini, and G.R. Mohtashami Borzadaran

Dedicated to sincere professor Mashaallah Mashinchi

Article type: Research Article

(Received: 27 April 2022, Received in revised form: 28 June 2022)

(Accepted: 09 August 2022, Published Online: 10 August 2022)

Abstract. In probability and statistics the earliest concept related to independence is the uncorrelatedness. It is well known that a pair of independent random variables are uncorrelated, but uncorrelated random variables may or may not be independent. The aim of this paper is to provide some new models for the joint distribution of the uncorrelated random variables that are not independent. The proposed models include a bivariate mixture structure, a transformation method, and copula method. Several examples illustrating the results are included.

Keywords: Copula, Dependent, Independent, Sub-independence, Uncorrelated.
2020 MSC: Primary 62H20, 62H10, 62E10.

1. Introduction

Independence is a basic notion in probability and statistics. The earliest concept related to independence is the uncorrelatedness defined by the condition $cov(X,Y) = E(XY) - E(X)E(Y) = 0$. As David [9] points out, it took a long time for statisticians to distinguish between zero correlation and statistical independence. Uncorrelatedness is a weaker condition than independence. It is well known that a pair of independent random variables are uncorrelated, but uncorrelated random variables may or may not be independent [25]. For example, if $X \sim N(0,1)$ and $Y = -X^2$, then, $cov(X,Y) = 0$, but X and Y are strongly dependent. When two random variables are bivariate normal, the uncorrelatedness implies independence. This result is not valid for the case that the univariate marginal distributions are normal and their joint distribution is non-normal. As the above example shows, since $X + Y = 0$, we note that (X,Y) are not jointly normal though they are marginally normal. More examples of non-normal bivariate distributions with univariate normal marginals can be found in [6,8,23]. Uncorrelated dependent random variables has very extensive applications. Recently, much attention has been paid to time series models with uncorrelated but dependent errors [4,13,14,26,27,29,30]. In

✉ adolati@yazd.ac.ir, ORCID: 0000-0003-3220-3171
DOI: 10.22103/jmmr.2022.19416.1245
Publisher: Shahid Bahonar University of Kerman
How to cite: A. Dolati, M. Amini and G.R. Mohtashami Borzadaran, *Some results on uncorrelated dependent random variables*, J. Mahani Math. Res. 2022; 11(3): 175-189.

financial markets, there is the fact that the returns of assets tend to be uncorrelated, but dependent [3]. In some regression models $Y = h(X) + \epsilon$, the standard assumption is that $h(X)$ and ϵ are uncorrelated, however, they may not be independent [3]. Motivated by these applications, many attempts have been made to construct and study the properties of uncorrelated dependent random variables [2, 5, 6, 8, 12, 16–18, 21, 24, 33, 35, 39]. In practice, one may fit models for the univariate marginal distribution of each variables and test for lack of correlation and independence between them. If tests reject the hypothesis of independence but not the uncorrelatedness, the uncorrelated dependent models can be appropriate in such cases. Since the dependency structure can be of many types, choosing the right model is a challenging task. Introducing new flexible models can be useful in choosing the right model for different situations. The aim of this paper, is to provide some simple methods for constructing uncorrelated dependent random variables. Section 2, generalizes the mixture method proposed in Behboodian [6]. In Section 3, a transformation approach is proposed. Section 4 is devoted to the copula methods for constructing uncorrelated dependent random variables. Section 5, provides some examples of seemingly dependent random variables, that is the independent random variables that seem dependent. Section 6 gives a brief conclusion.

2. Uncorrelated dependent via Mixtures

Behboodian [6] used a bivariate mixture to construct examples of uncorrelated dependent random variables. For $i = 1, 2$, let X_i and Y_i be two independent random variables with the univariate marginal distribution functions F_i (of X_i) and G_i (of Y_i). Consider the pair (X, Y) as the mixture of (X_1, Y_1) and (X_2, Y_2) with the joint distribution function

$$H(x, y) = pF_1(x)G_1(y) + qF_2(x)G_2(y), \tag{1}$$

where $0 < p < 1$ and $p + q = 1$. Behboodian [6] showed that X and Y are independent if, and only if

$$[F_1(x) - F_2(x)][G_1(y) - G_2(y)] = 0. \tag{2}$$

If X_1 and X_2 (or Y_1 and Y_2) are not identically distributed, then X and Y are dependent. If these random variables have finite second moments with $E(X_1) = E(X_2)$ or $E(Y_1) = E(Y_2)$, then X and Y are uncorrelated. A concept stronger than uncorrelatedness but weaker than independence, is sub-independence was first introduced by Durairajan [11] and studied by Hamedani (and co-authors) in various papers [19,20]. Two random variables X and Y are said to be sub-independent if the distribution of $X + Y$ is the convolution of the distributions of X and Y, i.e. $F_{X+Y}(t) = F_X \star F_Y(t) = \int_{-\infty}^{\infty} F_X(t - x)dF_Y(x)$. Recently Schennach [37] provided some characterizations of sub-independence and constructive methods to generate sub-independent random variables. The following result showed that sub-independent random variables are uncorrelated [18].

Proposition 2.1. *If X and Y are sub-independent with finite second moments, then they are uncorrelated.*

Proof. An equivalent definition for sub-independency of two random variables X and Y is that $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$, for all $t \in R$, where ϕ_X, ϕ_Y and ϕ_{X+Y} are the characteristic functions of X, Y and $X+Y$ respectively. Taking the first and second derivatives from both sides of this equality, gives the required result. □

The sub-independent analogue of equation (2) in terms of the characteristic functions is given by

$$[\phi_1(t) - \phi_2(t)][\psi_1(t) - \psi_2(t)] = 0, \quad t \in R,$$

where ϕ_1, ϕ_2, ψ_1 and ψ_2 are the characteristic functions of X_1, X_2, Y_1 and Y_2; see, e.g., [18]. The following result provides a generalization of the mixture structure (1).

