Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505
JMMR 2012

INFINITE MINIMAL HALF SYNCHRONIZING

M. Shahamat

Article type: Research Article
(Received: 16 May 2022, Received in revised form: 10 August 2022)
(Accepted: 27 August 2022, Published Online: 28 August 2022)

Abstract. Synchronized systems, have attracted much attention [2] and extension of them has been of interest since that notion was introduced [3]. One was via *half synchronized systems*, that is, systems having *half synchronizing* blocks. In fact, if for a left transitive ray such as $\cdots x_{-1}x_0m$ and mv any block in X one has again $\cdots x_{-1}x_0mv$ a left ray in X, then m is called half synchronizing. A block m is *minimal (half-)synchronizing*, whenever $w \subsetneq m$, w is not (half-)synchronizing. Examples with ℓ minimal (half-)synchronizing blocks have been given for $0 \leq \ell \leq \infty$. To do this we consider a β-shift and will replace 1 with some blocks u_i to have countable many new systems. Then, we will merge them.

Keywords: Minimal half synchronizing, Synchronizing, Entropy.
2020 MSC: Primary 37B10, 54H20, 37B40.

1. Introduction

One of the most studied dynamical systems is a subshift of finite type (SFT). An SFT is a system whose set of forbidden blocks is finite [5]. Equivalently, an SFT X is a subshift whose any block of length greater than a certain number M is synchronizing, that is, if m is *any* block with $|m| \geq M$ and if v_1m and mv_2 are both blocks of X, then v_1mv_2 is a block of X. If an irreducible system has at least one synchronizing block, then it is called a *synchronized system* and examples are *sofics*: factors of SFT's. Synchronized systems, has attracted much attention [2] and extension of them has been of interest since that notion was introduced [3]. One was via *half synchronized systems*, that is, systems having *half synchronizing* blocks. In fact, if for a left transitive ray such as $\cdots x_{-1}x_0m$ and mv any block in X one has again $\cdots x_{-1}x_0mv$ a left ray in X, then m is called half synchronizing [3]. Clearly any synchronized system is half synchronized. Dyke subshifts and certain β-shifts are non-synchronized but half synchronized systems.

Thomsen in [8] considers a synchronized component of a general subshift and investigates the approximation of entropy from inside of this synchronized component by some certain SFT's. In [6], we gave a new proof for that result, and in particular, our approach was a constructive approach. Our main tool was that we consider the minimal synchronizing blocks and minimal half

m.shahamat@iaud.ac.ir, ORCID: 0000-0003-3646-0458
DOI: 10.22103/jmmr.2022.19511.1266

Publisher: Shahid Bahonar University of Kerman
How to cite: M. Shahamat, *Infinite Minimal Half Synchronizing*, J. Mahani Math. Res. 2023; 12(2): 105-113.

synchronizing blocks. Any half synchronized system is coded system. A subshift which has a cover (i.e., it admits some irreducible countable generating graph) is called a coded system. Examples of coded systems with infinitely many essentially different synchronizing blocks are introduced.

In this note, we give an example of a coded system with many essentially different half synchronizing blocks. We do this by some alterations on a no-synchronized beta shift.

2. **Background and definitions**

This section is devoted to some basic definitions in symbolic dynamics. The notations have been taken from [3] and [5] for the relevant concepts.

First, we present some elementary concept from [5]. In this trend we take attention of the readers to look at the realated papers [1], [7] and [8]. Let $\mathcal{A}$ be an alphabet, that is, a non-empty finite set. The full shift $\mathcal{A}$-shift, denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. Equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology. A *block* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the *empty block* which is denoted by ε. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a block of length $j-i+1$ by $x_{[i,\,j]} = x_i x_{i+1} \ldots x_j$. If $n \geq 1$, then u^n denotes the concatenation of n copies of u, and put $u^0 = \varepsilon$. The *shift map* σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate is $y_i = x_{i+1}$. By our topology, σ is a homeomorphism. Let $\mathcal{F}$ be the collection of all forbidden blocks over $\mathcal{A}$. For a full shift $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any block from $\mathcal{F}$. A *shift space* or a *subshift* is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks.

