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ANOTHER LOOK AT INHERITANCE OF UNIFORM CONTINUITY OF 1-DIMENSIONAL AGGREGATION FUNCTIONS BY THEIR SUPER-ADDITIVE TRANSFORMATIONS

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Dedicated to sincere professor Mashaallah Mashinchi

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Abstract. In an earlier paper by Šeliga, Širáň and the second author (J. Mahani Math. Res. Center 8 (2019) 37–51) on lifting continuity properties of aggregation functions to their super-additive and sub-additive transformations it was shown that uniform continuity is preserved by super-additive transformations in dimension 1. We give a shorter and more direct proof of this result and of a related linear bound on uniformly continuous aggregation functions.

Keywords: Aggregation function, Sub-additive and Super-additive transformation, Uniform continuity
2020 MSC: 47S40

1. Introduction

An n-dimensional *aggregation function* is known to be an arbitrary mapping $A:\ [0,\infty[^n \to [0,\infty[$ that is monotone in every coordinate and has zero value at the origin, that is, $A(0,\ldots,0)=0$. Aggregation functions have been studied extensively both from the point of view of theory and applications; we refer here to the monograph [1] and to the outlook paper [5] for various aspects of the subject. A particularly fruitful stream of their study was initiated by associating with every aggregation function A as above its super-additive and sub-additive transformation, defined in [2] by

$$A^*(\mathbf{x}) = \sup\left\{\sum_{j=1}^{k} A(\mathbf{x}^{(j)})\ ;\ \ \mathbf{x}^{(j)} \in [0.\infty[^n,\ \sum_{j=1}^{k}\mathbf{x}^{(j)} = \mathbf{x}\right\}\ ,\quad \text{and}$$

$$A_*(\mathbf{x}) = \inf\left\{\sum_{j=1}^{k} A(\mathbf{x}^{(j)})\ ;\ \ \mathbf{x}^{(j)} \in [0,\infty[^n,\ \sum_{j=1}^{k}\mathbf{x}^{(j)} = \mathbf{x}\right\}$$

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for any $\mathbf{x} \in [0,\infty[^n$. In what follows we will only be considering the super-additive transformation, and note that the supremum in its definition may turn out to be infinite. However, by an appropriately modified claim from [3], if that happens for some point in $[0,\infty[^n$ with all non-zero coordinates, then the values of the super-additive transformation are infinite at every point in $[0,\infty[^n$ except at the origin. Motivated by this, an aggregation function A will be said to have a *non-escaping cover* if all values of its super-additive transformation A^* are finite, i.e., if A^* is well-defined everywhere on $[0,\infty[^n$.

In [7] the authors considered the natural question of which continuity-type properties of an aggregation function carry over to its super- and/or sub-additive transformation, which was also motivated by continuity conditions appearing in some results of [4] on aggregation functions with prescribed super-additive and sub-additive transformations. The most intriguing case appears to be the one of inheritance of uniform continuity of aggregation functions by their super-additive transformations, which was in [7] resolved only in dimension $n = 1$. The purpose of this note is to give a much simpler and more direct proof of this result (which was stated as Theorem 4 in [7]), simplifying also the proof of an important bound (stated as Proposition 1 in [7]) in dimension 1 with the help of an interesting result on 1-dimensional uniformly continuous functions on unbounded domains [6]. Regarding our restriction to the case $n = 1$ it should be noted that results on one-dimensional aggregation function proved to be important in the development of more sophisticated methods applicable to higher dimensions, cf. e.g. [8].

2. **Results**

As indicated, we begin by giving an alternative proof of a useful linear bound on the values of a uniformly continuous increasing function with zero value at the origin. We actually prove the bound under much milder assumptions and with the help of a somewhat neglected result – Theorem 3.1 of [6] – on uniformly continuous functions on unbounded intervals.

Proposition 1. Let $A : [0,\infty[\to [0,\infty[$ be a continuous aggregation function with a non-escaping cover. Assume that there exist positive real numbers c and d such that $A(x) - A(y) \le d$ whenever $x \ge y$ and $x - y \le c$; in particular, this holds if A is uniformly continuous on $[0,\infty[$. Then there exists a positive real number α_A such that $A(z) \le \alpha_A z$ for every $z \in [0,\infty[$.