Proposition 2.2. *For $i = 1, 2$, let (X_i, Y_i) has the joint distribution function H_i and the univariate marginal distribution functions F_i (of X_i) and G_i (of Y_i). For $p \in [0, 1]$, $q = 1 - p$, consider the joint distribution function*

$$H(x, y) = pH_1(x, y) + qH_2(x, y). \tag{3}$$

Then a pair (X, Y) with the joint distribution function H is uncorrelated dependent if $pCov(X_1, Y_1) + qCov(X_2, Y_2) = 0$ and $E(X_1) = E(X_2)$ or $E(Y_1) = E(Y_2)$.

Proof. The univariate marginal distribution functions F (of X) and G (of Y) are given by

$$F(x) = pF_1(x) + qF_2(x) \quad \text{and} \quad G(y) = pG_1(y) + qG_2(y).$$

From $H(x, y) = F(x)G(y)$, X and Y are independent if and only if

$$\begin{aligned} p[H_1(x, y) - F_1(x)G_1(y)] &+ q[H_2(x, y) - F_2(x)G_2(y)] \\ &+ pq[F_1(x) - F_2(x)][G_1(y) - G_2(y)] = 0. \end{aligned} \tag{4}$$

By Hoeffding's identity [34], from (4) we have

$$Cov(X, Y) = pCov(X_1, Y_1) + qCov(X_2, Y_2) + pq[E(X_1) - E(X_2)][E(Y_1) - E(Y_2],$$

which is the required result. □

In the following we provide some examples.

Example 2.3. *Let $(X_1, Y_1) \sim N_2(\mu_1, \mu_2, 4, 4, \frac{1}{4})$ and $(X_2, Y_2) \sim N_2(\theta_1, \theta_2, 9, 9, -\frac{1}{27})$. Then for a pair (X, Y) with the joint distribution*

$$H(x, y) = \frac{1}{4}H_{X_1,Y_1}(x, y) + \frac{3}{4}H_{X_2,Y_2}(x, y),$$

we have $\frac{1}{4}Cov(X_1, Y_1) + \frac{3}{4}Cov(X_2, Y_2) = 0$. If $\mu_1 = \theta_1$ or $\mu_2 = \theta_2$ then (X, Y) is uncorrelated dependent.

Note that, Proposition 2 can be used to construct the distributions of both discrete and continuous uncorrelated dependent random variables. The following example, provides an example of the discrete uncorrelated dependent variables.

Example 2.4. *Consider the joint distribution function*

$$H(x,y) = \frac{H_1(x,y) + H_2(x,y)}{2},$$

where H_1 and H_2 have the joint density functions

$$h_1(x,y) = \frac{16!}{x!y!(16-x-y)!}(\frac{1}{4})^{x+y}(\frac{1}{2})^{16-x-y}, \quad x,y \in \{0,1,...,16\}, x+y = 16,$$

and

$$h_2(x,y) = \frac{e^{-2}}{y!(x-y)!}, \quad y = 0,1,...,x, x = 0,1,2,...$$

It is easy to check that for a pair $(X_i, Y_i) \sim H_i$, $i = 1,2$, we have that $E(Y_1) = E(Y_2) = 4$ and $Cov(X_1, Y_1) = -Cov(X_2, Y_2)$. Thus the pair X and Y are uncorrelated dependent.

In Proposition 2.2 if (X_1, Y_1) and (X_2, Y_2) are uncorrelated and $E(X_1) = E(X_2)$ (or $E(Y_1) = E(Y_2)$), then the pair (X, Y) distributed as (3) is uncorrelated dependent. We also note that if $(X_2, Y_2) =^d (-X_1, Y_1)$ (or $(X_2, Y_2) =^d (X_1, -Y_1)$), then $Cov(X_2, Y_2) = -Cov(X_1, Y_1)$ and $Y_1 =^d Y_2$ (or $X_1 =^d X_2$). In this case, for $p = q = \frac{1}{2}$, the pair (X, Y) in terms of (3) distributed as

$$H(x,y) = \frac{1}{2}H_1(x,y) + \frac{1}{2}H_1^*(x,y),$$

where $H_1^*(x,y) = G_1(y) - H_1(-x,y)$ is the distribution of the pair $(-X_1, Y_1)$. A sub-independence version of the condition (4) in terms of the characteristic functions of X_1, X_2, Y_1 and Y_2 also could be used. It is easy to see that, $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$, for all t if and only if

$$\begin{aligned} p[\phi_{X_1+Y_1}(t) - \phi_{X_1}(t)\phi_{Y_1}(t)] &+ q[\phi_{X_2+Y_2}(t) - \phi_{X_2}(t)\phi_{Y_2}(t)] \\ &= pq[\phi_{X_1}(t) - \phi_{X_2}(t)][\phi_{Y_2}(t) - \phi_{Y_1}(t)]. \end{aligned} \tag{5}$$

A sufficient condition for (5) is that both of (X_1, Y_1) and (X_2, Y_2) be sub-independent and $X_1 =^d X_2$ (or $Y_1 =^d Y_2$). The following example, adopted from [19] illustrates this result.

