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ANALYTICAL EXPRESSION FOR THE EXACT CURVED SURFACE AREA AND VOLUME OF HYPERBOLOID OF TWO SHEETS VIA MELLIN-BARNES TYPE CONTOUR INTEGRATION

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Abstract. In this article, we aim at obtaining the analytical expression (**not previously found and recorded in the literature**) for the exact curved surface area of a hyperboloid of two sheets in terms of Appell's double hypergeometric function of second kind and triple hypergeometric function of Srivastava. The derivation is based on Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$, Meijer's G-function and series manipulation technique. Further, we also obtain the formula for the volume of hyperboloid of two sheets. The closed forms for the exact curved surface area and volume of the hyperboloid of two sheets are also verified numerically by using *Mathematica Program*.

Keywords: Meijer's G-function; Mellin-Barnes type contour integrals; Hyperboloid of two sheets; General triple hypergeometric function of Srivastava; Appell's function of second kind; Mathematica Program.
2020 MSC: 33C20, 33C70, 97G30, 97G40

1. Introduction and preliminaries

For the definition of Pochhammer symbols, power series form of generalized hypergeometric function ${}_pF_q(z)$ and several related results, we refer the beautiful monographs (see, e.g., [1, 5, 7, 15, 16, 19])

Analytic continuation formula [7, p.63, Eq.(19), [18], p.459, Eq.(81)]:

$$\frac{1}{(A+m)_\ell}\,{}_2F_1\left[\begin{array}{c} A, A+m; \\ A+m+\ell; \end{array} z\right] = (-1)^{m+\ell}(-z)^{-A-m}\sum_{n=\ell}^{\infty}\frac{(A)_{n+m}(n-\ell)!}{(n+m)!\; n!\; z^n}+$$

$$+\frac{(-z)^{-A-m}}{(\ell+m-1)!}\sum_{n=0}^{\ell-1}\frac{(A)_{n+m}(1-m-\ell)_{n+m}}{(n+m)!\; n!\; z^n}\times$$

$$\times\left\{\log(-z)+\psi(1+m+n)+\psi(1+n)-\psi(A+m+n)-\psi(\ell-n)\right\}+$$

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$$+(-z)^{-A}\sum_{n=0}^{m-1}\frac{(A)_n(m-n-1)!}{(m+\ell-n-1)!\ n!\ z^n}, \tag{1}$$

where $A\neq 0,-1,-2,...;\ |\arg(-z)|<\pi;\ |z|>1;\ \ell\in\mathbb{N}_0$ and $m\in\mathbb{N}_0$.
$\{\text{Gamma of non-positive integer}\}^{-1}=0$

Theorem 1.1. *When* $p=0,1,2,...,$ *then*

$$\frac{1}{\Gamma(-\mathbf{p})}\ {}_4F_3\left[\begin{array}{c}A,B,C,D;\\ \\ E,G,-\mathbf{p};\end{array}\ z\right]=\frac{(A)_{p+1}(B)_{p+1}(C)_{p+1}(D)_{p+1}z^{p+1}}{(E)_{p+1}(G)_{p+1}(p+1)!}\times$$

$$\times\ {}_4F_3\left[\begin{array}{c}A+p+1,B+p+1,C+p+1,D+p+1;\\ \\ E+p+1,\ G+p+1,\ 2+p;\end{array}\ z\right];\ |z|<1. \tag{2}$$

Proof. Consider the left hand side of the assertion (2)

$$\frac{1}{\Gamma(-p)}\ {}_4F_3\left[\begin{array}{c}A,B,C,D;\\ \\ E,G,-p;\end{array}\ z\right]=\sum_{r=0}^{\infty}\frac{(A)_r(B)_r(C)_r(D)_r z^r}{(E)_r(G)_r\ \Gamma(-p+r)\ r!}$$

$$=\sum_{r=p+1}^{\infty}\frac{(A)_r(B)_r(C)_r(D)_r z^r}{(E)_r(G)_r\ \Gamma(-p+r)\ r!}.$$

Replacing r by $r+p+1$, we get

$$\frac{1}{\Gamma(-p)}{}_4F_3\left[\begin{array}{c}A,B,C,D;\\ E,G,-p;\end{array}\ z\right]=\sum_{r=0}^{\infty}\frac{(A)_{r+p+1}(B)_{r+p+1}(C)_{r+p+1}(D)_{r+p+1}z^{r+p+1}}{(E)_{r+p+1}(G)_{r+p+1}\ (r+p+1)!\ r!}$$

$$=\frac{(A)_{p+1}(B)_{p+1}(C)_{p+1}(D)_{p+1}z^{p+1}}{(E)_{p+1}(G)_{p+1}(2)_p}\times$$

$$\times\sum_{r=0}^{\infty}\frac{(A+p+1)_r(B+p+1)_r(C+p+1)_r(D+p+1)_r z^r}{(E+p+1)_r(G+p+1)_r\ (2+p)_r\ r!}. \tag{3}$$

Expressing the result in the form of generalized hypergeometric function, we arrive at the result (2).

□

Appell's function of second kind [26, p.53, Eq.(5)] is defined as:

$$F_2\left[\ a;\ b,c;\ d,g;\ x,y\ \right]=F^{1:1;1}_{0:1;1}\left[\begin{array}{c}a:\ b;\ c;\\ \\ -:\ d;\ g;\end{array}\ x,y\right]$$

(4)

$$=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n\ x^m\ y^n}{(d)_m(g)_n\ m!\ n!}=\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\ y^n}{(g)_n\ n!}\ {}_2F_1\left[\begin{array}{c}a+n,\ b;\\ \\ d;\end{array}\ x\right].$$

Convergence conditions of Appell's double series F_2:

(i) Appell's function F_2 is convergent when $|x|+|y|<1$; $a,b,c,d,g\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

(ii) Appell's function F_2 is absolutely convergent when $|x|+|y|=1$; $x\neq 0,\ y\neq 0$; $a,b,c,d,g\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\Re(a+b+c-d-g)<0$.

(iii) When a is a negative integer, then Appell's series F_2 will be a polynomial, $b,c,d,g\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

(iv) When b,c are negative integers, then Appell's series F_2 will be a polynomial, $a,d,g\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

For absolutely and conditionally convergence of Appell's double series F_2, we refer a beautiful paper of Hài *et al.* [8].

Mellin-Barnes type contour integral representation of binomial function ${}_1F_0(z)$:
(5)
$$(1-z)^{-a}={}_1F_0\left[\begin{array}{cc} a; & \\ & z \\ -; & \end{array}\right]=\frac{1}{2\pi i\ \Gamma(a)}\int_{-i\infty}^{+i\infty}\Gamma(a+s)\Gamma(-s)(-z)^s\,ds:\ z\neq 0,$$
where $|\arg(-z)|<\pi, a\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $i=\sqrt{(-1)}$.

Mellin-Barnes type contour integral representation of ${}_pF_q(z)$ [26, p.43, Eq.(6)]:
$${}_pF_q\left[\begin{array}{cc} \alpha_1,\alpha_2,...,\alpha_p; & \\ & z \\ \beta_1,\beta_2,...,\beta_q; & \end{array}\right]$$
$$=\frac{\Gamma(\beta_1)\Gamma(\beta_2)...\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)...\Gamma(\alpha_p)}.\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\alpha_1+\xi)...\Gamma(\alpha_p+\xi)\Gamma(-\xi)(-z)^\xi}{\Gamma(\beta_1+\xi)...\Gamma(\beta_q+\xi)}\,d\xi, \tag{6}$$
where $z\neq 0$.

Convergence conditions:
If $p=q+1$, then $|\arg(-z)|<\pi$.
If $p=q$, then $|\arg(-z)|<\frac{\pi}{2}$,
and no α_i $(i=1,2,...,p)$ is zero or negative integer but some of β_j $(j=1,2,...,q)$ may be zero or negative integers.

Mellin-Barnes type contour integral representation of Meijer's G-function ([26, p.45, Eq.(1)], see also [7,16]):
When $p\leq q$ and $1\leq m\leq q,\ 0\leq n\leq p$, then
$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{array}{c}\alpha_1,\alpha_2,\alpha_3,...,\alpha_n;\alpha_{n+1},...,\alpha_p\\ \beta_1,\beta_2,\beta_3,...,\beta_m;\beta_{m+1},...,\beta_q\end{array}\right)=$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\prod_{j=1}^{m}\Gamma(\beta_j-s)\prod_{j=1}^{n}\Gamma(1-\alpha_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+s)\prod_{j=n+1}^{p}\Gamma(\alpha_j-s)}(z)^s\,ds$$

(7)

$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\beta_1 - s)...\Gamma(\beta_m - s)\Gamma(1-\alpha_1+s)...\Gamma(1-\alpha_n+s)}{\Gamma(1-\beta_{m+1}+s)...\Gamma(1-\beta_q+s)\Gamma(\alpha_{n+1}-s)...\Gamma(\alpha_p - s)}(z)^s\, ds,$$

where $z \neq 0, (\alpha_i - \beta_j) \neq$ positive integers, $i = 1,2,3,...,n;\ j = 1,2,3,...,m$. For details of contours, see [7, p.207, [16], p.144].

Convergence conditions of Meijer's G-function:
When $\Lambda = m + n - \left(\frac{p+q}{2}\right),\ \nu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j$, then

(i) The integral (7) is convergent when $|arg(z)| < \Lambda\pi$ and $\Lambda > 0$.

(ii) If $|arg(z)| = \Lambda\pi$ and $\Lambda \geq 0$, then the integral (7) is absolutely convergent when $p = q$ and $\Re(\nu) < -1$.