Let $W_n(X)$ denote the set of all admissible n-blocks. The *language* of X is the collection $W(X) = \cup_n W_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks u, $v \in W(X)$ there is a block $w \in W(X)$ so that $uwv \in W(X)$. A shift space X is called a *shift of finite type* (SFT) if there is a finite set $\mathcal{F}$ of forbidden blocks such that $X = X_{\mathcal{F}}$. A shift of *sofic* is the image of an SFT by a factor code (an onto sliding block code). Every SFT is sofic [5, Theorem 3.1.5], but the converse is not true [5, Page 67].

Let G be a graph with edge set $\mathcal{E} = \mathcal{E}(G)$ and the set of vertices $\mathcal{V} = \mathcal{V}(G)$. The *edge shift* X_G is the shift space over the alphabet $\mathcal{A} = \mathcal{E}$ defined by

$$X_G = \left\{ \xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1}) \right\}.$$

Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A labeled graph is a pair $\mathcal{G} = (G, \mathcal{L})$, where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E}(G) \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$ from the finite alphabet $\mathcal{A}$. For a path $\pi = \pi_0 \ldots \pi_k$, $\mathcal{L}(\pi) = \mathcal{L}(\pi_0) \ldots \mathcal{L}(\pi_k)$ is the label of π. π_u means a path labeled u.

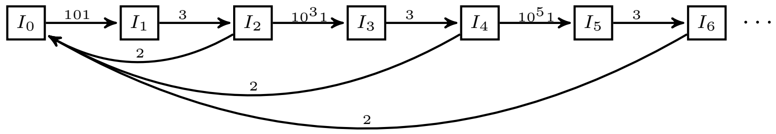

FIGURE 1. The cover G for a synchronized system with infinitely many essentially different synchronizing blocks, say $10^{2i-1}1, i \in \mathbb{N}$.

Let $\mathcal{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path ξ in G and set

$$X_\mathcal{G} := \{\mathcal{L}_\infty(\xi) :\ \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* or a *cover* of $X = \overline{X_\mathcal{G}}$. In particular, X is sofic if and only if $X = X_\mathcal{G}$ for a finite graph G [5, Proposition 3.2.10].

In this part we collect some information from [3]. Let X be a subshift and $x \in X$. Then, $x_+ = (x_i)_{i \in Z^+}$ (resp., $x_- = (x_i)_{i \leq 0}$) is called right (resp., left) infinite X-ray. Let $X^+ := \{x_+ : x \in X\}$ and $X^- := \{x_- : x \in X\}$. For a left infinite X-ray, say x_-, its follower set is $w_+(x_-) = \{x_+ \in X^+ : x_- x_+ \in X\}$ and for $m \in W(X)$ its follower set is $w_+(m) = \{x_+ \in X^+ : m x_+ \in X^+\}$. Analogously, we define predecessor sets $w_-(x_+) := \{x_- \in X^- : x_- x_+ \in X\}$ and $w_-(m) := \{x_- \in X^- : x_- m \in X^-\}$. A block $m \in W(X)$ is *synchronizing* if whenever um and mv are in $W(X)$, we have $umv \in W(X)$. An irreducible shift space X is *synchronized system* if it has a synchronizing block. A block $m \in W(X)$ is *half synchronizing* if there is a left transitive point $x \in X$ such that $x_{[-|m|+1,\,0]} = m$ and $w_+(x_{(-\infty,\,0]}) = w_+(m)$. A shift space that is the closure of the set of sequences obtained by freely concatenating the blocks in a list of countable blocks, called the set of generators, is a *coded system* [5].

3. Infinite minimal half synchronized systems

Figure 1 gives a cover of a coded system with infinitely many essentially different synchronizing blocks. We devote this section to bring such an example for a half synchronized system. We do this by some alterations on a no-synchronized beta shift and hence we start with recalling some basic facts about beta shifts.