Example 2.5. *For $i = 1,2$, consider the pair (X_i, Y_i) with the joint density function*

$$f_{X_i,Y_i}(x,y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}[1 - 16\alpha_i G(x,y)e^{-\frac{1}{2}(x^2+y^2)}], \quad x,y \in R,$$

where $G(x,y) = 6xy - 2x^2 - 2y^2 + 4x^2y^2 - 2x^3y - 2xy^3 + 1$, and the corresponding characteristic function

$$\phi_{X_i,Y_i}(t,s) = e^{-\frac{1}{2}(t^2+s^2)}[1 + \alpha_i ts(t-s)^2 e^{\frac{1}{4}(t^2+s^2)}].$$

Then for $i = 1, 2$,

$$\phi_{X_i+Y_i}(t) = \phi_{X_i,Y_i}(t,t) = e^{-t^2}, \quad \phi_{X_i}(t) = \phi_{X_i,Y_i}(t,0) = e^{-\frac{1}{2}t^2},$$
$$\phi_{Y_i}(s) = \phi_{X_i,Y_i}(0,s) = e^{-\frac{1}{2}s^2}.$$

A pair (X, Y) *distributed as (3) is sub-independent and thus uncorrelated dependent. Note that*

$$\phi_{X,Y}(t,s) = e^{-\frac{1}{2}(t^2+s^2)} + (p\alpha_1 + q\alpha_2)ts(t-s)^2 e^{\frac{1}{4}(t^2+s^2)},$$

and thus X *and* Y *are independent, if and only if,* $p\alpha_1 + q\alpha_2 = 0$.

3. Uncorrelated dependent random variables via transformation

Another strategy for constructing uncorrelated dependent random variables is the use of suitable transformations of arbitrary random variables. Behboodian [5] shows that if $g(.)$ be an odd and $h(.)$ an even real-valued function, then for a symmetric random variable X, the random variables $Y = g(X)$ and $Z = h(X)$ are uncorrelated, provided that Y and Z are non-degenerate (or non-constant) and the expectations $E(Y^2)$ and $E(Z^2)$ exist. For example, if U is a uniform $(-1, 1)$ random variable and $P(V = U^2) = 1$, then U and V are uncorrelated dependent. In the following we provide a transformation method for constructing uncorrelated dependent random variables.

Proposition 3.1. *Let* U *and* V *be two independent uniform* $(0, 1)$ *random variables and let*

$$X = \frac{g(U)}{g(U) + g(V)}, \quad Y = g(U) + g(V), \tag{6}$$

where g *is a non-negative non-increasing function with* $E(g(U)) < \infty$. *Then* X *and* Y *are uncorrelated.*

Proof. Since U and V are identically distributed, we have $\frac{g(U)}{g(U)+g(V)} =^d \frac{g(V)}{g(U)+g(V)}$ and $E\left(\frac{g(U)}{g(U)+g(V)}\right) = \frac{1}{2}$. Thus

$$\begin{aligned} Cov(X,Y) &= E[g(U)] - \frac{1}{2}E[g(U) + g(V)] \\ &= E[g(U)] - \frac{1}{2} \times 2E[g(U)] = 0. \end{aligned}$$

□

To show that X and Y are not independent in general, let $S = g(U)$ and $T = g(V)$. Then S and T has the common density function $f(t) = \frac{-1}{g'(g^{-1}(t))}$. The joint density function of the uncorrelated dependent random variables $X = \frac{S}{S+T}$ and $Y = S + T$ is given by

$$f_{X,Y}(x,y) = yf(xy)f((1-x)y)), \quad 0 < x < 1, y > 0. \tag{7}$$

As the following example shows, in general, $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$.

Example 3.2. *Let $g(t) = (-\ln(t))^\theta$, $\theta \geq 1$. Then the joint distribution function of the random variables X and Y in (6) is given by*

$$f_{X,Y}(x,y) = \frac{1}{\theta^2}(x(1-x))^{\frac{1}{\theta}-1} y^{\frac{2}{\theta}-1} e^{-y^{\frac{1}{\theta}}[x^{\frac{1}{\theta}}+(1-x)^{\frac{1}{\theta}}]}, \quad 0 < x < 1, y > 0.$$

The pair (X,Y) are dependent for $\theta \neq 1$. Since $S = -\log(U)$ and $T = -\log(V)$ are independent exponential random variables, we have $Y = S+T \sim gamma(2,1)$ and $X = \frac{S}{S+T} \sim U(0,1)$ are independent. It is a special case of the known result that the independent random variables S and T are gamma distributed (with the same scale parameter) if and only if $S+T$ and $\frac{S}{S+T}$ are independent (see [28]).

The following result provides another transformation method for constructing uncorrelated dependent random variables.

Proposition 3.3. *Let (X_1,Y_1) and (X_2,Y_2) be two sub-independent pairs with the finite second moments. If (X_1,Y_1) and (X_2,Y_2) are independent, then the pair (X_1+X_2, Y_1+Y_2) is sub-independent and thus uncorrelated dependent.*

Proof. Since (X_1,Y_1) and (X_2,Y_2) are independent and for $i = 1,2$, (X_i,Y_i) is sub-independent, then the characteristic function of X_1+X_2, Y_1+Y_2 satisfies

$$\begin{aligned} \phi_{X_1+X_2}(t)\phi_{Y_1+Y_2}(t) &= \phi_{X_1}(t)\phi_{X_2}(t)\phi_{Y_1}(t)\phi_{Y_2}(t) \\ &= \phi_{X_1+Y_1}(t)\phi_{X_2+Y_2}(t) \\ &= \phi_{X_1+X_2+Y_1+Y_2}(t), \end{aligned}$$

which is the required result. □

Example 3.4. *For $i = 1,2$, let (X_i,Y_i) be the sub-independent pairs considered in Example 2.5. If (X_1,Y_1) and (X_2,Y_2) are independent, then the joint characteristic function of X_1+X_2 and Y_1+Y_2 is given by*

$$\phi_{X_1+X_2,Y_1+Y_2}(t,s) = e^{-(t^2+s^2)}[1+\alpha_1 ts(t-s)^2 e^{-\frac{1}{4}(t^2+s^2)}][1+\alpha_2 ts(t-s)^2 e^{-\frac{1}{4}(t^2+s^2)}].$$

Clearly, X_1+X_2 and Y_1+Y_2 are dependent when $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$, but they are sub-independent and thus uncorrelated.