(iii) If $|arg(z)| = \Lambda\pi$ and $\Lambda \geq 0$, then the integral (7) is also absolutely convergent, when $p \neq q$, $(q-p)\sigma > \Re(\nu) + 1 - \left(\frac{q-p}{2}\right)$ and $s = \sigma + ik$, where σ and k are real. σ is chosen so that for $k \to \pm\infty$.

Relations between Meijer's G- function and ${}_2F_1(z)$ [17, p.61, [28], p.77, Eq.(1)]:

$$G^{2\,2}_{2\,2}\left(z\,\middle|\,\begin{matrix}1-a, 1-b; -\\ 0, c-a-b; -\end{matrix}\right) = \frac{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1\left[\begin{matrix}a,\ b;\\ & 1-z\\ c;\end{matrix}\right],$$

where $|1-z| < 1$ and $c-a, c-b \neq 0, -1, -2, ...$

$$G^{2\,2}_{2\,2}\left(z\,\middle|\,\begin{matrix}a_1, a_2; -\\ b_1, b_2; -\end{matrix}\right) = \frac{\Gamma(1-a_1+b_1)\Gamma(1-a_1+b_2)\Gamma(1-a_2+b_1)}{\Gamma(2-a_1-a_2+b_1+b_2)}\times$$

$$(8)\quad \times\Gamma(1-a_2+b_2)z^{b_1}\,{}_2F_1\left[\begin{matrix}1-a_1+b_1, 1-a_2+b_1;\\ & 1-z\\ 2-a_1-a_2+b_1+b_2;\end{matrix}\right];\ |1-z| < 1.$$

Just as the Gaussian ${}_2F_1$ function was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 [26, p.53, Eq.(4), Eq.(5), Eq.(6) and Eq.(7)] and their seven confluent forms $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ given by Humbert ([9,11], see also [10, pp.75-76]) were unified and generalized by Kampé de Fériet [13] who defined a general hypergeometric function of two variables.

The notation introduced by Kampé de Fériet for his double hypergeometric function [2, p.150, Eq.(26)] of superior order was subsequently abbreviated by Burchnall and Chaundy [3, p.112]. We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet)

in a slightly modified notation of Srivastava and Panda [27, p.423, Eq.(26)] :
(9)

$$F^{p:\ q;\ k}_{\ell:\ m;\ n}\left[\begin{matrix}(a_p):(b_q);(c_k);\\(\alpha_\ell):(\beta_m);(\gamma_n);\end{matrix}\ x,\ y\right]=\sum_{r,s=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_{r+s}\prod\limits_{j=1}^{q}(b_j)_r\prod\limits_{j=1}^{k}(c_j)_s}{\prod\limits_{j=1}^{\ell}(\alpha_j)_{r+s}\prod\limits_{j=1}^{m}(\beta_j)_r\prod\limits_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!},$$

where, for convergence [27, p.424, Eq.(27)]

$$(10)\qquad (i)\ p+q<\ell+m+1,\quad p+k<\ell+n+1,\quad |x|<\infty,\quad |y|<\infty,\ \text{or}$$

$$(ii)\ p+q=\ell+m+1,\quad p+k=\ell+n+1\ \text{and}$$

$$(11)\qquad |x|^{1/(p-\ell)}+|y|^{1/(p-\ell)}<1,\ \text{if}\ p>\ell,$$

$$(iii)\ p+q=\ell+m+1,\quad p+k=\ell+n+1\ \text{and}$$

$$(12)\qquad \max\{|x|,|y|\}<1,\text{if}\ p\le\ell.$$

For absolutely and conditionally convergence of double series (9), we refer to a research paper of Hài et al. [8, pp.106-107, Th.(1), Th.(2) and Th.(3)].

In the year 1967, a unification of Lauricella's fourteen triple hypergeometric functions $F_1, F_2, ..., F_{14}$ [14], including ten functions in Saran's notations F_E, F_F, F_G, F_K,
F_M, F_N, F_P, F_R, F_S and F_T [20, 21] and the three additional triple hypergeometric functions H_A, H_B, H_C [22, 23] was introduced by Srivastava [24] who defined a general triple hypergeometric series $F^{(3)}[x, y, z]$ ([24, p.428], see also [4, p.156, [6], p.40]):

$$F^{(3)}[x,y,z]=F^{(3)}\left[\begin{matrix}(a_A)::(b_B);(b'_{B'});\ (b''_{B''}):(c_C);(c'_{C'});(c''_{C''});\\(e_E)::(g_G);(g'_{G'});(g''_{G''}):(h_H);(h'_{H'});(h''_{H''});\end{matrix}\ x,\ y,\ z\right]$$

$$(13)\qquad =\sum_{m,n,p=0}^{\infty}\Lambda(m,n,p)\ \frac{x^m}{m!}\ \frac{y^n}{n!}\ \frac{z^p}{p!},$$

where, for convenience,

$$\Lambda(m,n,p)=\frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\ \prod_{j=1}^{B}(b_j)_{m+n}\ \prod_{j=1}^{B'}(b'_j)_{n+p}\ \prod_{j=1}^{B''}(b''_j)_{m+p}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\ \prod_{j=1}^{G}(g_j)_{m+n}\ \prod_{j=1}^{G'}(g'_j)_{n+p}\ \prod_{j=1}^{G''}(g''_j)_{m+p}}\times$$

$$(14)\qquad \times\frac{\prod_{j=1}^{C}(c_j)_m\ \prod_{j=1}^{C'}(c'_j)_n\ \prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\ \prod_{j=1}^{H'}(h'_j)_n\ \prod_{j=1}^{H''}(h''_j)_p}$$

where (a_A) abbreviates the array of A parameters $a_1, a_2, ..., a_A$, with similar interpretations for $(b_B), (b'_{B'}), (b''_{B''})$, *et cetera*. The triple hypergeometric series in (13) converges when

$$\begin{cases} 1+E+G+G''+H-A-B-B''-C\geq 0, \\ 1+E+G+G'+H'-A-B-B'-C'\geq 0, \\ 1+E+G'+G''+H''-A-B'-B''-C''\geq 0, \end{cases}$$

where the equalities hold true for suitably constrained values of $|x|, |y|$ and $|z|$.

The logarithmic derivative of the Gamma function also known as psi function or Digamma function [19, p.10, Eq.(1), [25], p.24, Eq.(2), [16], p.12, Eq.(1)], is defined as:

$$\psi(z) = \frac{d}{dz}\ell n\{\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)};\ z \neq 0, -1, -2, -3, ... \tag{15}$$

$$\psi(z) = -\gamma - \sum_{n=0}^{\infty}\left\{\frac{1}{(z+n)} - \frac{1}{(n+1)}\right\};\ z \neq 0, -1, -2, -3, ..., \tag{16}$$

where γ is Euler-Mascheroni constant and $\gamma \cong 0.577215664901532860606512....$

Suppose $\phi(x, y) = 0$ is the projection of the curved surface of three dimensional figure $z = f(x, y)$ over the x-y plane, then curved surface area is given by

$$\hat{S} = \underbrace{\int\int}_{\substack{\text{over the area}\\ \phi(x,y)=0}} \sqrt{\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}}\,dx\ dy. \tag{17}$$

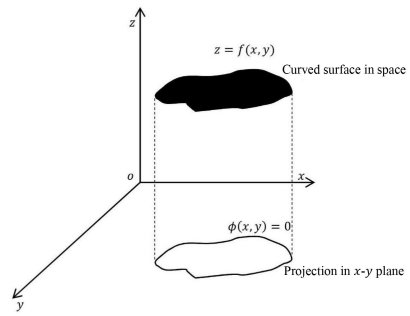

Figure 1. Projection of curved surface in x-y plane.

A definite integral:

$$\int_{\theta=0}^{\frac{\pi}{2}} \sin^{\alpha}\theta\cos^{\beta}\theta\ d\theta = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{2\Gamma\left(\frac{\alpha+\beta+2}{2}\right)}, \tag{18}$$

where $\Re(\alpha) > -1,\ \Re(\beta) > -1$.

Gauss-Legendre six points formula for numerical integration [12, p.361, Eq.(5.83) and p.364, Table(5.3)]

$$\int_{-1}^{+1} G(x)dx \approx 0.4679139346\, G(0.2386191861) + 0.4679139346\, G(-0.2386191861) +$$

$$+0.3607615730\, G(0.6612093865) + 0.3607615730\, G(-0.6612093865) +$$

$$+ 0.1713244924\, G(0.9324695142) + 0.1713244924\, G(-0.9324695142). \quad (19)$$

Note: $\int_{x=\alpha}^{\beta} F\{x\}\ dx = \left(\frac{\beta-\alpha}{2}\right) \int_{t=-1}^{+1} F\left\{\frac{(\beta-\alpha)t + (\beta+\alpha)}{2}\right\}\ dt.$

Special functions arise in the solution of various classical problems of physics, generally involving the flow of electromagnetic, acoustic or thermal energy. In the study of propagation of heat in a metallic bar, one could consider a bar with a rectangular cross-section, a round cross-section, an elliptic cross-section or even more complicated cross-sections. In these situations while dealing with this kind of problem, leads to somewhat different mathematical equations. These equations to be solved are ordinary differential equations. The solution of these ordinary differential equations form the majority of special functions of mathematical physics. In order to determine the allowable frequencies of oscillation of the waves on the surface of fluid put in a circular dish, hypergeometric series are required to terminate and determine the allowable frequencies. An important integral which is used in acoustical problems is Weber-Schaft Heitlin integral which is evaluated through the hypergeometric approach and this integral is also involved in problems on modulation products. The hypergeometric functions have been turned out to have a variety of applications in a wide range of research subjects. In this article an attempt is made to find the applications of the hypergeometric functions for determining the exact curved surface area of hyperboloid of two sheets because the problem of approximating the surface area of a triaxial hyperboloid does not appear to have been addressed i.e, no closed form expression exists for the surface area of a hyperboloid of two sheets. This situation stems from the fact that it is impossible to execute the integration in the expression for the surface area in closed form for the most general case of three unequal axes.

