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L_k-BIHARMONIC HYPERSURFACES IN THE 3-OR 4-DIMENSIONAL LORENTZ-MINKOWSKI SPACES

R. Hosseinoughli and A. Mohammadpouri ✉

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Abstract. A hypersurface M^n in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ is called L_k-biharmonic if the position vector ψ satisfies the condition $L_k^2\psi = 0$, where L_k is the linearized operator of the $(k+1)$-th mean curvature of M for $k = 0, 1, \ldots, n-1$. This definition is a natural generalization of the concept of a biharmonic hypersurface. We prove that any L_k-biharmonic surface in $\mathbb{L}^3$ is k-maximal. We also prove that any L_k-biharmonic hypersurface in $\mathbb{L}^4$ with constant k-th mean curvature is k-maximal. These results give a partial answer to the Chen's conjecture for L_k-operator that L_k-biharmonicity implies L_k-maximality.

Keywords: Linearized operator L_k, L_k-biharmonic hypersurface, k-maximal hypersurface, k-th mean curvature.
2020 MSC: Primary: 53A05; Secondary: 53B20, 53C21.

1. Introduction

B. Y. Chen in the middle of 1980s began the study of biharmonic submanifolds in his investigation of finite type submanifolds. In a different setting, in [11], G. Y. Jiang studied the biharmonic submanifold by a spectral variational principal; namely as critical points of bi-energy functional.

A well-known conjecture due to Chen [4] asserts that every biharmonic submanifold of the Euclidean space must be minimal. Chen's conjecture has extensively studied by many geometers for more than three decades. So far only partial answers have been obtained, and the conjecture remains open. The reader can refer to [24] for updated information on the Chen's conjecture and the generalizations.

The conjecture generally does not hold for submanifolds in a pseudo-Euclidean space $\mathbb{E}^n_s$ in contrast to the Euclidean space. Many examples of non-minimal submanifolds in a pseudo-Euclidean space have been produced in [5]. Although, Chen's conjecture holds for hypersurfaces in a pseudo-Euclidean space for the following specific cases:

- surfaces in $\mathbb{E}^3_s$ $(s = 1, 2)$ [5];

✉ pouri@tabrizu.ac.ir, ORCID: 0000-0002-8546-6445
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- nondegenerate hypersurfaces in $\mathbb{E}_s^n$ with diagonalizable shape operator and at most two distinct principal curvatures [12];
- nondegenerate hypersurfaces in $\mathbb{E}_s^4$ $(s = 1, 2, 3)$ with diagonalizable shape operator [8];
- Lorentz hypersurfaces in $\mathbb{E}_1^4$ [3];
- nondegenerate hypersurfaces in $\mathbb{E}_2^4$ with index 2 [20];
- nondegenerate hypersurfaces in $\mathbb{E}_s^5$ with diagonalizable shape operator and constant scalar curvature [7];
- nondegenerate hypersurfaces in $\mathbb{E}_s^5$ with diagonalizable shape operator and three distinct principal curvatures [9];
- Lorentz hypersurfaces in $\mathbb{E}_1^{n+1}$ with non-diagonalizable shape operator, and three distinct principal curvatures having minimal polynomial $(t-k_1)^2(t-k_2)(t-k_n)$ [6].

As we know, the operator L_k of a hypersurface M immersed into $\mathbb{R}^{n+1}$, as an extension of the Laplacian differential operator $L_0 = \Delta$, is the second order linear differential operator, related to the classical Newton transformation P_k is given by $L_k f = \mathrm{tr}(P_k \circ \mathrm{hessf})$ for any smooth function f on M and $k = 1, \dots, n-1$ (for more details we refer reader to [22]). Very recently, inspired by the idea of biharmonic submanifolds and Chen's conjecture, Kashani and subsequently, the second author, et al. [1, 2, 18, 19], by using the L_k-operators have studied hypersurfaces whose position vector ψ satisfies the general condition $L_k^2\psi = 0$ and called them L_k-biharmonic hypersurfaces. In particular, when $k = 0$, the L_0-biharmonic hypersurface is nothing but the biharmonic hypersurface.