4. The copula of uncorrelated dependent random variables

A bivariate copula is the restriction to $[0,1]^2$ of the cumulative distribution function of a vector (U,V) of uniform $(0,1)$ random variables, i.e., for all $u_1,u_2 \in [0,1]$, $C(u_1,u_2) = P(U_1 \leq u, U_2 \leq u_2)$ and $P(U_i \leq u_i) = u_i$, $i = 1,2$. In statistics, copulas are used as a tool for modelling dependence between random variables; see, e.g., [22, 34]. Let X and Y be two continuous random variables with the joint distribution function H and marginal distribution functions F and G, respectively. Sklar [41] showed that, there exists a unique copula $C : [0,1]^2 \to [0,1]$ such that $H(x,y) = C(F(x), G(y))$ for all $x,y \in \mathbb{R}$. The main reason for using copulas in multivariate data modeling is that they allow different dependency structures for the data. In practice, by using a goodness-of-fit test procedure, first the appropriate margin distribution may be fitted separately to each of the variables. Then, by finding a suitable copula [15], appropriate model for bivariate data is proposed, by using the Sklar's theorem. Let Π denote the copula of independent random variables, i.e., $\Pi(u,v) = uv$. We let M and W denote the Fréchet-Hoeffding upper and lower bound copulas, respectively, which, for any copula C, satisfy: $\max(u+v-1,0) = W(u,v) \leq C(u,v) \leq M(u,v) = \min(u,v)$ for every $(u,v) \in [0,1]^2$; see, [34].

Proposition 4.1. *Let X and Y be two continuous random variables with the marginal distribution functions F and $G(y) = 1 - G(-y)$, respectively, and the associated copula C. If $C(u,v) = C^*(u,v)$ for all $u, v \in [0,1]$, where $C^*(u,v) = u - C(u, 1-v)$, then (X,Y) is uncorrelated dependent.*

Proof. By Hoeffding's identity [34] we have

$$\begin{aligned} Cov(X,Y) &= \int_0^\infty \int_0^\infty [C^*(F(x),G(y)) - F(x)G(y)]dxdy \\ &= \int_0^\infty \int_0^\infty [F(x) - C(F(x), 1-G(y)) - F(x)G(y)]dxdy \\ &= -\int_0^\infty \int_0^\infty [C(F(x),G(-y)) - F(x)G(-y)]dxdy \\ &= -\int_0^\infty \int_0^\infty [C(F(x),G(y)) - F(x)G(y)]dxdy \\ &= -Cov(X,Y). \end{aligned}$$

Thus $Cov(X,Y) = 0$. ☐

The above result provides a partial answer to the question, what is the copula of the uncorrelated dependent variables? In practice, if tests reject the hypothesis of independence but not the uncorrelatedness, an uncorrelated dependent copula can be appropriate in such cases. In the following some method for constructing copulas of uncorrelated dependent variables are given.

Example 4.2. *For a given copula D and $\alpha \in (0, \frac{1}{2})$, let*

$$C_\alpha(u,v) = \alpha D(u,v) + (1-2\alpha)\Pi(u,v) + \alpha D^*(u,v),$$

where $D^*(u,v) = u - D(u, 1-v)$ and $\Pi(u,v) = uv$. Note that if $D = M$ and $\alpha = \frac{1}{2}$, then $D^* = W$ and $C = \frac{M+W}{2}$. It is easy to check that the copula C_α satisfies $C_\alpha = C_\alpha^*$, and thus a pair (X,Y) having this copula structure is uncorrelated dependent. To see this, we first rewrite C_α as

$$C_\alpha(u,v) - \Pi(u,v) = \alpha[(D(u,v) - \Pi(u,v)] + \alpha[D^*(u,v) - \Pi(u,v)].$$

For two continuous random variables X and Y with the copula C_α and marginal distributions $F(x)$ and $G(y)$, respectively, we have

$$\begin{aligned} Cov(X,Y) &= \alpha \int_0^\infty \int_0^\infty [D(F(x),G(y)) - F(x)G(y)]dxdy \\ &+ \alpha \int_0^\infty \int_0^\infty [D^*(F(x),G(y)) - F(x)G(y)]dxdy. \end{aligned}$$

If $G(y) = 1 - G(-y)$, then

$$\int_0^\infty \int_0^\infty [D^*(F(x),G(y))-F(x)G(y)]dxdy = -\int_0^\infty \int_0^\infty [D(F(x),G(y))-F(x)G(y)]dxdy,$$

and thus $Cov(X,Y) = 0$.