The article is organized as follows. In section 2, we evaluated some important integrals $\int_{\theta=-\pi}^{\pi} \left(\frac{\cos^2\theta}{\beta^2} + \frac{\sin^2\theta}{\lambda^2}\right)^s d\theta$ and $\int_{r=0}^{\delta} \frac{r^{2s+1}}{(1+r^2)^s} dr$ with suitable convergence conditions by using Mellin-Barnes type contour integral representations of binomial function ${}_1F_0(z)$, Meijer's G-function, series manipulation technique, an analytic continuation formula of Gauss function and Psi function $\psi(.)$. In section 3, we evaluated a useful Mellin-Barnes type contour integral involving Digamma function by using Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$; in terms of double hypergeometric function of Kampé de Fériet. All the above integrals are useful and help us in the derivation of closed form for the exact curved surface area of a hyperboloid of two sheets. In section 4, we derive the closed form for obtaining the exact curved surface area of an upper sheet of a hyperboloid of two sheets by using Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$ and series manipulation technique, in terms of Appell's double hypergeometric function of second kind and general triple hypergeometric series of Srivastava. In section 5, for justification and verification, we obtain the curved surface areas of some hyperboloids of two sheets for particular values of the parameters (a, b, c, δ and H) in the form of numerical examples. In section 6, we derive the formula for the volume of hyperboloid of two sheets.

2. Evaluation of some useful definite integrals

The following definite integrals hold true associated with suitable convergence conditions:

Theorem 2.1.

$$\int_{\theta=-\pi}^{\pi}\left(\frac{\cos^2\theta}{\beta^2}+\frac{\sin^2\theta}{\lambda^2}\right)^s d\theta=\frac{2\pi\lambda}{\beta^{1+2s}}\ {}_2F_1\left[\begin{array}{c}\frac{1}{2},1+s;\\ \\ 1;\end{array}\ 1-\frac{\lambda^2}{\beta^2}\right], \tag{20}$$

where $\beta\geq\lambda>0$ *and it is obvious that* $0\leq(1-\frac{\lambda^2}{\beta^2})<1$.

Theorem 2.2.

$$\int_{\theta=-\pi}^{\pi}\left(\frac{\cos^2\theta}{\beta^2}+\frac{\sin^2\theta}{\lambda^2}\right)^s d\theta=\frac{2\pi\beta}{\lambda^{1+2s}}\ {}_2F_1\left[\begin{array}{c}\frac{1}{2},1+s;\\ \\ 1;\end{array}\ 1-\frac{\beta^2}{\lambda^2}\right], \tag{21}$$

where $\lambda\geq\beta>0$ *and it is obvious that* $0\leq(1-\frac{\beta^2}{\lambda^2})<1$.

Proof. For the independent demonstration of the assertions (20) and (21)

$$\text{Suppose } I_1=\int_{\theta=-\pi}^{\pi}\left(\frac{\cos^2\theta}{\beta^2}+\frac{\sin^2\theta}{\lambda^2}\right)^s d\theta.$$

$$=\frac{4}{\lambda^{2s}}\int_{\theta=0}^{\frac{\pi}{2}}\left(\sin^2\theta\right)^s\left\{1+\frac{\lambda^2\cos^2\theta}{\beta^2\sin^2\theta}\right\}^s d\theta$$

$$=\frac{4}{\lambda^{2s}}\int_{\theta=0}^{\frac{\pi}{2}}\sin^{2s}\theta\ {}_1F_0\left[\begin{array}{c}-s;\\ \\ -\ ;\end{array}\ \frac{-\lambda^2\cos^2\theta}{\beta^2\sin^2\theta}\right]d\theta. \tag{22}$$

Employing the contour integral (5) of ${}_1F_0(.)$, we get

(23)

$$I_1 = \frac{2}{\pi i \Gamma(-s)\ \lambda^{2s}} \int_{\theta=0}^{\frac{\pi}{2}} \sin^{2s} \left\{ \int_{\zeta=-i\infty}^{+i\infty} \Gamma(-\zeta)\Gamma(-s+\zeta) \left(\frac{\lambda^2 \cos^2\theta}{\beta^2 \sin^2\theta} \right)^{\zeta} d\zeta \right\} d\theta.$$

Interchanging the order of integration in double integral of (23), we get
(24)

$$I_1 = \tfrac{2}{\pi i \Gamma(-s)\ \lambda^{2s}} \int_{\zeta=-i\infty}^{+i\infty} \Gamma(-\zeta)\Gamma(-s+\zeta) \left(\tfrac{\lambda^2}{\beta^2}\right)^{\zeta} \left\{ \int_{\theta=0}^{\frac{\pi}{2}} \sin^{2s-2\zeta}\theta\ \cos^{2\zeta}\theta\ d\theta \right\} d\zeta.$$

Using the integral formula (18), we get

$$I_1 = \frac{1}{\pi i \Gamma(-s)\Gamma(1+s)\ \lambda^{2s}} \times$$

$$\times \int_{\zeta=-i\infty}^{+i\infty} \Gamma(-\zeta)\Gamma(-s+\zeta)\Gamma\left(\frac{1}{2}+s-\zeta\right)\Gamma\left(\frac{1}{2}+\zeta\right)\left(\frac{\lambda^2}{\beta^2}\right)^{\zeta} d\zeta$$

$$= \frac{1}{\pi i \Gamma(-s)\Gamma(1+s)\ \lambda^{2s}} \times$$

(25)

$$\times \int_{\zeta=-i\infty}^{+i\infty} \Gamma(0-\zeta)\Gamma(1-(s+1)+\zeta)\Gamma\left(\frac{1}{2}+s-\zeta\right)\Gamma\left(1-\frac{1}{2}+\zeta\right)\left(\frac{\lambda^2}{\beta^2}\right)^{\zeta} d\zeta.$$

Applying the definition (7) of Meijer's G-function, we get

$$I_1 = \frac{2}{\Gamma(-s)\Gamma(1+s)\ \lambda^{2s}}\ G^{2\ 2}_{2\ 2}\left(\frac{\lambda^2}{\beta^2} \,\middle|\, \begin{matrix} s+1, \frac{1}{2}; - \\ 0, \frac{1}{2}+s; - \end{matrix} \right). \tag{26}$$

Employing the conversion formula (8) in equation (26), and after further simplification, we arrive at the result (20).

The proof of the result (21) follows the same steps as in the proof of (20). So we omit the details here. □

Theorem 2.3. *When $0 < \delta \leq 1$, then*

$$\int_{r=0}^{\delta} \frac{r^{2s+1}}{(1+r^2)^s}\ dr = \frac{\delta^{2+2s}}{2(1+s)}\ {}_2F_1\left[\begin{matrix} s, 1+s; \\ 2+s; \end{matrix}\ -\delta^2 \right], \tag{27}$$

and when $\delta = 1$, then $\Re(s) < 2$.

Theorem 2.4. *When $\delta \geq 1$ or $\frac{1}{\delta} \leq 1$, then*

$$\int_{r=0}^{\delta} \frac{r^{2s+1}}{(1+r^2)^s}\ dr = \frac{-s(1+s)}{4\delta^2}\ {}_3F_2\left[\begin{matrix} 1, 1, 2+s; \\ 2,\ 3; \end{matrix}\ -\tfrac{1}{\delta^2} \right] -$$

$$-\frac{s}{2}\left[2\log(\delta)+(1-\gamma)-\psi(1+s)\right]+\frac{\delta^2}{2}, \tag{28}$$

where γ is Euler-Mascheroni constant, $\gamma \cong 0.5772156649015328606065 12...$ and $\psi(.)$ is digamma function.

Proof. For the independent demonstration of the assertions (27) and (28)

$$\text{Suppose } I_2=\int_{r=0}^{\delta}\frac{r^{2s+1}}{(1+r^2)^s}\,dr=\int_0^{\delta} r^{2s+1}(1+r^2)^{-s}\,dr$$

$$=\int_0^{\delta} r^{2s+1}\,{}_1F_0\left[\begin{matrix} s\,; \\ \\ -; \end{matrix}\; -r^2\right]dr \tag{29}$$

Employing the contour integral (5) of ${}_1F_0(.)$, we get

$$I_2=\int_0^{\delta} r^{2s+1}\left\{\frac{1}{2\pi i\Gamma(s)}\int_{\zeta=-i\infty}^{+i\infty}\Gamma(s+\zeta)\Gamma(-\zeta)\, r^{2\zeta}\, d\zeta\right\}dr. \tag{30}$$

Interchanging the order of integration in double integral of (30), we get

$$I_2=\frac{1}{2\pi i\Gamma(s)}\int_{\zeta=-i\infty}^{+i\infty}\Gamma(s+\zeta)\Gamma(-\zeta)\left\{\int_0^{\delta} r^{2\zeta+2s+1}\,dr\right\}d\zeta. \tag{31}$$

$$=\frac{\delta^{2+2s}}{4\pi i\Gamma(s)}\int_{\zeta=-i\infty}^{+i\infty}\frac{\Gamma(s+\zeta)\Gamma(-\zeta)\,\gamma^{2\zeta}}{(1+s+\zeta)}\,d\zeta \tag{32}$$

$$=\frac{\gamma^{2+2s}}{4\pi i\Gamma(s)}\int_{\zeta=-i\infty}^{+i\infty}\frac{\Gamma(s+\zeta)\Gamma(1+s+\zeta)\Gamma(-\zeta)\,\gamma^{2\zeta}}{\Gamma(2+s+\zeta)}\,d\zeta. \tag{33}$$

Employing the Mellin-Barnes contour integral representation (6) of ${}_pF_q(.)$ and after further simplification, we get

$$I_2=\frac{\delta^{2+2s}}{2(1+s)}\,{}_2F_1\left[\begin{matrix} s, 1+s; \\ \\ 2+s; \end{matrix}\; -\delta^2\right], \tag{34}$$

and hence the result (27), where $\delta < 1$.