Let β be a real number greater than 1. Set

$$1_\beta := a_1 a_2 a_3 \cdots \in \{0, 1, 2, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}},$$

where $a_1 = \lfloor \beta \rfloor$ and

$$a_i = \lfloor \beta^i (1 - a_1 \beta^{-1} - a_2 \beta^{-2} - \cdots - a_{i-1} \beta^{-i+1}) \rfloor$$

and $i \geq 2$. The sequence 1_β is the expansion of 1 in the base β, that is, $1 = \sum_{i=1}^{\infty} a_i \beta^{-i}$. Let $\leq$ be the lexiographic ordering of $\{\mathbb{N} \cup \{0\}\}^{\mathbb{N}}$. The sequence 1_β has the property that

$$\sigma^k 1_\beta \leq 1_\beta, k \in \mathbb{N}. \tag{1}$$

where σ denotes the shift map on $\{\mathbb{N} \cup \{0\}\}^{\mathbb{N}}$, which is the β-expansion of 1 for some $\beta \geq 1$. Furthermore, it follows from (1) that

$$X_\beta = \{x \in \{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{Z}} : x_{[i,\infty)} < 1_\beta, i \in \mathbb{Z}\}$$

is a shift space of $\{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{Z}}$, called the β-shift.

We construct an infinite labeled graph $\mathcal{G}_\beta$ as follows. Take a countable infinite set of vertices $I_0, I_1, \ldots$. Let I_0 be the base point. Edges an defined as follows. First, for all $i \geq 0$ there is an edge labeled a_i with initial vertex I_i and terminal vertex I_{i+1}. Also, for every i, for every $0 \leq c < a_i$ there is an edge labeled c with initial vertex I_i and terminal vertex I_0.

These shifts are symbolic spaces with rich structures. For a more detailed treatment, see [4].

Theorem 3.1. *[4]*

(1) If there exists $v \in W(X)$ such that $v \nsubseteq 1_\beta$, then X_β is a synchronized system.

(2) $x_+ \in (X_\beta)^+$ if and only if the graph contains an infinite path with labeled x_+.

Definition 3.2. Let m be an arbitrary half synchronizing block for X. A block m is *minimal half synchronizing block*, whenever $w \subsetneq m$, then w is not half synchronizing. If a shift space X has finitely many minimal half synchronizing blocks, then we say that X is a *FmHalfSyn* system; otherwise, it is called an *ImHalfSyn* system.

Example 3.3. *(1) The block 1 is minimal half synchronizing for any S-gap shift $X(S)$ and no other minimal half synchronizing block exist which means that $X(S)$ is a FmHalfSyn.*

(2) The Dyke system and beta shifts are FmHalfSyn [3, Example 0.10]. For these systems, any block is half synchronizing and so any character is a minimal half synchronizing block. (See [3] for definition of Dyke system).

3.1. **Constructing an example with infinitely many minimal half synchronizing blocks.** To do this we consider a β-shift on $\{0, 1\}$ and will replace 1 with some blocks u_i, $i \in \mathbb{N}$ to have countable many new systems. Then, we will merge them in a way explicitly explained below.

For $n \in \mathbb{N}$, let $\mathcal{G}_n = (G_n, \mathcal{L}_n)$ be a labeled graph with edge set $\mathcal{E}_n$ and vertex set $\mathcal{V}_n$. Let V be a countable partition on $\cup\mathcal{V}_n$. Let $\mathcal{G} = \vee_n \mathcal{G}_n$ be a graph whose vertex set is V. If there is an edge labeled a from I to J for some I, $J \in \mathcal{V}_n$, then put an edge labeled a from $[I]$ to $[J]$ in V. Denote by $\vee_n X_n$

FIGURE 2. From left to right $\mathcal{G}_1, \mathcal{G}_2$.

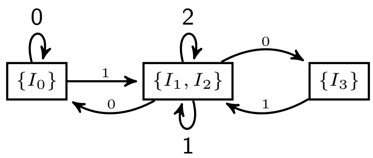

FIGURE 3. The merged graph $\mathcal{G}_1 \vee \mathcal{G}_2$.

the coded space whose cover is $\vee_n \mathcal{G}_n$. See Figure 2 for $\mathcal{V}_1 \cup \mathcal{V}_2 = \{I_0, I_1, I_2, I_3\}$ and $V = \{\{I_0\}, \{I_1, I_2\}, \{I_3\}\}$.

Let X be a shift space over $\mathcal{A}$ and let $\mathcal{B}$ be a finite subset of $\mathbb{N} \cup \{0\}$ such that $\mathcal{B} \cap \mathcal{A} = \emptyset$. Fix $a \in \mathcal{A}$ and $u \in W(\mathcal{B}^{\mathbb{Z}})$.