Example 4.3. *Let C_θ be the copula defined by*

$$C_\theta(u,v) = uv + \theta\phi(u)\phi(v), \tag{8}$$

where $\theta \in [-1,1]$ and ϕ is a function on $[0,1]$ with $\phi(0) = \phi(1) = 0$ and $|\phi(u) - \phi(v)| \le |u - v|$ for all $u, v \in [0,1]$ [1]. If $\phi(u) + \phi(1-u) = 0$ for all $u \in [0,1]$, then $C = C^*$. For instance, consider the function $\phi(u) = u(1-u)(1-2u)$, for all $u \in [0,1]$. Then C_θ is a member of the parametric family of copulas with cubic sections studied in [1, 34]

Starting from a uniform $(0,1)$ random variable, the following result provides a method for construction uncorrelated dependent random variables.

Proposition 4.4. *Let U be a uniform $(0,1)$ random variable and let $g : [0,1] \to R$ be a non-decreasing function such that $g(u) + g(1-u) \in \{0,1\}$. Then, the pair $(g(U), g(V))$, where $V = 2\min(U, 1-U)$, is uncorrelated dependent.*

Proof. Since $V = 2\min(U, 1-U) = 1 - |2U - 1|$ has uniform $(0,1)$ distribution, then $E(g(U)) = E(g(V)) = \int_0^1 g(u)du$. On the other hand we have

$$\begin{aligned} E\left(g(U)g(V)\right) &= \int_0^{\frac{1}{2}} g(u)g(2u)du + \int_{\frac{1}{2}}^1 g(u)g(2(1-u))du \\ &= \int_0^{\frac{1}{2}} g(2u)[g(u) + g(1-u)]du. \end{aligned} \tag{9}$$

If $g(u)+g(1-u) = 0$, then $\int_0^1 [g(u)+g(1-u)]du = 2\int_0^1 g(u)du = 2E(g(U)) = 0$. Therefore, $Cov(g(U), g(V)) = 0$. If $g(u) + g(1-u) = 1$, then $\int_0^1 [g(u) + g(1-u)]du = 2\int_0^1 g(u)du = 1$, or equivalently, $E(g(U)) = E(g(V)) = \frac{1}{2}$. Since $E(g(U)g(V)) = \int_0^{\frac{1}{2}} g(2u)du = \frac{1}{4}$, the result follows. □

Remark 4.5. Note that the function $g(u) = u$, $u \in [0,1]$, satisfies the condition $g(u) + g(1-u) = 1$. Thus the vector (U,V), with $V = 2\min(U, 1-U)$, is uncorrelated. It is easy to check that the copula of (U,V) is given by

$$A(u,v) = \frac{M(u,v) + W(u,v)}{2},$$

where, $M(u,v) = \min(u,v)$ and $W(u,v) = \max(u+v-1,0)$. Since copulas are invariant under increasing transformation of random variables, then the copula of $(g(U), g(V))$ is A. We also note that if U is a uniform $(0,1)$ random variable and $V = \min\left(\frac{U}{\theta}, \frac{1-U}{1-\theta}\right)$, for $\theta \in (0,1)$, then V has uniform $(0,1)$ distribution and the joint distribution of the pair (U,V) is a shuffle of min copula in the sense of [32] given by

$$C_\theta(u,v) = \begin{cases} u, & 0 \leq u \leq \theta v \leq \theta, \\ \theta v, & 0 \leq \theta v < u < 1-(1-\theta)v, \\ u+v-1, & \theta \leq 1-(1-\theta)v \leq u \leq 1, \end{cases} \tag{10}$$

which is a singular copula whose probability mass is spread uniformly on two line segments joining $(0,0)$ to $(\theta,1)$ and $(\theta,1)$ to $(1,0)$. For this copula, $C_{\frac{1}{2}} = A(u,v)$. In fact, the pair (U,V) is uncorrelated dependent only if, $\theta = \frac{1}{2}$.

In the following some examples of uncorrelated dependent random variables provided by using Proposition 4.4.

Example 4.6. *Let F be the cumulative distribution function of a symmetric random variable around zero. Since $F^{-1}(u)+F^{-1}(1-u) = 0$, for all $u \in (0,1)$, then $g(u) = F^{-1}(u)$ is a generator of the uncorrelated random variables in Proposition 4.4. For example, if $\Phi(.)$ is the cumulative distribution function of standard normal random variable, then $X = \Phi^{-1}(U)$ and $Y = \Phi^{-1}(1-|2U-1|)$ are uncorrelated random variables having standard normal distribution. We note that the joint distribution of (X,Y) is a singular normal distribution.*

Example 4.7. *Let $D(.,.)$ be an arbitrary copula. For $\alpha \in (0,1)$, consider the function $g_\alpha(t) = \frac{\alpha - D(1-t,\alpha)+D(t,\alpha)}{2\alpha}$, $t \in [0,1]$. Then, $g_\alpha(.)$ is non-decreasing in t and satisfies $g_\alpha(t)+g_\alpha(1-t) = 1$. For example, let $D(u,v) = \min(u,v)$, then*

$$\begin{aligned} g_\alpha(t) &= \frac{\alpha - \min(1-t,\alpha) + \min(t,\alpha)}{2\alpha} \\ &= \frac{W(t,\alpha) + M(t,\alpha)}{2\alpha} \\ &= \frac{1}{\alpha} A(t,\alpha), \end{aligned}$$

generate the uncorrelated dependent vector $(\frac{1}{\alpha}A(U,\alpha), \frac{1}{\alpha}A(1-|2U-1|,\alpha)$.