If $\delta > 1$, then employing the analytic continuation formula (1) in equation (34), we get

$$I_2=\frac{\delta^{2+2s}}{2(1+s)}\left\{\frac{(1+s)}{\delta^{2+2s}}\sum_{n=1}^{\infty}\frac{(s)_{1+n}(n-1)!}{(n+1)!\; n!\; (-\delta^2)^n}+\right.$$

$$\left.+\frac{-s(1+s)}{\delta^{2+2s}}\left[\log(\delta^2)+\psi(2)-\psi(1+s)\right]+\frac{(1+s)}{\delta^{2s}}\right\} \tag{35}$$

$$(36)\qquad = \frac{s}{2}\sum_{n=1}^{\infty}\frac{(1+s)_n}{n(n+1)!\,(-\delta^2)^n} - \frac{s}{2}\left[\log(\delta^2)+\psi(2)-\psi(1+s)\right] + \frac{\delta^2}{2}.$$

Replacing n by $n+1$ in equation (36), we get

(37)

$$I_2 = \frac{s}{2}\sum_{n=0}^{\infty}\frac{(1+s)_{n+1}}{(n+1)(n+2)!\,(-\delta^2)^{(n+1)}} - \frac{s}{2}\left[\log(\delta^2)+\psi(2)-\psi(1+s)\right] + \frac{\delta^2}{2}$$

$$(38)\qquad = \frac{-s(1+s)}{4\delta^2}\sum_{n=0}^{\infty}\frac{(2+s)_n(1)_n}{(2)_n(3)_n\,(-\delta^2)^n} - \frac{s}{2}\left[\log(2\delta)+\psi(2)-\psi(1+s)\right] + \frac{\delta^2}{2}.$$

Employing the definition of generalized hypergeometric function ${}_pF_q(.)$, we arrive at the result (28).

□

3. Evaluation of a useful Mellin-Barnes type contour integral involving Digamma function

Theorem 3.1. *The following Mellin Barnes type contour integral holds true, when* $|\frac{c^2}{a^2}| < 1$*:*

$$\int_{s=-i\infty}^{+i\infty}\frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+p+s)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(s)}\psi(1+s)\,ds =$$

$$= \frac{2\pi\sqrt{\pi}\;i\Gamma(2+p)c^2}{a^2}\left\{\gamma\;{}_2F_1\left[\begin{matrix}\frac{1}{2},2+p;\\ \\ 2;\end{matrix}\quad -\frac{c^2}{a^2}\right] -\right.$$

$$(39)\qquad \left. -\frac{1}{2}F^{1:2;2}_{1:1;1}\left[\begin{matrix}\mathbf{2:}\ \frac{1}{2},2+p;1,1\mathbf{;}\\ \\ \mathbf{3:}\ 1;\ 2\mathbf{;}\end{matrix}\quad -\frac{c^2}{a^2},1\right]\right\},$$

where γ *is Euler-Mascheroni constant.*

Proof. Consider the left hand side of (39) and using the equation (16), we get

$$\text{Let } I_3 = \int_{s=-i\infty}^{+i\infty}\frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+p+s)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(s)}\psi(1+s)\,ds =$$

$$= \int_{s=-i\infty}^{+i\infty}\frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+p+s)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(s)}\times$$

$$\times\left\{-\gamma-\sum_{n=0}^{\infty}\left(\frac{1}{(1+s+n)}-\frac{1}{(n+1)}\right)\right\}\,ds,$$

where γ is Euler-Mascheroni constant.

$$\text{Therefore } I_3 = -\gamma \int_{s=-i\infty}^{+i\infty} \frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+p+s)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(s)} ds +$$

$$+\sum_{n=0}^{\infty}\frac{(1)_n}{(2)_n}\int_{s=-i\infty}^{+i\infty}\frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+p+s)\Gamma(1+n+s)\Gamma(1+s)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(s)\Gamma(s)\Gamma(2+n+s)} ds$$

Employing the Mellin-Barnes contour integral representation (6) of ${}_pF_q(.)$, we get

$$I_3 = 4\pi\sqrt{\pi}\Gamma(1+p)\ i\frac{\gamma}{\Gamma(0)}\ {}_2F_1\left[\begin{array}{c} -\frac{1}{2}, 1+p; \\ \\ 0\ ; \end{array} \frac{-c^2}{a^2}\right] -$$

(40)

$$-\ 4\pi\sqrt{\pi}\Gamma(1+p)\ i\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(2)_n(2)_n\Gamma(0)\Gamma(0)}\ {}_4F_3\left[\begin{array}{c} -\frac{1}{2}, 1+p, 1+n, 1; \\ \\ 0, 0,\ 2+n; \end{array} \frac{-c^2}{a^2}\right],$$

where $|\frac{c^2}{a^2}| < 1$.
Employing the formula (2) in equation (40), we get

$$I_3 = \frac{2\pi\sqrt{\pi}\Gamma(2+p)c^2\gamma\ i}{a^2}\ {}_2F_1\left[\begin{array}{c} \frac{1}{2}, 2+p; \\ \\ 2\ ; \end{array} \frac{-c^2}{a^2}\right] -$$

(41)

$$-\ 4\pi\sqrt{\pi}\Gamma(1+p)\ i\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(2)_n(2)_n}\sum_{m=1}^{\infty}\frac{\left(-\frac{1}{2}\right)_m(1+p)_m(1)_m(1+n)_m\left(\frac{-c^2}{a^2}\right)^m}{\Gamma(m)\Gamma(m)(2+n)_m\ m!}.$$

Replacing m by $m+1$, we get

$$I_3 = \frac{2\pi\sqrt{\pi}\Gamma(2+p)c^2\ i}{a^2}\left\{\gamma\ {}_2F_1\left[\begin{array}{c} \frac{1}{2}, 2+p; \\ \\ 2\ ; \end{array} \frac{-c^2}{a^2}\right] -\right.$$

$$\left. -\ \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2)_{m+n}\left(\frac{1}{2}\right)_m(2+p)_m(1)_n\left(\frac{-c^2}{a^2}\right)^m}{(3)_{m+n}(1)_m(2)_n\ m!}\right\}. \tag{42}$$

Employing the definition of double hypergeometric function (9) of Kampé de Fériet, we arrive at the result (39). □

Throughout the paper, the semi-axes of the hyperboloid of two sheets $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ are assumed in the form of $a > b > 0$ and $c > 0$.

4. Closed forms for the curved surface area of any one surface of a hyperboloid of two sheets

Theorem 4.1. *The curved surface area of an upper sheet of a hyperboloid of two sheets (whose axis is positive direction of z-axis) i.e,* $z = c\left(1+\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{1}{2}}$; $a > b > 0$ *and* $c > 0$ *bounded by the plane* $z = H$ $(0 < c < H)$ *is given by*

Case I: *When* $0 < \left(\delta = \sqrt{\left(\frac{H^2}{c^2}-1\right)}\right) < 1$, *then curved surface area of an upper sheet of a hyperboloid of two sheets is*

$$\hat{S} = b^2\delta^2\pi\ F_2\left[1;\ \frac{1}{2},\frac{-1}{2}\ ;\ 1,2\ ;1-\frac{b^2}{a^2},\frac{-c^2\delta^2}{a^2}\right]-$$

$$-\frac{b^2c^2\delta^6\pi}{6a^2}\ F^{(3)}\left[\begin{matrix} -::-;2,3;\ 2:\frac{1}{2}\ ;\ 1\ ;\ \frac{1}{2}\ ; \\ -::-;\ 4\ ;-:1\ ;\ 2\ ;\ 2,2\ ; \end{matrix}\ 1-\frac{b^2}{a^2},-\delta^2,\frac{-c^2\delta^2}{a^2}\right], \tag{43}$$

where $|1-\frac{b^2}{a^2}| < 1$; $|\frac{-c^2\delta^2}{a^2}| < 1$; $|1-\frac{b^2}{a^2}|+|\frac{-c^2\delta^2}{a^2}| < 1; a > b > 0$ *and* $c > 0$.

Case II: *When* $\left(\delta = \sqrt{\left(\frac{H^2}{c^2}-1\right)}\right) > 1$, *then curved surface area of an upper sheet of a hyperboloid of two sheets is*

$$\hat{S} = b^2\delta^2\pi\ F_2\left[1;\ \frac{1}{2},\frac{-1}{2}\ ;\ 1,1\ ;1-\frac{b^2}{a^2},\frac{-c^2}{a^2}\right]-$$

$$-\frac{b^2c^2\pi}{2a^2}\left(1+2\log(\delta)\right)\ F_2\left[2;\ \frac{1}{2},\frac{1}{2}\ ;\ 1,2\ ;1-\frac{b^2}{a^2},\frac{-c^2}{a^2}\right]-$$

$$-\frac{b^2c^2\pi}{2a^2\delta^2}\ F^{(3)}\left[\begin{matrix} -::-;3\ ;\ 2:\frac{1}{2}\ ;\ 1,1\ ;\ \frac{1}{2}\ ; \\ -::-;\ -;-:1\ ;\ 2,3\ ;\ 2,2\ ; \end{matrix}\ 1-\frac{b^2}{a^2},-\frac{1}{\delta^2},\frac{-c^2}{a^2}\right]+$$

$$+\frac{b^2c^2\pi}{4a^2}\ F^{(3)}\left[\begin{matrix} -::2\ ;2\ ;-:\frac{1}{2}\ ;\ \frac{1}{2}\ ;1,1\ ; \\ -::-;\ 3;-:1\ ;\ 1\ ;\ 2\ ; \end{matrix}\ 1-\frac{b^2}{a^2},\frac{-c^2}{a^2},1\right] \tag{44}$$

where $|1-\frac{b^2}{a^2}| < 1; |\frac{-c^2}{a^2}| < 1; |1-\frac{b^2}{a^2}|+|\frac{-c^2}{a^2}| < 1; a > b > 0$ *and* $c > 0$.