We construct a new shift space from X denoted by $X_{u \hookrightarrow a}$ by replacing u for a whenever a appears in $x \in X$ or block in X. For instance, if

$$m = m_1 \dots m_{i-1} a m_{i+1} \dots m_l \in W(X)$$

then

$$m_1 \dots m_{i-1} u m_{i+1} \dots m_l \in W(X_{u \hookrightarrow a})$$

and is denoted by $m_{u \hookrightarrow a}$. The same can be done for $x \in X$ and so $x_{u \hookrightarrow a} \in X_{u \hookrightarrow a}$ is well defined. Also in a natural way from a cover $\mathcal{G}$ (resp., G) for X, we can define $\mathcal{G}_{u \hookrightarrow a}$(resp., $G_{u \hookrightarrow a}$). For instance, Figure 4 is a cover for a β-shift and Figure 5 is its u_i-merge cover.

Example 3.4. *Let* $1_\beta = 110100\dots$. *Pick* $a = 1$ *and* $u_i = 54^{2i-1}523^{2i-1}2$. *Then, Figures 4 and 5 show* $\mathcal{G} := \mathcal{G}_\beta$ *and* $\mathcal{G}_{u_i \hookrightarrow a}$.

At this stage we have all the requirements for establishing our ImHalfsyn example. Pick $\beta > 1$ such that $X_\beta = X_{\mathcal{G}}$ is not synchronized and $1_\beta \subseteq \{0, 1\}^{\mathbb{N}}$. Let $a = 1, u_i = 54^{2i-1}523^{2i-1}2$ and $X_i := (X_\beta)_{u_i \hookrightarrow a} = X_{\mathcal{G}_{u_i \hookrightarrow a}}$. We shall need the following lemma.

Lemma 3.5. *For all* $i \in \mathbb{N}$, X_i *is a non-synchronized system.*

Proof. Let $m_{u_i \hookrightarrow a}$ is a synchronizing block for X_i. Choose $m' \subseteq m$ such that $m_{u_i \hookrightarrow a} \subseteq u_i m'_{u_i \hookrightarrow a} u_i = (am'a)_{u_i \hookrightarrow a}$. Set $m'' := am'a$. This $m''_{u_i \hookrightarrow a}$ is a synchronizing block for X_i.

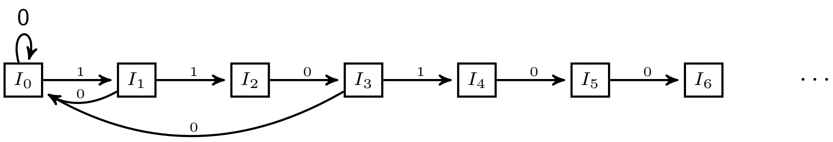

FIGURE 4. The graph $\mathcal{G}$ of the beta-shift with $1_\beta = 110100\ldots$.

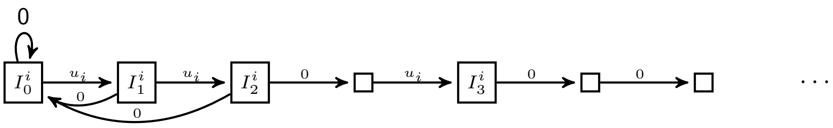

FIGURE 5. The graph $\mathcal{G}_{u_i\hookrightarrow a}$. Here I_j^i, is the terminating vertex for the jth occurrence of u_i.

Let $vm'', m''w \in W(X_\beta)$. Then,

$$v_{u_i\hookrightarrow a}m''_{u_i\hookrightarrow a} = (vm'')_{u_i\hookrightarrow a},\ m''_{u_i\hookrightarrow a}w_{u_i\hookrightarrow a} = (m''w)_{u_i\hookrightarrow a} \in W(X_i)$$

and so

$$(vm''w)_{u_i\hookrightarrow a} = v_{u_i\hookrightarrow a}m''_{u_i\hookrightarrow a}w_{u_i\hookrightarrow a} \in W(X_i).$$

This means $vm''w \in W(X_\beta)$. So m'' is a synchronizing block for X_β that is absurd. □

By Lemma 3.5, X_i is not synchronized. Let I_0^i be the base point for $\mathcal{G}_{u_i\hookrightarrow a}$ as I_0^i in Figure 5 and Let $A_i = \{I_1^i, I_2^i, \ldots\}$ be the set of all vertices in $\mathcal{G}_{u_i\hookrightarrow a}$ which are the terminal vertex of a path labeled u_i as I_j^i, $j \geq 1$ in Figure 5.