Remark 4.8. For two copulas C_1 and C_2, we say C_2 is more concordant than C_1 (written $C_1 \prec_c C_2$) if $C_1(u,v) \leq C_2(u,v)$ for all $(u,v) \in [0,1]^2$. A copula C is positively quadrant dependent (PQD) if $\Pi \prec_c C$. By reversing the sense of this inequality we have negatively quadrant dependence (NQD) concept [34]. Note that, the copula of uncorrelated dependent random variables, for example $A(u,v) = (M(u,v) + W(u,v))/2$, is not PQD nor NQD.

The population version of three of the most common nonparametric measures of association between the components of a continuous random pair

(X, Y) are *Kendall's tau* (τ), *Spearman's rho* (ρ), and *Gini's gamma* (γ). Such measures are called measures of concordance since they satisfy a set of axioms due to Scarsini [36], depend only on the copula C of the pair (X, Y), and are given by

$$\tau(X,Y) = -1 + 4\int_0^1\int_0^1 C(x,y)\,dC(x,y), \tag{11}$$

$$\rho(X,Y) = -3 + 12\int_0^1\int_0^1 C(x,y)dxdy, \tag{12}$$

and

$$\gamma(X,Y) = -2 + 8\int_0^1\int_0^1 C(x,y)dA(x,y), \tag{13}$$

respectively, where $A(.,.)$ denotes the copula $(M+W)/2$.

Proposition 4.9. *Let (X, Y) be a random vector with the copula C satisfies $C = C^*$. Then $\tau(X, Y) = \rho(X, Y) = \gamma(X, Y) = 0$.*

Proof. Since for a copula C satisfies $C(u, v) = u - C(u, 1 - v)$, we have

$$\int_0^1\int_0^1 C(u,v)dC(u,v) = \frac{1}{2} - \int_0^1\int_0^1 C(u,v)dC(u,v),$$

$$\int_0^1\int_0^1 uvdC(u,v) = \frac{1}{2} - \int_0^1\int_0^1 uvdC(u,v),$$

and

$$\int_0^1\int_0^1 C(u,v)dA(u,v) = \frac{1}{2} - \int_0^1\int_0^1 C(u,v)dA(u,v).$$

Now, the result follows from (11), (12) and (13). □

Thus, for uncorrelated dependent random variables, the concordance measures cannot capture the dependence structures. For example for two uncorrelated dependent random variables, both or one of the measures Kendall's τ and Spearman's ρ can be zero without the variables being independent. Note that the converse of the Proposition 4.9 is not true. That is, for a dependent pair (U, V) with the copula C, the value of some measures of concordance could be zero, while C does not satisfy $C = C^*$, as the following example shows.

Example 4.10. *Let C be the copula defined by*

$$C(u,v) = \min\{u, v, \max(u - \frac{1}{2}, v - \frac{1}{2}, u + v - 1, 0)\}.$$

It is easy to see that this copula satisfies $C \neq C^*$, but $\tau(C) = 0$ and $\rho(C) = -\frac{1}{2}$.

There are several ways to measure the dependence between uncorrelated random variables. A useful index is the Schweizer and Wolff's σ [38] defined by

$$\sigma(X,Y) = 12\int_0^1\int_0^1 |C(u,v)-uv|dudv. \tag{14}$$

This measure belongs to the class of monotone dependence measures studied in [7]. For other measures of dependence between uncorrelated random variables see [12, 17].

Example 4.11. *For the shuffle of min copula C_θ given by (10) we have that $\rho(C_\theta) = \tau(C_\theta) = -1+2\theta$ and $\sigma(C_\theta) = 1-2\theta(1-\theta)$. Thus, $\rho(C_{\frac{1}{2}}) = \tau(C_{\frac{1}{2}}) = 0$ but $\sigma(C_{\frac{1}{2}}) = \frac{1}{2}$. For copula (4.3)*

$$\sigma(C_\theta) = 12|\theta|\left(\int_0^1 \phi(t)dt\right)^2.$$

For $\phi(u) = u(1-u)(1-2u)$, we have that $\sigma(C) = \frac{|\theta|}{8}$.

5. Seemingly dependent random variables

We say that two random variables Y and Z are *seemingly dependent* if they are independent but there exists a function h such that $Z = h(Y)$. Let $[X]$ (the largest integer that does not exceed X) and $\{X\} = X - [X]$, be the integer part and the fractional part of a random variable X. In some cases the accuracy of an observation is limited, that is, the integer part of $[X]$ is observable but the $\{X\}$ is unobservable. The integer part of an exponential distribution is a geometric distribution and it is known that $[X]$ and $\{X\}$ are independent [10, 31, 40, 42] and in our notion, they are seemingly dependent. Since for exponential random variable X, $P(\{X\} \le t|[X] = k) = P(\{X\} \le t)$, the distribution of $\{X\}$ is sufficient to estimate the precise value of X under the condition that $[X]$ is given. In the following, we provide some examples of seemingly dependent exponential random variables.

Proposition 5.1. *Let X be a positive stable random variable with the Laplace-Stieltjes transform $L_X(s) = E[e^{-sX}] = e^{-s^\alpha}, s > 0,\ \alpha \in (0,1)$ and let $Y \sim \exp(1)$ be independent of X. Then the random variables $[T]$ and $T-[T]$ are seemingly dependent, where $T = \left(\frac{Y}{X}\right)^\alpha$.*

Proof. Since $\left(\frac{Y}{X}\right)^\alpha \sim \exp(1)$, the result follows. □

Proposition 5.2. *Let $\{X_n, n \ge 1\}$ be a sequence of independent exponential random variables and let N be a random variable independent of X_is, with the geometric distribution. Then the random variables $[\sum_{i=1}^N X_i]$ and $\sum_{i=1}^N X_i - [\sum_{i=1}^N X_i]$ are seemingly dependent.*

Proof. The result follows from the fact that the random sum $\sum_{i=1}^{N} X_i$ has an exponential distribution. □

The following example provides an application of the integer valued seemingly dependent random variables. Estimating the number of migratory birds in a territory of wind power farms is an important problem. The main threat to birds is the possibility of their collision with the turbines located in places of their movement; see, e.g., [43]. Let N be the number of migratory birds and suppose that a bird dies in route with probability p and lives otherwise independent of other total birds, then $\sum_{i=1}^{N} X_i$ and $N - \sum_{i=1}^{N} X_i$, where $X_i \sim Ber(p)$, denote the number of birds died and live in the route, respectively. The following result shows that these functionally dependent random variable are independent and information about one of them is sufficient to estimate the other.