Remark: The above formulas (43) and (44)are verified numerically through *Mathematica program.*

Proof. Equation of a hyperboloid of two sheets

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1,\ a > b > 0 \text{ and } c > 0. \tag{45}$$

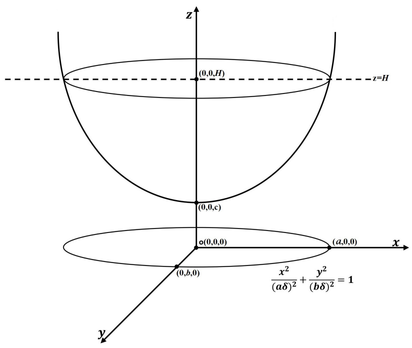

FIGURE 2. Upper sheet of hyperboloid of two sheets $z = c\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{1}{2}}$, when $z > c > 0$

Therefore,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1. \tag{46}$$

The intersection of the surface (46) with the plane $z = H$, where $0 < c < H$ (parallel to x-y plane, lying above x-y plane) will be an ellipse (in space) lying in the plane $z = H$ as well as lying on the surface of a hyperboloid of two sheets. The projection of that ellipse in x-y plane will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{H^2}{c^2} - 1 = \delta^2, \tag{47}$$

where $\delta = \sqrt{\left(\frac{H^2}{c^2} - 1\right)}$.

Again from equation (46) the equation of an upper sheet of a hyperboloid of two sheets will be

$$z = c\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{1}{2}}, \tag{48}$$

$$\frac{\partial z}{\partial x} = \frac{cx}{a^2\sqrt{\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}}, \tag{49}$$

$$\frac{\partial z}{\partial y} = \frac{cy}{b^2\sqrt{\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}}. \tag{50}$$

Substitute the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in equation (17). Therefore curved surface area of an upper sheet of a hyperboloid of two sheets bounded by the plane $z = H$ is

(51)

$$\hat{S} = \underbrace{\iint}_{\substack{\text{over the area of ellipse} \\ \frac{x^2}{(a\delta)^2}+\frac{y^2}{(b\delta)^2}=1}} \sqrt{\left\{1 + \frac{c^2x^2}{a^4\left(1+\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)} + \frac{c^2y^2}{b^4\left(1+\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}\right\}}\, dx\ dy.$$

Put $x = aX,\ y = bY$, then

$$\hat{S} = ab \underbrace{\iint}_{\substack{\text{over the area of circle} \\ X^2+Y^2=\delta^2}} \sqrt{\left\{1 + \frac{c^2}{(1+X^2+Y^2)}\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}\right)\right\}}\, dX\ dY. \quad (52)$$

Put $X = r\cos\theta,\ Y = r\sin\theta$, then $dX\ dY = r\ dr\ d\theta$.

(53)

$$\text{Therefore}\ \hat{S} = ab \int_{\theta=-\pi}^{\pi} \left(\int_{r=0}^{\delta} \left\{1 + \frac{c^2r^2}{(1+r^2)}\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\right\}^{\frac{1}{2}} rdr \right) d\theta.$$

Remark: Since we have no standard formula of definite/indefinite integrals in the literature of integral calculus for the integration with respect to "r" and "θ" in double integral (53). Therefore we can solve such integrals exactly through hypergeometric function approach.

(54)

$$\text{Therefore}\ \hat{S} = ab \int_{\theta=-\pi}^{\pi} \int_{r=0}^{\delta} {}_1F_0\left[\begin{array}{c} -\frac{1}{2}; \\ \ \\ -\ ; \end{array} \frac{-c^2r^2}{(1+r^2)}\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\right] rdr\ d\theta.$$

Since there is uncertainty about the argument of ${}_1F_0$ in equation (54), because the argument of ${}_1F_0$ in equation (54) may be greater than 1. Therefore applying contour integral (5) of ${}_1F_0(.)$ in equation (54), we get

$$\hat{S} = ab \int_{\theta=-\pi}^{\pi} \int_{r=0}^{\delta} \left[\frac{1}{2\pi i\Gamma\left(\frac{-1}{2}\right)} \int_{s=-i\infty}^{+\infty} \Gamma(-s)\Gamma(\frac{-1}{2}+s) \times \right.$$

$$\left. \times \left\{ \frac{c^2r^2}{(1+r^2)} \left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\right\}^s ds \right] rdrd\theta, \quad (55)$$

where $|\arg\left\{c^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\right\}| < \pi$ and $i = \sqrt{(-1)}$.

$$\hat{S} = \frac{-ab}{4\pi\sqrt{\pi}\ i} \int_{s=-i\infty}^{+\infty} \Gamma(-s)\Gamma(\frac{-1}{2}+s)\ c^{2s} \left\{ \int_{\theta=-\pi}^{\pi} \left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)^s d\theta \right\} \times$$

$$\times\left\{\int_{r=0}^{\delta}\frac{r^{2s+1}}{(1+r^2)^s}\,dr\right\}ds. \tag{56}$$

First special case of the triple integral (56) or derivation of the result (43)

When $a>b$ and $0<\left(\delta=\sqrt{(\frac{H^2}{c^2}-1)}\right)<1$, then employing the formulas (20) and (27) in (56), we get

$$\hat{S}=\frac{-b^2\delta^2}{4\sqrt{\pi}\;i}\int_{s=-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma(\frac{-1}{2}+s)}{(1+s)}\left(\frac{c\delta}{a}\right)^{2s}{}_2F_1\left[\begin{array}{c}\frac{1}{2},1+s;\\ \\ 1;\end{array}1-\frac{b^2}{a^2}\right]\times$$

$$\times\;{}_2F_1\left[\begin{array}{c}s,1+s;\\ \\ 2+s;\end{array}-\delta^2\right]ds, \tag{57}$$

where $|\delta^2|<1$ and $|1-\frac{b^2}{a^2}|<1$.

$$\text{Therefore }\hat{S}=\frac{-b^2\delta^2}{4\sqrt{\pi}\;i}\int_{s=-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma(\frac{-1}{2}+s)}{(1+s)}\left(\frac{c\delta}{a}\right)^{2s}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m(1+s)_m\left(1-\frac{b^2}{a^2}\right)^m}{(1)_m\;m!}\left(1+\sum_{n=1}^{\infty}\frac{(s)_n(1+s)_n\;(-\delta^2)^n}{(2+s)_n\;n!}\right)ds \tag{58}$$

$$=\frac{-b^2\delta^2}{4\sqrt{\pi}\;i}\int_{s=-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma(\frac{-1}{2}+s)}{(1+s)}\left(\frac{c\delta}{a}\right)^{2s}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m(1+s)_m\left(1-\frac{b^2}{a^2}\right)^m}{(1)_m\;m!}ds-$$

$$-\frac{b^2\delta^2}{4\sqrt{\pi}\;i}\int_{s=-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma(\frac{-1}{2}+s)}{(1+s)}\left(\frac{c\delta}{a}\right)^{2s}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m(1+s)_m\left(1-\frac{b^2}{a^2}\right)^m}{(1)_m\;m!}\times$$

$$\times\sum_{n=1}^{\infty}\frac{(s)_n(1+s)_n\;(-\delta^2)^n}{(2+s)_n\;n!}\;ds. \tag{59}$$

Interchanging the order of summation and integration, we get

$$\hat{S}=\frac{-b^2\delta^2}{4\sqrt{\pi}\;i}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m\left(1-\frac{b^2}{a^2}\right)^m}{(1)_m\;m!}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\frac{-1}{2}+s)\Gamma(1+m+s)\Gamma(-s)}{\Gamma(2+s)}\times$$

$$\left(\frac{c^2\delta^2}{a^2}\right)^s ds-\frac{b^2\delta^2}{4\sqrt{\pi}\;i}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_m\left(1-\frac{b^2}{a^2}\right)^m(-\delta^2)^n}{(1)_m\;m!\;n!}\times$$

$$(60)\quad \times \int_{-i\infty}^{+i\infty} \frac{\Gamma(\frac{-1}{2}+s)\Gamma(1+m+s)\Gamma(n+s)\Gamma(1+n+s)\Gamma(-s)}{\Gamma(0+s)\Gamma(1+s)\Gamma(2+n+s)} \left(\frac{c^2\delta^2}{a^2}\right)^s ds.$$

Replacing n by $n+1$ in equation (60), we get

$$\hat{S} = \frac{-b^2\delta^2}{4\sqrt{\pi}\ i} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m}{(1)_m\ m!} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\frac{-1}{2}+s)\Gamma(1+m+s)\Gamma(-s)}{\Gamma(2+s)}$$

$$\left(\frac{c^2\delta^2}{a^2}\right)^s ds - \frac{b^2\delta^2}{4\sqrt{\pi}\ i} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m (-\delta^2)^{n+1}}{(1)_m\ m!\ (n+1)!} \times$$

$$(61)\quad \times \int_{-i\infty}^{+i\infty} \frac{\Gamma(\frac{-1}{2}+s)\Gamma(1+m+s)\Gamma(1+n+s)\Gamma(2+n+s)\Gamma(-s)}{\Gamma(0+s)\Gamma(1+s)\Gamma(3+n+s)} \left(\frac{c^2\delta^2}{a^2}\right)^s ds.$$