Let $\psi : M_s^n \to \mathbb{E}_t^{n+1}$ be an isometric immersion of an n-dimensional connected orientable hypersurface M_s^n of index s into the pseudo-Euclidean space $\mathbb{E}_t^{n+1}$ with Gauss map N, $\langle N, N\rangle = \varepsilon = \pm 1$ and the shape operator S with respect to N. In [14], it has been proved that

$$\begin{aligned} L_k^2\psi = &- \varepsilon c_k C_k H_{k+1}\nabla H_{k+1} - 2c_k(S \circ P_k)\nabla H_{k+1} \\ &- [\varepsilon C_k H_{k+1}(nHH_{k+1} - (n-k-1)H_{k+2}) - L_k H_{k+1}]N, \end{aligned} \tag{1}$$

where $0 \le k \le n-1$, $(k+1)C_k = c_k = (-\varepsilon)^k(n-k)\binom{n}{k}$ and H_{k+1} is $(k+1)$-th mean curvature of M_s^n. From the above formula, it is obvious that any hypersurface with vanishing $(k+1)$-th mean curvature, namely, k-maximal hypersurface (in the Euclidean case are called k-minimal), is the L_k-biharmonic. The affirmative answer to the converse of this fact is not trivial. Related to this problem, Aminian and Kashani in [1] stated the generalization Chen's conjecture for the L_k-operator in Euclidean cases. The L_k-conjecture states that all L_k-biharmonic hypersurfaces in Euclidean space are k-minimal.

In the same paper, they proved the L_k-conjecture for Euclidean hypersurfaces with at most two principal curvatures. This result shows that L_1-conjecture is also true for Euclidean surfaces. Following them, the second author with co-authors proved that (1) L_k-conjecture holds for hypersurfaces in Euclidian space $\mathbb{E}^4$ with constant k-th mean curvature [18], (2) L_1-conjecture holds for hypersurfaces in Euclidian space $\mathbb{E}^n$ with at most three principal curvatures and constant mean curvature [19], (3) L_k-conjecture holds for weakly convex Euclidean hypersurfaces [17].

In view of the above history, it seems natural to investigate the L_k-conjecture for indefinite metrics states that all L_k-biharmonic hypersurfaces in pseudo-Euclidean space $\mathbb{E}^n_s$ are k-maximal. Maximal hypersurfaces in Lorentzian geometry are directly analogous to minimal hypersurfaces in Riemannian geometry. The difference in terminology between the two settings has to do with the fact that small regions in maximal hypersurfaces are local maximizers of the area functional, while small regions in minimal hypersurfaces are local minimizers of the area functional. As we know, a shape operator of pseudo-Riemannian hypersurfaces is not diagonalizable always unlike the Riemannian hypersurfaces, this fact makes work harder in the indefinite case. Here, we study the L_k-biharmonic hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ of smallest possible dimension, i.e. $n = 2$ or $n = 3$. It remains as an open problem to investigate the L_k-conjecture for indefinite metrics of dimension $n > 4$. Here we obtain the following theorems to guarantee that the L_k-conjecture is true for indefinite metrics when $n = 2$ or $n = 3$.

Theorem 1.1. *Let $\psi : M \to \mathbb{L}^3$ be an orientable surface immersed into the Lorentz-Minkowski space L^3 and the immersion satisfies the condition $L_k^2\psi = 0$ for some $k = 0, 1$. Then $(k+1)$-th mean curvature is zero, or equivalently, the immersion is k-maximal.*

Theorem 1.2. *Let $\psi : M \to \mathbb{L}^4$ be an orientable hypersurface immersed into the Lorentz-Minkowski space $\mathbb{L}^4$ with constant k-th mean curvature and the immersion satisfies the condition $L_k^2\psi = 0$ for some $k = 0, 1, 2$. Then $(k+1)$-th mean curvature is zero, or equivalently, the immersion is k-maximal.*

In Theorem 1.1, the case $k = 0$ has been proved by Chen and Ishikawa in [5], so in order to prove the result, we consider the case $k = 1$. The proof is easily obtained from the formula (1) as follows:

Since the immersion satisfies the condition $L_1^2\psi = 0$, by identifying tangent and normal parts of $L_k^2\psi$ in (1) with zero and setting $k = 1$, we find that

$$(S \circ P_1)\nabla H_2 = \frac{1}{2}H_2\nabla H_2, \tag{2}$$

and

$$L_1 H_2 = -2HH_2^2. \tag{3}$$

Since $P_2 = 0$, we also have $(S \circ P_1)\nabla H_2 = -H_2\nabla H_2$ ([14]). Comparing this with (2) yields H_2 is constant. If H_2 is zero, the proof is finished. If H_2 is nonzero, then by using (3), we get that M is maximal, i.e., $H = 0$, which is a contradiction because any isoparametric surface in $\mathbb{L}^3$ with constant nonzero H_2 is not maximal [10].

Therefore, our main objective here is to prove Theorem 1.2. The case $k = 0$ has been proved by Defever et al. [3, 8], so in order to prove Theorem 1.2, we consider the cases $k = 1$ and $k = 2$, respectively, in the subsections 3.1 and 3.2. Note that the proof of Theorem 1.2 is harder and uses some different tools compared with methods was used in the references [3, 8].

2. **Preliminaries**

In this section, we recall some basic definitions and prerequisites from [13, 16, 23].

Let $\mathbb{L}^4$ denote the 4-dimensional Lorentz-Minkowski space, i.e., the space $\mathbb{R}^4$ equipped with the following metric

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) stands for the natural coordinates of $\mathbb{R}^4$.