Proposition 5.3. *Let $\{X_n, n \geq 1\}$ be a sequence of independent Bernoulli random variables and let N be a random variable independent of X_is, with the Poisson distribution. Then $\sum_{i=1}^{N} X_i$ and $N - \sum_{i=1}^{N} X_i$ are seemingly dependent.*

Proof. Let $X_i \sim Ber(p)$ and let $N \sim Poisson(\lambda)$. Then, $S_N^1 = \sum_{i=1}^{N} X_i \sim Poisson(\lambda p)$. Let $S_N^2 = N - S_N^1$. The joint probability generating function of S_N^1 and S_N^2 is given by

$$\begin{aligned}
g_{S_N^1, S_N^2}(t_1, t_2) &= E(t_1^{S_N^1} t_2^{S_N^2}) \\
&= E[(t_1 t_2^{-1})^{S_N^1} t_2^{S_N^2}] \\
&= E[E(t_1 t_2^{-1})^{S_N^1} t_2^{S_N^2} | N] \\
&= E[t_2^N \left(g_{X_i}(t_1 t_2^{-1})\right)^N] \\
&= E[t_2^N (q + p t_1 t_2^{-1})^N] \\
&= E[(q t_2 + p t_1)^N] \\
&= \exp\{\lambda(p t_1 + q t_2 - 1)\} \\
&= \exp\{\lambda p(t_1 - 1)\} \exp\{\lambda q(t_2 - 1)\} \\
&= g_{S_N^1}(t_1) g_{S_N^2}(t_2),
\end{aligned}$$

which is the required result. □

6. Conclusions

In this paper, we first provided a generalization of the mixture method proposed in [6] for constructing uncorrelated dependent random variables. We also proposed a method based on transformations of the uniform $(0, 1)$ random variables. Nowadays, copulas are used as a tool for modelling dependence between random variables. We developed some copula based methods for constructing uncorrelated dependent random variables. A related concept to uncorrelatedness is the seemingly dependence concept. We provided several examples of

such variables. Several questions arise for further study. We present a few:
1) Are there general approaches for constructing seemingly dependent random variables?
2) It is useful to develop empirical tests of the assumption of uncorrelated dependence, similar to the case for testing independence. A test for the assumption of sub-independence is proposed in [37].
3) As we mentioned in Remark 2, the copula of uncorrelated dependent random variables does not satisfy the usual concordance ordering. There is the question of whether there are inequalities for uncorrelated dependence random variables, like those resulting from concordance ordering for dependence.
4) An application of the dependent uncorrelated models is in time series data analysis as the distribution of the error term; see, e.g., [4, 13, 14, 26, 27, 29, 30]. Comparing the proposed models as the error distribution in time series models, is another topic for further research.

7. Aknowledgement

The authors would like to thank the anonymous reviewers for their careful reading and their insightful comments and suggestions.

References

[1] C. Amblard and S. Girard, *Symmetry and dependence properties within a semiparametric family of bivariate copulas*, Journal of Nonparametric Statistics, 14, (2002), 715-727.
[2] M. Amini, H.R Nili Sani and A. Bozorgnia, *Aspects of negative dependence structures*, Communications in Statistics-Theory and Methods, 42, (2013), 907-917.
[3] H. Bazargan, H. Bahai, and A. Aminzadeh-Gohari, *Calculating the return value using a mathematical model of significant wave height*, Journal of Marine Science and Technology, 12, (2007), 34-42.
[4] B. Beckers, H. Herwartz, and M. Seidel, *Risk forecasting in TGARCH models with uncorrelated dependent innovations*, Quantitative Finance, 17(1), (2017), 121-137.
[5] J. Behboodian, *Uncorrelated dependent random variables*, Mathematics Magazine, 51(5), (1978), 303-304.
[6] J. Behboodian, *Examples of uncorrelated dependent random variables using a bivariate mixture*, The American Statistician, 44(3), (1990), 218.
[7] J. Behboodian, A. Dolati and M. Úbeda-Flores, *Measures of association based on average quadrant dependence*, Journal of Probability and Statistics, 3, (2005), 161-173.
[8] J. D. Broffitt, *Zero correlation, independence and normality*, The American Statistician, 40(4), (1986), 276-277.
[9] H. A. David, *A historical note on zero correlation and independence*, The American Statistician, 63(2), (2009), 185-186.
[10] L. Y. Deng and R.S. Chhikara, *On the characterization of the exponential distribution by the independence of its integer and fractional parts*, Statistica Neerlandica, 44(2), (1990), 83-85.
[11] T. M. Durairajan, *A classroom note on sub-independence*, Gujrat Statistical Research, 6, (1979), 17-18.
[12] N. Ebrahimi, G. G. Hamedani E. S Soofi and H. Volkmer, *A class of models for uncorrelated random variables*, Journal of Multivariate Analysis, 101(8), (2010), 1859-1871.