Applying contour integral (6) of ${}_pF_q(z)$ in both contour integrals of right hand side of equation (61), we get

$$\hat{S} = b^2\delta^2\pi \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m}{(1)_m}\ {}_2F_1\left[\begin{array}{l} \frac{-1}{2}, 1+m; \\ \\ 2\ ; \end{array} \frac{-c^2\delta^2}{a^2}\right] -$$

$$(62)\quad - b^2\delta^4\pi \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m (-\delta^2)^n\ \Gamma(1+n)}{(1)_m\ \Gamma(3+n)} \times$$

$$(63)\quad \times \frac{1}{\Gamma(0)}\ {}_4F_3\left[\begin{array}{l} \frac{-1}{2}, 1+m, 1+n, 2+n; \\ \\ 0,\ 1,\ 3+n\ ; \end{array} \frac{-c^2\delta^2}{a^2}\right];\ \frac{c^2\delta^2}{a^2} < 1.$$

Now applying the result (2) (with $p=0$) in ${}_4F_3\left(\frac{-c^2\delta^2}{a^2}\right)$ of right hand side of equation (63), we get

$$\hat{S} = b^2\delta^2\pi \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m}{(1)_m} \sum_{n=0}^{\infty} \frac{\left(\frac{-1}{2}\right)_n (1+m)_n (\frac{-c^2\delta^2}{a^2})^n}{(2)_n\ n!} -$$

$$-\frac{b^2c^2\delta^6\pi}{2a^2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m (1+m)(1+n)(2+n)\ \Gamma(1+n) \left(1-\frac{b^2}{a^2}\right)^m (-\delta^2)^n}{(1)_m\ (3+n)\Gamma(3+n)} \times$$

$$(64)\quad \times\ {}_4F_3\left[\begin{array}{l} \frac{1}{2}, 2+m, 2+n, 3+n; \\ \\ 2,\ 2,\ 4+n\ ; \end{array} \frac{-c^2\delta^2}{a^2}\right]$$

$$= b^2\delta^2\pi \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1+m)_n \left(\frac{1}{2}\right)_m \left(\frac{-1}{2}\right)_n \left(1-\frac{b^2}{a^2}\right)^m (\frac{-c^2\delta^2}{a^2})^n}{(2)_n\ m!\ n!} - \frac{b^2c^2\delta^6\pi}{6a^2}\times$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m (2)_m (3)_n \left(1-\frac{b^2}{a^2}\right)^m (-\delta^2)^n}{(1)_m (4)_n\ m!}\times$$

$$\times \sum_{p=0}^{\infty} \frac{\left(\frac{1}{2}\right)_p (2+m)_p (2+n)_p\ (3+n)_p \left(\frac{-c^2\delta^2}{a^2}\right)^p}{(2)_p\ (2)_p\ (4+n)_p\ p!} \tag{65}$$

$$= b^2\delta^2\pi \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_{m+n} \left(\frac{1}{2}\right)_m (\frac{-1}{2})_n \left(1-\frac{b^2}{a^2}\right)^m (\frac{-c^2\delta^2}{a^2})^n}{(1)_m\ (2)_n\ m!\ n!} - \frac{b^2c^2\delta^6\pi}{6a^2}\times$$

(66)

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(2)_{n+p}(3)_{n+p}(2)_{p+m} \left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_p \left(1-\frac{b^2}{a^2}\right)^m\ (-\delta^2)^n \left(\frac{-c^2\delta^2}{a^2}\right)^p}{(4)_{n+p}\ (1)_m\ (2)_n\ (2)_p\ (2)_p\ m!\ p!}.$$

Employing the definition of Appell's double hypergeometric function of second kind (4) and general triple hypergeometric series (13), we arrive at the result (43).

Second special case of the triple integral (56) or derivation of the result (44)

When $a > b$ and $\left(\delta = \sqrt{(\frac{H^2}{c^2} - 1)}\right) > 1$, then employing the formulas (20) and (28) in (56), we get

$$\hat{S} = \frac{-b^2}{2\sqrt{\pi}\ i} \int_{s=-i\infty}^{+\infty} \Gamma(-s)\Gamma(\frac{-1}{2}+s) \left(\frac{c^2}{a^2}\right)^s {}_2F_1\left[\begin{array}{c} \frac{1}{2}, 1+s; \\ \\ 1; \end{array} 1-\frac{b^2}{a^2}\right]\times$$

$$\times \left\{ \frac{-s(1+s)}{4\delta^2}\ {}_3F_2\left[\begin{array}{c} 1,1,2+s; \\ \\ 2,3; \end{array} -\frac{1}{\delta^2}\right] - \right.$$

$$\left. -\frac{s}{2}\left[2\log(\delta) + \psi(2) - \psi(1+s)\right] + \frac{\delta^2}{2}\right\} ds;\ |1-\frac{b^2}{a^2}| < 1 \tag{67}$$

$$= \frac{-b^2}{8\delta^2\sqrt{\pi}\ i} \int_{-i\infty}^{+\infty} \Gamma(-s)\Gamma(\frac{-1}{2}+s) \left(\frac{c^2}{a^2}\right)^s {}_2F_1\left[\begin{array}{c} \frac{1}{2}, 1+s; \\ \\ 1; \end{array} 1-\frac{b^2}{a^2}\right]\times$$

$$\times\{-s(1+s)\}\ {}_3F_2\left[\begin{matrix}1,1,2+s; & \\ & -\frac{1}{\delta^2}\\ 2,3; & \end{matrix}\right] ds-$$

$$-\frac{b^2}{2\sqrt{\pi}\ i}\int_{-i\infty}^{+\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)\left(\frac{c^2}{a^2}\right)^s {}_2F_1\left[\begin{matrix}\frac{1}{2},1+s; & 1-\frac{b^2}{a^2}\\ 1; & \end{matrix}\right]\{-s\log(\delta)\}\,ds-$$

$$-\frac{b^2}{2\sqrt{\pi}\ i}\int_{-i\infty}^{+\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)\left(\frac{c^2}{a^2}\right)^s {}_2F_1\left[\begin{matrix}\frac{1}{2},1+s; & 1-\frac{b^2}{a^2}\\ 1; & \end{matrix}\right]\left\{\frac{-s\psi(2)}{2}\right\}ds-$$

$$-\frac{b^2}{2\sqrt{\pi}\ i}\int_{-i\infty}^{+\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)\left(\frac{c^2}{a^2}\right)^s {}_2F_1\left[\begin{matrix}\frac{1}{2},1+s; & 1-\frac{b^2}{a^2}\\ 1; & \end{matrix}\right]\left\{\frac{s\psi(1+s)}{2}\right\}ds-$$

(68)

$$-\frac{b^2}{2\sqrt{\pi}\ i}\int_{-i\infty}^{+\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)\left(\frac{c^2}{a^2}\right)^s {}_2F_1\left[\begin{matrix}\frac{1}{2},1+s; & \\ & 1-\frac{b^2}{a^2}\\ 1; & \end{matrix}\right]\left\{\frac{\delta^2}{2}\right\}ds.$$

$$=\frac{b^2}{8\delta^2\sqrt{\pi}\ i}\int_{-i\infty}^{+\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)s(1+s)\left(\frac{c^2}{a^2}\right)^s\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p(1+s)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\times$$

$$\times\sum_{m=0}^{\infty}\frac{(1)_m(2+s)_m\left(\frac{-1}{\delta^2}\right)^m}{(2)_m(3)_m}\ ds+$$

$$+\frac{b^2\log(\delta)}{2\sqrt{\pi}\ i}\int_{-i\infty}^{+i\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)s\left(\frac{c^2}{a^2}\right)^s\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p(1+s)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}ds+$$

$$+\frac{b^2\psi(2)}{4\sqrt{\pi}\ i}\int_{-i\infty}^{+i\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)s\left(\frac{c^2}{a^2}\right)^s\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p(1+s)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}ds-$$

$$-\frac{b^2}{4\sqrt{\pi}\ i}\int_{-i\infty}^{+i\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)s\left(\frac{c^2}{a^2}\right)^s\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p(1+s)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\psi(1+s)ds-$$

(69) $$-\frac{b^2\delta^2}{4\sqrt{\pi}\ i}\int_{-i\infty}^{+i\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)\left(\frac{c^2}{a^2}\right)^s\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p(1+s)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}ds.$$

Interchanging the order of summation and integration, we get

$$\hat{S} = \frac{b^2}{8\delta^2\sqrt{\pi}\ i}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p(1)_m\left(\frac{-1}{\delta^2}\right)^m}{(1)_p(2)_m(3)_m\ p!}\times$$

$$\times\int_{-i\infty}^{+i\infty}\frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+s+p)\Gamma(2+s+m)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(s)\Gamma(1+s)}ds+$$

$$+\frac{b^2\log(\delta)}{2\sqrt{\pi}\ i}\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\int_{-i\infty}^{+i\infty}\frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+s+p)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(s)}ds+$$

$$+\frac{b^2(1-\gamma)}{4\sqrt{\pi}\ i}\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\int_{-i\infty}^{+i\infty}\frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+s+p)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(s)}ds-$$

$$-\frac{b^2}{4\sqrt{\pi}\ i}\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\int_{-i\infty}^{+i\infty}\frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+s+p)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(s)}\psi(1+s)ds-$$