Proof. Let c and d be as in the above statement. The fact that the ratio $A(z)/z$ is bounded above for $z \in]0, c[$ for any positive real number c follows from [3] and also from [7] by the assumption that A has a non-escaping cover; the upper bound in general depends on c and so can be written in the form $A(z) \le a(c)$ for some positive real number $a(c)$ but this will not be an issue here. Further, by a straightforward adaptation of Theorem 3.1 of [6], there exists a positive real number $b = b(c, d)$ (depending in general on c and d) such that

$A(z) \leq bz$ for every $z \in [c, \infty[$. From the two upper bounds $A(z) \leq a(c)z$ for $z \in [0, c[$ and $A(z) \leq b(c,d)z$ one easily obtains the existence of some positive real α_A such that $A(z) \leq \alpha_A z$ for every $z \in [0, \infty[$. □

With this tool we are now ready to give a simpler and more direct proof of inheritance of uniform continuity of increasing functions on $[0, \infty[$ with zero value at the origin by their super-additive transformations.

Theorem 1. Let $A : [0, \infty[\to [0, \infty[$ be an aggregation function with a non-escaping cover. If A is uniformly continuous, then so is its super-additive transformation A^*.

Proof. We need to show that for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that for an arbitrary pair $x, y \in [0, \infty[$ such that $x \geq y$ and $x - y < \delta$ it holds that $A^*(x) - A^*(y) < \varepsilon$. Thus, let an arbitrary $\varepsilon > 0$ be given. By the assumed uniform continuity of A, to the positive real $\varepsilon/4$ there exists a $\delta_0 > 0$ such that for any non-negative real $x \geq y$, the inequality $x - y < \delta_0$ implies $A(x) - A(y) < \varepsilon/4$. Further, let α_A be the positive real number from Proposition 1 associated with A, i.e., with the property that $A(z) \leq \alpha_A z$ for every $z \in [0, \infty[$.

Set now $\delta = \min\{\varepsilon/(2\alpha_A), \delta_0\}$ and let x, y be non-negative real numbers with $x > y$ and $x - y < \delta$. By the definition of the super-additive transformation A^* of A, there is a sequence (x_j), $1 \leq j \leq n$, of positive real numbers (for some n) such that

$$A^*(x) < \varepsilon/4 + \sum_{j=1}^{n} A(x_j) \tag{1}$$

Let m be the smallest positive integer such that $x_1 + \cdots + x_m > y$. (We note that we do not assume any ordering among the members of the sequence (x_j)). Define a new sequence (y_j), $1 \leq j \leq m$, of non-negative real numbers by letting $y_j = x_j$ for every j such that $1 \leq j \leq m - 1$, and $y_m = y - (x_1 + \cdots + x_{m-1})$. Observe that $y_1 + \cdots + y_m = y$, and $x_m - y_m = (x_1 + \cdots + x_m) - y \leq x - y < \delta$. Furthermore, if $n > m$, then we also have $x_{m+1} + \cdots + x_n < x - y < \delta$. In such a case, again by definition of A^* and then by application of Proposition 1 and by the way the parameter δ has been introduced one has

$$\sum_{j=m+1}^{n} A(x_j) \leq A^*(\delta) \leq \alpha_A \delta < \varepsilon/2 \tag{2}$$

But then, irrespective of whether $m = n$ or $m < n$, summing up the above facts together with (1) and (2) and merging them with the obvious inequality $A^*(y) \geq A(y_1) + \cdots + A(y_m)$ and taking into the account that $x_j = y_j$ for

$1 \le j \le m$ we obtain

$$A^*(x) - A^*(y) < \varepsilon/4 + \sum_{j=1}^{n} A(x_j) - \sum_{j=1}^{m} A(y_j) = \varepsilon/4 + A(x_m) - A(y_m) + \sum_{j=m+1}^{n} A(x_j) < \varepsilon$$

because $A(x_m) - A(y_m) < \varepsilon/4$; note that the last sum above is void if $m = n$. This completes the proof. □

The converse to Theorem 1 is obviously false; if A^* is uniformly continuous then A does not even have to be continuous. For example, if $A(x) = x$ for $x \in [0, 2] \cup [4, \infty[$ and $A(x) = 1 + x/2$ for $x \in [2, 4[$, then trivially $A^*(x) = x$ for $x \in [0, \infty[$ while A is discontinuous.

3. Summary

In this note we have given a simplified proof of inheritance of uniform continuity of 1-dimensional aggregation functions on $[0, \infty[$ to their super-additive transformations, based on a new and shorter proof of a linear upper bound on 1-dimensional uniformly continuous aggregation functions (even under a much milder conditions), extending and improving thereby some results of [7].

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