[13] C. Francq, R. Roy, and J. M. Zakoïan, *Diagnostic checking in ARMA models with uncorrelated errors*, Journal of the American Statistical Association, 100(470), (2005), 532-544.

[14] C. Francq and H. Raïssi, *Multivariate portmanteau test for autoregressive models with uncorrelated but nonindependent errors*, Journal of Time Series Analysis, 28(3), (2007), 454-470.

[15] C. Genest, B. Rémillard and D. Beaudoin, *Goodness-of-fit tests for copulas: A review and a power study*, Insurance: Mathematics and economics, 44(2), (2009), 199-213.

[16] J. D. Gibbons, *Mutually exclusive events, independence and zero correlation*, The American Statistician, 22, (1968), 31-32.

[17] A. K. Gupta, D. Song, and G. Székely, *The dependence of uncorrelated statistics*, Applied Mathematics Letters, 7(5), (1994), 29-32.

[18] G. G. Hamedani, H. Volkmer, and J. Behboodian, *A note on sub-independent random variables and a class of bivariate mixtures*, Studia Scientiarum Mathematicarum Hungarica, 49(1), (2012), 19-25.

[19] G. G. Hamedani, *Sub-independence: An expository perspective*, Communications in Statistics-Theory and Methods, 42(20), (2013), 3615-3638.

[20] G. G. Hamedani M. Maadooliat, *Sub-Independence: A Useful Concept.* Nova Science Publishers, 2015.

[21] K. Joag-Dev, *Independence via uncorrelatedness under certain dependence structures*, The Annals of Probability, 11, (1983), 1037-1041.

[22] H. Joe, *Dependence Modeling with Copulas.* CRC press, 2014.

[23] C. J. Kowalski, *Non-normal bivariate distributions with normal marginals*, The American Statistician, 27(3), (1973), 103-106.

[24] A. Krajka and D. Szynal, *On measuring the dependence of uncorrelated random variables*, Journal of Mathematical Sciences, 76(2), (1995), 2269-2274.

[25] H. O. Lancaster, *Zero correlation and independence*, Australian Journal of Statistics, 21, (1959), 53-56.

[26] T. H. Lee and X. Long, *Copula-based multivariate GARCH model with uncorrelated dependent errors*, Journal of Econometrics, 150(2), (2009), 207-218.

[27] I. N. Lobato, *Testing that a dependent process is uncorrelated*, Journal of the American Statistical Association, 96(455), (2001), 1066-1076.

[28] E. Lukacs, *A characterization of the gamma distribution*, Ann. Math. Statist., 26, (1955), 319–324.

[29] Y. B. Maïnassara, *Multivariate portmanteau test for structural VARMA models with uncorrelated but non-independent error terms*, Journal of Statistical Planning and Inference, 141(8), (2011), 2961-2975.

[30] Y. B. Maïnassara and H. Raïssi, *Semi-strong linearity testing in linear models with dependent but uncorrelated errors*, Statistics and Probability Letters, 103, (2015), 110-115.

[31] G. Marsaglia, *Random variables with independent integer and fractional parts*, Statistica Neerlandica, 49(2), (1995), 133-137.

[32] P. Mikusiński, H. Sherwood, and M. D. Taylor, *Shuffles of min*, Stochastica, 13, (1992), 61–74.

[33] T. F. Móri and G. J. Székely, *Representations by uncorrelated random variables*, Mathematical Methods of Statistics, 26(2), (2017), 149-153.

[34] R. B. Nelsen, *An Introduction to Copulas.* Springer Science & Business Media, 2007.

[35] E. Pollak, *A comment on zero correlation and independence*, The American Statistician, (1971), 25, 53.

[36] M. Scarsini, *On measures of concordance*, Stochastica, 8, (1984), 201–218.

[37] S. M. Schennach, *Convolution without independence*, Journal of Econometrics, 211(1), (2019), 308-318.

[38] B. Schweizer E. F. Wolff, *On nonparametric measures of dependence for random variables*, Annals of Statistics, 9, (1981), 879-885.

[39] W. Shih, *More on zero correlation and independence*, The American Statistician, (1971), 25, 62.

[40] T. Shimura, *Limit distribution of a roundoff error*, Statistics and Probability Letters, 82(4), (2012), 713-719.

[41] A. Sklar, *Fonctions de repartition á n dimensions et leurs marges*, Publications de l'Institut de statistique de l'Université de Paris, 8, (1959), 229-231.

[42] F. W. Steutel and J. G. F Thiemann, *On the independence of integer and fractional parts*, Statistica Neerlandica, **43(1)**, (1989), 53-59.

[43] V. Yeremieiev, *Forecasting of the number of bird collisions with turbines in the territory of Pre-Azov region wind park using the route census method*, Proceedings of E3S Web of Conferences Vol. 280, (EDP Sciences, 2021), p. 06010.

A. Dolati[1,2]
Orcid number: 0000-0003-3220-3171
[1] Department of Statistics
Ferdowsi University of Mashhad
Mashhad, Iran
[2] Department of Statistics
Yazd University
Yazd, Iran
Email address: adolati@yazd.ac.ir, dolati50@yahoo.com

M. Amini
Orcid number: 0000-0002-8336-201X
Department of Statistics
Ferdowsi University of Mashhad
Mashhad, Iran
Email address: m-amini@um.ac.ir

G.R. Mohtashami Borzadaran
Orcid number: 0000-0002-8841-1386
Department of Statistics
Ferdowsi University of Mashhad
Mashhad, Iran
Email address: grmohtashami@um.ac.ir