(70)

$$-\frac{b^2\delta^2}{4\sqrt{\pi}\ i}\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\int_{-i\infty}^{+i\infty}\frac{\Gamma\left(\frac{-1}{2}+s\right)\Gamma(1+s+p)\Gamma(-s)\left(\frac{c^2}{a^2}\right)^s}{\Gamma(1+s)}ds.$$

Employing the Mellin-Barnes contour integral representation (6) of ${}_pF_q(.)$ in I, II, III and V integrals and using equation (39) in the IVth Mellin-Barnes contour integral in the right hand side of equation (70), we get

$$\hat{S} = -\frac{b^2\pi}{2\delta^2}\sum_{p,m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p(1)_m\left(\frac{-1}{\delta^2}\right)^m}{(3)_m\ p!}\frac{1}{\Gamma(0)}{}_3F_2\left[\begin{matrix}\frac{-1}{2},1+p,2+m; \\ 0,\ 1;\end{matrix}\ \frac{-c^2}{a^2}\right]-$$

$$-2b^2\pi\log(\delta)\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p}{p!}\left(\frac{1}{\Gamma(0)}\ {}_2F_1\left[\begin{matrix}\frac{-1}{2},1+p; \\ 0;\end{matrix}\ \frac{-c^2}{a^2}\right]\right)-$$

$$-b^2(1-\gamma)\pi\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p}{p!}\left(\frac{1}{\Gamma(0)}\ {}_2F_1\left[\begin{matrix}\frac{-1}{2},1+p; \\ 0;\end{matrix}\ \frac{-c^2}{a^2}\right]\right)-$$

$$-\frac{b^2}{2}\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\left\{\frac{\pi(2)_pc^2}{a^2}\left(\gamma\ {}_2F_1\left[\begin{matrix}\frac{1}{2},2+p; \\ 2;\end{matrix}\ -\frac{c^2}{a^2}\right]-\right.\right.$$

$$-\frac{1}{2}F^{1:2;2}_{1:1;1}\left[\begin{array}{lr} 2: \frac{1}{2}, 2+p\ ; 1, 1\ ; & \\ & -\frac{c^2}{a^2}, 1 \\ 3: 1\ ;\ 2\ ; & \end{array}\right]\Bigg)\Bigg)\Bigg\}+$$

(71)
$$+b^2\delta^2\pi\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p}{p!}\ {}_2F_1\left[\begin{array}{lr} -\frac{1}{2}, 1+p; & \\ & \frac{-c^2}{a^2} \\ 1; & \end{array}\right],$$

where $\frac{c^2}{a^2}<1$.

Employing the formula (2) in equation (71), we get

$$\hat{S}=-\frac{b^2c^2\pi}{2a^2\delta^2}\sum_{p,m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p(2)_p(1)_m\left(1-\frac{b^2}{a^2}\right)^p\left(\frac{-1}{\delta^2}\right)^m}{(1)_p(2)_m\ p!}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n(2+p)_n(3+m)_n\left(\frac{-c^2}{a^2}\right)^n}{(2)_n(2)_n\ n!}-$$

$$-\frac{b^2c^2\pi\log(\delta)}{a^2}\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p(2)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m(2+p)_m\left(\frac{-c^2}{a^2}\right)^m}{(2)_m\ m!}-$$

$$-\frac{b^2c^2\pi}{2a^2}\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p(2)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m(2+p)_m\left(\frac{-c^2}{a^2}\right)^m}{(2)_m\ m!}+$$

$$+\frac{b^2c^2\pi}{4a^2}\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p(2)_p\left(1-\frac{b^2}{a^2}\right)^p}{(1)_p\ p!}\sum_{m,n=0}^{\infty}\frac{(2)_{m+n}\left(\frac{1}{2}\right)_m(2+p)_m(1)_n(1)_n\left(\frac{-c^2}{a^2}\right)^m}{(3)_{m+n}(1)_m(2)_n\ m!\ n!}+$$

$$+b^2\delta^2\pi\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}\right)_p\left(1-\frac{b^2}{a^2}\right)^p}{p!}\sum_{m=0}^{\infty}\frac{\left(\frac{-1}{2}\right)_m(1+p)_m\left(\frac{-c^2}{a^2}\right)^m}{(1)_m\ m!}$$

$$=-\frac{b^2c^2\pi}{2a^2\delta^2}\sum_{p,m,n=0}^{\infty}\frac{(3)_{m+n}(2)_{n+p}\left(\frac{1}{2}\right)_p(1)_m\left(\frac{1}{2}\right)_n\left(1-\frac{b^2}{a^2}\right)^p\left(\frac{-1}{\delta^2}\right)^m\left(\frac{-c^2}{a^2}\right)^n}{(1)_p(2)_m(3)_m(2)_n(2)_n\ p!\ n!}-$$

$$-\frac{b^2c^2\pi\log(\delta)}{a^2}\sum_{p,m=0}^{\infty}\frac{(2)_{p+m}\left(\frac{1}{2}\right)_p\left(\frac{1}{2}\right)_m\left(1-\frac{b^2}{a^2}\right)^p\left(\frac{-c^2}{a^2}\right)^m}{(1)_p(2)_m\ p!\ m!}-$$

$$-\frac{b^2c^2\pi}{2a^2}\sum_{p,m=0}^{\infty}\frac{(2)_{p+m}\left(\frac{1}{2}\right)_p\left(\frac{1}{2}\right)_m\left(1-\frac{b^2}{a^2}\right)^p\left(\frac{-c^2}{a^2}\right)^m}{(1)_p(2)_m\ p!\ m!}+$$

$$+\frac{b^2c^2\pi}{4a^2}\sum_{p,m,n=0}^{\infty}\frac{(2)_{p+m}(2)_{m+n}\left(\frac{1}{2}\right)_p\left(\frac{1}{2}\right)_m(1)_n(1)_n\left(1-\frac{b^2}{a^2}\right)^p\left(\frac{-c^2}{a^2}\right)^m}{(3)_{m+n}(1)_p(1)_m(2)_n\ p!\ m!n!}+$$

$$+ b^2\delta^2\pi \sum_{p,m=0}^{\infty} \frac{(1)_{p+m}\left(\frac{1}{2}\right)_p\left(\frac{-1}{2}\right)_m\left(1-\frac{b^2}{a^2}\right)^p\left(\frac{-c^2}{a^2}\right)^m}{(1)_p(1)_m\ p!\ m!}. \tag{72}$$

Employing the definition of general triple hypergeometric series (13) and Appell's double hypergeometric function of second kind (4), we arrive at the result (44).

□

5. Verification and applications of the closed forms (43), (44) and the double integral (53); A comparative study

To understand the subject matter, we shall discuss some numerical problems:

Use of formula (43) for exact curved surface area of a hyperboloid of two sheets.

Problem 1. Find the exact curved surface area of an upper sheet of a hyperboloid of two sheets $-\frac{x^2}{25}-\frac{y^2}{9}+\frac{z^2}{2}=1$ bounded by the plane $z=\sqrt{3}$.

Solution: We have $a=5,\ b=3,\ c=\sqrt{2}=1.414213562...,\ H=\sqrt{3}=1.732050808...$ then $\delta=\frac{1}{\sqrt{2}}=0.7071067812...<1$, therefore $a>b>0; c>0$ and $|1-\frac{b^2}{a^2}|+|\frac{-c^2\delta^2}{a^2}|<1$.

Substituting the above values of a,b,c,H and δ in the closed form (43), we have

$$\hat{S}=(4.5\pi)\ F_2\left[\ 1;\ \tfrac{1}{2},\ \tfrac{-1}{2};\ 1,\ 2;\ 0.64,\ -0.04\ \right]-$$

$$-\,(0.015\pi)\ F^{(3)}\left[\begin{matrix} -::-;2,3;\ 2:\frac{1}{2}\ ;\ 1\ ;\ \frac{1}{2}\ ; \\ \\ -::-;\ 4\ ;-:1\ ;\ 2\ ;\ 2,2\ ;\end{matrix}\ 0.64,-0.50,-0.04\right], \tag{73}$$

Now using the Mathematica program to calculate the sum of the double and triple infinite series in equation (73), we get the **exact** curved surface area of an upper sheet of a hyperboloid of two sheets $-\frac{x^2}{25}-\frac{y^2}{9}+\frac{z^2}{2}=1$ bounded by the plane $z=\sqrt{3}$ is $\hat{S}=23.89526308...$ square units.

Since we have no standard formula of definite/indefinite integrals in the literature of integral calculus for the integration with respect to "r" and "θ" in double integral (53). Therefore we can solve such integrals **approximately** with the help of Gauss-Legendre six points formula (19).

Now substituting the above values of a,b,c,H and δ in the double integral (53), we get

$$\hat{S}\approx 15\int_{\theta=-\pi}^{\pi}\left(\int_{r=0}^{\frac{1}{\sqrt{2}}}\left\{r^2+\frac{c^2r^4}{(1+r^2)}\left(\frac{\cos^2\theta}{25}+\frac{\sin^2\theta}{9}\right)\right\}^{\frac{1}{2}}dr\right)d\theta. \tag{74}$$

Put $r = \frac{(t+1)}{2\sqrt{2}}$ in equation (74), when $r = 0$, then $t = -1$ and when $r = \frac{1}{\sqrt{2}}$, then $t = +1$, therefore

(75)

$$\hat{S} \approx \frac{15}{2\sqrt{2}} \int_{\theta=-\pi}^{\pi} \left(\int_{t=-1}^{+1} \left\{ \frac{(t+1)^2}{8} + \frac{(t+1)^4}{4\{8+(t+1)^2\}} \left(\frac{\cos^2\theta}{25} + \frac{\sin^2\theta}{9} \right) \right\}^{\frac{1}{2}} dt \right) d\theta.$$

Now put $\theta = \pi u$ in equation (75),, when $\theta = -\pi$, then $u = -1$ and when $\theta = \pi$, then $u = +1$, therefore we get

(76)

$$\hat{S} \approx \frac{15\pi}{2\sqrt{2}} \int_{u=-1}^{+1} \left(\int_{t=-1}^{+1} \left\{ \frac{(t+1)^2}{8} + \frac{(t+1)^4}{4\{8+(t+1)^2\}} \left(\frac{\cos^2(\pi u)}{25} + \frac{\sin^2(\pi u)}{9} \right) \right\}^{\frac{1}{2}} dt \right) du.$$

Now using Gauss-Legendre six points formula (19) for inner integral with respect to "t" and outer integral with respect to "u", we arrive at the **approximate** surface area $\hat{S} \approx 23.8956...$ square units.