Let $\psi : M \to \mathbb{L}^4$ be an isometric immersion from a 3-dimensional orientable manifold M to $\mathbb{L}^4$ with Gauss map N, $\langle N, N\rangle = \varepsilon$, where $\varepsilon = 1$ or $\varepsilon = -1$ according to M is equipped with a Lorentzian or a Riemannian metric, respectively. Denote by S, ∇, and $\bar{\nabla}$ the shape operator with respect to N, the Levi-Civita connection on M, and the usual flat connection on $\mathbb{L}^4$, respectively.

We say that a basis $\{e_1, e_2, e_3\}$ of T_pM is a pseudo-orthornormal, if it satisfies in the following properties

$$\langle e_1, e_2\rangle = -1,\ \langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0,\ \text{and } \langle e_3, e_3\rangle = 1.$$

It is well-known that the shape operator S of M can be expressed by one of the following four forms, with respect to a suitable frame [15]:

$$\text{I. } S \approx \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}; \qquad \text{II. } S \approx \begin{pmatrix} \mu & -\nu & 0 \\ \nu & \mu & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},\ \nu \neq 0;$$

$$\text{III. } S \approx \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}; \qquad \text{IV. } S \approx \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ -1 & 0 & \lambda \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda, \mu, \nu \in \mathbb{R}$.
The frame that represent S in the cases I and II is orthonormal whereas in the cases III and IV is the pseudo-orthonormal.

The characteristic polynomial $Q_s(t)$ of the shape operator S is given by

$$Q_s(t) = \det(tI - S) = \sum_{k=0}^{3} a_k t^{3-k},\ \text{with } a_0 = 1.$$

We easily see that the coefficients of $Q_s(t)$, for S of type I, are given by

$$a_1 = -(\lambda_1 + \lambda_2 + \lambda_3),\ a_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,\ a_3 = -\lambda_1\lambda_2\lambda_3. \tag{4}$$

For S of type II, they are given by

$$a_1 = -(2\mu + \lambda_3),\ a_2 = \mu^2 + \nu^2 + 2\mu\lambda_3,\ a_3 = -(\mu^2 + \nu^2)\lambda_3. \tag{5}$$

For S of type III, they are given by

$$a_1 = -(2\lambda + \lambda_3),\ a_2 = \lambda^2 + 2\lambda\lambda_3,\ a_3 = -\lambda^2\lambda_3. \tag{6}$$

Finally, for S of type IV, they are given by

$$a_1 = -3\lambda,\ a_2 = 3\lambda^2,\ a_3 = -\lambda^3. \tag{7}$$

The k-th mean curvature of M is defined by

$$\binom{3}{k} H_k = (-\varepsilon)^k a_k,\ H_0 = 1. \tag{8}$$

In particular, when $k = 1$,

$$3H_1 = -\varepsilon a_1 = \varepsilon \mathrm{tr}(S),$$

we see that H_1 is nothing but the mean curvature H of M. When $k = n$, $H_n = (-\varepsilon)^n a_n = (-\varepsilon)^n \det(S)$ is called the Gauss-Kronecker curvature of M. A hypersurface M in the Lorentz-Minkowski space $\mathbb{L}^4$ is said to be k-maximal if $H_{k+1} \equiv 0$, a 0-maximal hypersurface is nothing but a maximal hypersurface in $\mathbb{L}^4$.

The k-th Newton transformation of M is the operator $P_k : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$P_k = \sum_{j=0}^{k} a_{k-j} S^j.$$

Therefore, we see that

$$P_0 = I,\ P_1 = -3\varepsilon H I + S,\ P_2 = 3H_2 I + S \circ P_1,$$

and

$$P_3 = -\varepsilon H_3 I + S \circ P_2 = 0 \text{ (by Cayley-Hamilton theorem).}$$

Related to the Newton transformation P_k, the second-order linear differential operator $L_k : C^\infty(M) \to C^\infty(M)$ is defined by

$$L_k(f) = \mathrm{tr}(P_k \circ \nabla^2 f), \tag{9}$$

where, $\nabla^2 f : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by

$$\langle \nabla^2 f(V), W \rangle = \langle \nabla_V (\nabla f), W \rangle,\ V, W \in \mathfrak{X}(M).$$

We can naturally extend the definition of the operator L_k from functions to vector functions $F = (f_1, f_2, f_3, f_4),\ f_i \in C^\infty(M)$, as follows

$$L_k F = (L_k f_1, L_k f_2, L_k f_3, L_k f_4).$$

Then $L_k\psi$ can be phrased as follows

$$L_k\psi = (-L_k\langle \psi, e_1 \rangle, L_k\langle \psi, e_2 \rangle, L_k\langle \psi, e_3 \rangle, L_k\langle \psi, e_4 \rangle),$$

where $\{e_1, e_2, e_3, e_4\}$ is a standard basis of $\mathbb{L}^4$.