Now we introduce the cover $\mathcal{H}$ of our system. To picture out $\mathcal{H}$, in Figure 4, at any I_j, $j \geq 1$ paste all the base point of $(\mathcal{G})_{u_i\hookrightarrow 1}$, $i \geq 1$. We call $\mathcal{H}'$ the cover consisting of all $(\mathcal{G})_{u_i\hookrightarrow a}$, $i \geq 1$ glued at their base points and call the glued vertex B. Hence, $\mathcal{H}$ consists of $\mathcal{G}$ where at each I_j, $j \geq 1$. $\mathcal{H}'$ is pasted by identifying I_j, $j \geq 1$ and B. Figure 6 shows $\mathcal{H}$ for $1_\beta = 110100\ldots$ and $i = 1$, 2.

Theorem 3.6. *Let $\mathcal{H}$ be as above. Then, $X_\mathcal{H}$ is a non-synchronized but an ImHalfSyn system.*

Proof. Let $u_i = 45^{2i-1}423^{2i-1}2$ be as in Example 3.4. We prove the proposition by showing that

(1) Any block in $A = \{23^{2i-1}2 \subseteq u_i : i \in \mathbb{N}\}$ is a minimal half synchronizing block.
(2) $X_\mathcal{H}$ is not a synchronized system.

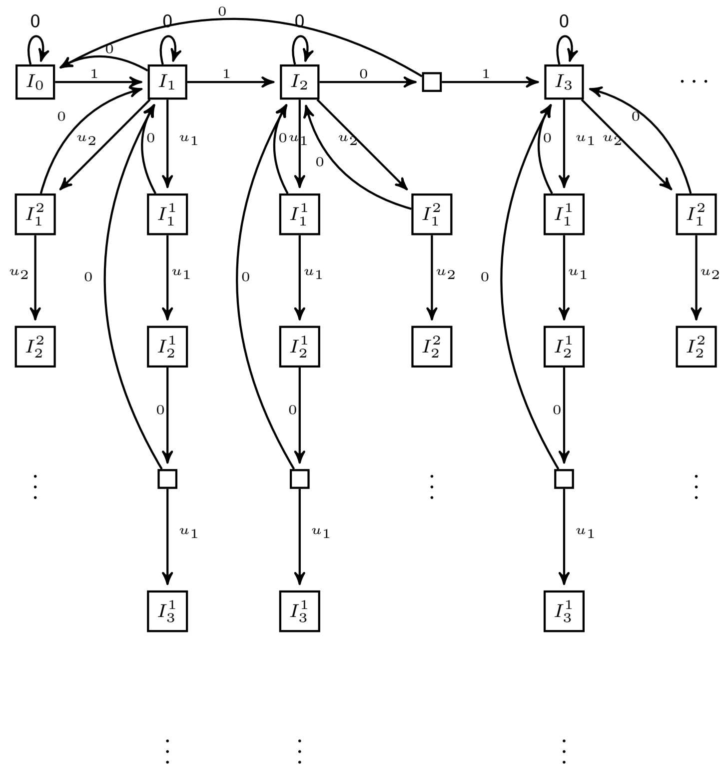

FIGURE 6. The cover $\mathcal{H}$ for a half synchronized system with infinitely many minimal half synchronizing blocks, say 0, 1, 5 and $23^{2i-1}2 \subseteq u_i$, $i = 1, 2$.

Set

$$G := \{\mathcal{L}(\pi) : \pi \text{ is a path in } \mathcal{H} \text{ and } i(\pi) = t(\pi) = I_1 \in \mathcal{V}_{\mathcal{G}_\beta} \subseteq \mathcal{V}_{\mathcal{H}}\}$$

and let $G^* := \{v_1, v_2, \ldots\}$ be the set of all finite concatenation of paths in G. Set $x_- := \cdots v_l v_{l-1} \cdots v_2 v_1 54^{2i-1}523^{2i-1}2$, $v_l \in G^*$. Note that x_- is left transitive. By our construction which rely on the properties of beta shifts, $w_+(x_-) = w_+(23^{2i-1}2)$. Hence $u_i' = 23^{2i-1}2$ is a half synchronizing block for $X_{\mathcal{H}}$.