Use of formula (44) for exact curved surface area of a hyperboloid of two sheets.

Problem 2. Find the exact curved surface area of an upper sheet of a hyperboloid of two sheets $-\frac{x^2}{25} - \frac{y^2}{9} + \frac{z^2}{5} = 1$ bounded by the plane $z = 5$.

Solution: We have $a = 5,\ b = 3,\ c = \sqrt{5} = 2.236067977...,\ H = 5$ then $\delta = 2 > 1$, therefore $a > b > 0; c > 0$ and $|1 - \frac{b^2}{a^2}| + |\frac{-c^2}{a^2}| < 1$.
Substituting the above values of a, b, c, H and δ in the closed form (44), we have

$$\hat{S} = (36\pi)\ F_2 \left[\ 1;\ \tfrac{1}{2},\ \tfrac{-1}{2};\ 1,\ 1;\ 0.64,\ -0.20\ \right] -$$

$$-(2.147664925\pi)\ F_2 \left[\ 2;\ \tfrac{1}{2},\ \tfrac{1}{2};\ 1,\ 2;\ 0.64,\ -0.20\ \right] -$$

$$-(0.225\pi)\ F^{(3)} \left[\begin{array}{c} -::-;\ 3;\ 2 : \tfrac{1}{2}\ ;\ 1,1\ ;\ \tfrac{1}{2}\ ; \\ -::-;\ -\ ;-: 1\ ;\ 2,3\ ;\ 2,2\ ; \end{array} 0.64, -0.25, -0.20 \right] +$$

(77)
$$+ (0.45\pi)\ F^{(3)} \left[\begin{array}{c} -:: 2\ ;\ 2\ ;\ - : \tfrac{1}{2}\ ;\ \tfrac{1}{2}\ ;\ 1,1\ ; \\ -::-;\ 3\ ;\ - : 1\ ;\ 1\ ;\ 2\ ; \end{array} 0.64, -0.20, 1 \right].$$

Now using the Mathematica program to calculate the sum of the double and triple infinite series in equation (77), we get the **exact** curved surface area of an upper sheet of a hyperboloid of two sheets $-\frac{x^2}{25} - \frac{y^2}{9} + \frac{z^2}{5} = 1$ bounded by the plane $z = 5$ is $\hat{S} = 208.4897195...$ square units.

Also substituting the above values of a, b, c, H and δ in the double integral (53), we get

$$\hat{S} \approx 15 \int_{\theta=-\pi}^{\pi} \left(\int_{r=0}^{2} \left\{ r^2 + \frac{c^2 r^4}{(1+r^2)} \left(\frac{\cos^2\theta}{25} + \frac{\sin^2\theta}{9} \right) \right\}^{\frac{1}{2}} dr \right) d\theta. \tag{78}$$

Put $r = (t+1)$ in equation (78), when $r = 0$, then $t = -1$ and when $r = 2$, then $t = +1$, therefore

(79)

$$\hat{S} \approx 15 \int_{\theta=-\pi}^{\pi} \left(\int_{t=-1}^{+1} \left\{ (t+1)^2 + \frac{5(t+1)^4}{\{1+(t+1)^2\}} \left(\frac{\cos^2\theta}{25} + \frac{\sin^2\theta}{9} \right) \right\}^{\frac{1}{2}} dt \right) d\theta.$$

Now put $\theta = \pi u$ in equation (79),, when $\theta = -\pi$, then $u = -1$ and when $\theta = \pi$, then $u = +1$, therefore we get

(80)

$$\hat{S} \approx (15\pi) \int_{u=-1}^{+1} \left(\int_{t=-1}^{+1} \left\{ (t+1)^2 + \frac{5(t+1)^4}{\{1+(t+1)^2\}} \left(\frac{\cos^2(\pi u)}{25} + \frac{\sin^2(\pi u)}{9} \right) \right\}^{\frac{1}{2}} dt \right) du.$$

Now using Gauss-Legendre six points formula (19) for inner integral with respect to "t" and outer integral with respect to "u", we arrive at the **approximate** surface area $\hat{S} \approx 208.555...$ square units.

6. Volume of an upper sheet of a hyperboloid of two sheets

Theorem 6.1. *The volume of an upper sheet of a hyperboloid of two sheets* $z = c\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{1}{2}}$; $a > b > 0$ *and* $c > 0$ *bounded by the plane* $z = H$ *is*

$$V = 2\pi ab \left[\frac{H}{2}(\delta^2) - \frac{c}{3} \left\{ (1+\delta^2)^{\frac{3}{2}} - 1 \right\} \right];\ 0 < c < H, \tag{81}$$

where $\delta \left(= \sqrt{\frac{H^2}{c^2} - 1} \right) > 0$

Proof. Equation of an upper sheet of a hyperboloid of two sheets

$$z = c\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{1}{2}};\ a > b > 0;\ c > 0 \tag{82}$$

Now volume of an upper sheet of a hyperboloid of two sheets bounded by the plane $z = H$ $(H > c > 0)$ is

$$V = \underbrace{\int\int}_{\text{projection}} \int_{z=\alpha}^{z=H} dz\, dy\, dx;\ \alpha = c\sqrt{\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}, \tag{83}$$

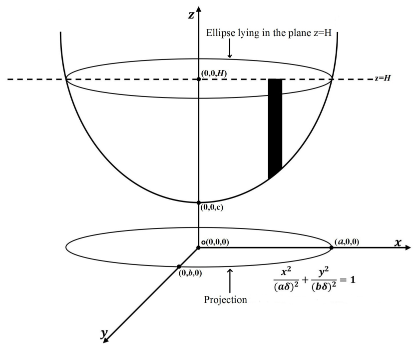

FIGURE 3. Upper sheet of hyperboloid of two sheets $z = c\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{1}{2}}$, when $z > c > 0$

where projection of an upper sheet of hyperboloid of two sheets bounded by the plane $z = H$ is taken on the x-y plane.

$$\text{Therefore } V = \underbrace{\iint}_{\substack{\text{over the area of ellipse} \\ \frac{x^2}{(a\delta)^2}+\frac{y^2}{(b\delta)^2}=1}} \left\{H - c\sqrt{\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}\right\} dy\ dx, \tag{84}$$

where $\delta = \sqrt{\frac{H^2}{c^2} - 1}$.

Put $x = aX,\ y = bY$, then

$$V = ab \underbrace{\iint}_{\substack{\text{over the area of circle} \\ X^2+Y^2=\delta^2}} \left\{H - c\sqrt{(1 + X^2 + Y^2)}\right\} dY\ dX. \tag{85}$$

Take $X = r\cos\theta,\ Y = r\sin\theta$, then

$$\begin{aligned} V &= ab\left(\int_{\theta=-\pi}^{\pi} d\theta\right)\left(\int_{r=0}^{\delta}\left\{H - c\sqrt{(1+r^2)}\right\} rdr\right) \\ &= 2\pi ab \int_{r=0}^{\delta}\left(Hr - cr\sqrt{(1+r^2)}\right) dr \\ &= 2\pi ab\left[\frac{Hr^2}{2} - \frac{c}{3}\left(1+r^2\right)^{\frac{3}{2}}\right]_0^{\delta} \\ &= 2\pi ab\left[\frac{H\delta^2}{2} - \frac{c}{3}\left\{(1+\delta^2)^{\frac{3}{2}} - 1\right\}\right]. \end{aligned} \tag{86}$$

On further simplifying, we arrive at the result (81). □

Problem 3. Find the volume of an upper sheet of a hyperboloid of two sheets $-\frac{x^2}{36} - \frac{y^2}{16} + \frac{z^2}{25} = 1$ bounded by the plane $z = 8$.
Solution: We have $a = 6,\ b = 4,\ c = 5,\ H = 8$ then $\delta = 1.248999600$. Substituting the above values of a, b, c, H and δ in the closed form (81) and after simplification, we obtain the volume of an upper sheet of a hyperboloid of two sheets $-\frac{x^2}{36} - \frac{y^2}{16} + \frac{z^2}{25} = 1$, bounded by the plane $z = 8$ is $V = 162.86$...cubic units.

7. Conclusion

In this paper, we obtained the closed form for the exact curved surface area of any one sheet of a hyperboloid of two sheets $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ intercepted by the plane $z = H$, through hypergeometric function approach i.e, by using series rearrangement technique, Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$ and Meijer's G-function; in terms of Appell's double hypergeometric function of second kind and triple hypergeometric function of Srivastava. These formulas are neither available in the literature of mathematics nor found in any mathematical tables. Moreover, we also derived the formula for the volume of hyperboloid of two sheets. We conclude that many formulas for curved surface areas of other three dimensional figures can be derived in an analogous manner, using Mellin-Barnes contour integration. Moreover, the results deduced above (presumably new), have potential applications in the fields of applied mathematics, statistics and engineering sciences.

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