To prove (i), we show that neither 23^j nor 3^j2 are half synchronizing. Let 23^j be a half synchronizing block where $j \geq 0$. So there is a left transitive point $x \in X_{\mathcal{H}}$ such that $x_{[-j,0]} = 23^j$ and $w_+(x_-) = w_+(23^j)$. But any left infinite path terminating at 23^j, must terminate at a path labeled $54^{2i-1}523^j$, $j \leq 2i-1$. Now let $k := 2i-1-j$. Then, $3^{k+2} \in w_+(23^j)$; such that

$3^{k+2} \notin w_+(x_-)$. Thus 23^i is not a half synchronizing block. Similar reasoning works for 3^j2.

For (ii), let m be a synchronizing block for $X_{\mathcal{H}}$. Then, there is a cycle C containing m and passing through I_0. Let m_C be the label of this cycle which is again synchronizing. Also there is a cycle $C' \subseteq C$ which completely lies in $\mathcal{G} := \mathcal{G}_\beta$, that is, C' does not contain any edge of any $\mathcal{H}'$ pasted at I_j, $j \geq 1$. Call $m_{C'}$ the labeled of C'. We will show that this $m_{C'}$ must be a synchronizing block for $X_\beta = X_{\mathcal{G}}$ which is absurd. Let $um_{C'}$ and $m_{C'}v$ be blocks in X_β. Our construction implies that whenever a path $\pi_{m_{C'}}$ labeled $m_{C'}$ appears in $\mathcal{G}$, then must be a path π_{m_C} labeled m_C appears in $\mathcal{H}$ containing $\pi_{m_{C'}}$ and conversely. Then, um_Cv as well as $um_{C'}v$ must be admissible. □

Note that in this paper, all figures were drawn by author.

4. Conclusion

Figure 6 gives a cover of a system with five set of minimal half synchronizing blocks. One can easily do this for finitely many such minimal set of blocks. In fact let $n_i \in \mathbb{N} \cup \{0, \infty\}$, $i = 1, 2$. Then, it is not hard to see that there is a system with n_1 minimal synchronizing blocks and n_2 minimal half synchronizing blocks. Proposition 3.6 gives an example for $n_1 = 0$ and $n_2 = \infty$. Now if we add cycles labeled $67^{2i-1}6$, $i \in \mathbb{N} \cup \{0, \infty\}$ and passing through the vertex I_0, the required minimal synchronizing blocks has been provided.

5. Aknowledgement

I would like to thank the reviewers for their thoughtful comments and efforts towards improving my manuscript.

References

[1] D. Ahmadi Dastjerdi and M. Shahamat, *Balanced Shifts*, Communications in Mathematics and Applications Science, 11 (2020), 415-424.

[2] F. Blanchard and G. Hansel, *Syst'emes cod ́es*, Comp. Sci. 44 (1986), 17-49.

[3] D. Fiebig and U. Fiebig, *Covers for coded systems*, Contemporary Mathematics, 135, 1992, 139-179.

[4] K. Johnson, *Beta-shift dynamical systems and their associated languages*, PhD Thesis, The University of North Carolina at Chapel Hill, 1999.

[5] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge Univ. Press. 1995.

[6] M. Shahamat, *synchronized components of a subshift*, J. Korean Math. Soc. 59 (2022), No. 1, pp. 1–12.

[7] K. Thomsen, *On the ergodic theory of synchronized systems*, Ergod. Th. Dynam. Sys. 356 (2006) 1235-1256.

[8] K. Thomsen, *On the structure of a sofic shift space*, American Mathematical Society, 356, Number 9 (2004), 3557-3619.

Manouchehr Shahamat
Orcid number: 0000-0003-3646-0458
Department of Mathematics
Dezful branch, Islamic Azad University
Dezful, Iran
Email address: m.shahamat@iaud.ac.ir