3. L_k-biharmonic hypersurfeces in $\mathbb{L}^4$

Invoking a formula [14, page 165], we have

$$
\begin{aligned}
L_k^2\psi = & -\varepsilon c_k C_k H_{k+1}\nabla H_{k+1} - 2c_k(S\circ P_k)\nabla H_{k+1} \\
& - [\varepsilon C_k H_{k+1}(3H_1H_{k+1} - (2-k)H_{k+2}) - L_k H_{k+1}]N,
\end{aligned} \tag{10}
$$

where $0 \le k \le 2$, $(k+1)C_k = c_k = (-\varepsilon)^k(3-k)\binom{3}{k}$ and H_{k+1} is $(k+1)$-th mean curvature of M.

By the definition of the L_k-biharmonic hypersurface, it is obvious that k-maximal hypersurfaces are trivially L_k-biharmonic. By identifying tangent and normal parts of $L_1^2\psi$ in (10) with zero, L_1-biharmonicity condition implies that

$$
(S\circ P_1)\nabla H_2 = \frac{3}{2}H_2\nabla H_2, \text{ and } L_1H_2 = 3H_2(H_3 - 3HH_2). \tag{11}
$$

In the same way, we deduce that the L_2-biharmonic hypersurfaces satisfy the following two conditions

$$
(S\circ P_2)\nabla H_3 = -\frac{1}{2}\varepsilon H_3\nabla H_3, \text{ and } L_2H_3 = 3\varepsilon HH_3^2. \tag{12}
$$

Now, we are going to classify the L_k-biharmonic isoparametric hypersurfaces in the Lorentz-Minkowski space. A hypersurface M in the Lorentz-Minkowski space is said to be isoparametric if the minimal polynomial of its shape operator is constant. A well-known result by Magid [15] states that the only isoparametric hypersurface in the Lorentz-Minkowski space $\mathbb{L}^4$ is (1) an open piece of the Euclidean space $\mathbb{R}^3$; (2) Lorentz-Minkowski space $\mathbb{L}^3$; (3) De Sitter space $\mathbb{S}_1^3(r)$; (4) anti De Sitter space $\mathbb{H}^3(-r)$; (5)generalized cylinder $\mathbb{S}_1^1(r)\times\mathbb{R}^2$, (6) $\mathbb{H}^1(-r)\times\mathbb{R}^2$, $\mathbb{S}_1^2(r)\times\mathbb{R}$, $\mathbb{H}^2(-r)\times\mathbb{R}$; or (7) $\mathbb{L}^1\times\mathbb{S}^2(r)$, $\mathbb{L}^2\times\mathbb{S}^1(r)$.
By making use of the table on [13, page 164] and (10), we compute the $(k+1)$-th mean curvature and $L_k^2\psi$ of the isoparametric hypersurface in the Lorentz-Minkowski space $\mathbb{L}^4$ and collect the results in the following table

Hypersurface	H_1	H_2	H_3	$L_1^2(\psi)$	$L_2^2(\psi)$
$\mathbb{S}_1^3(r)$	$1/r$	$1/r^2$	$1/r^3$	$-(36/r^5)N$	$-(9/r^7)N$
$\mathbb{H}^3(-r)$	$-1/r$	$1/r^2$	$-1/r^3$	$-(36/r^5)N$	$-(9/r^7)N$
$\mathbb{S}_1^1(r)\times\mathbb{R}^2$	$1/3r$	0	0	0	0
$\mathbb{H}^1(-r)\times\mathbb{R}^2$	$-1/3r$	0	0	0	0
$\mathbb{S}_1^2(r)\times\mathbb{R}$	$2/3r$	$1/3r^2$	0	$-(4/r^5)N$	0
$\mathbb{H}^2(-r)\times\mathbb{R}$	$-2/3r$	$1/3r^2$	0	$-(4/r^5)N$	0
$\mathbb{L}^1\times\mathbb{S}^2(r)$	$2/3r$	$1/3r^2$	0	$-(4/r^5)N$	0
$\mathbb{L}^2\times\mathbb{S}^1(r)$	$1/3r$	0	0	0	0

The Euclidean space $\mathbb{R}^3$ and the Lorentz-Minkowski space $\mathbb{L}^3$ are trivially L_k-biharmonic. This table shows that the L_k-conjecture is true for isoparametric hypersurfaces in $\mathbb{L}^4$; or equivalently, we have

Corollary 3.1. *The only L_k-biharmonic isoparametric hypersurfaces in $\mathbb{L}^4$ are k-maximal ones.*

3.1. **L_1-biharmonic hypersurfaces.** In order to prove Theorem 1.2 stated in the introduction, we will consider the case $k=1$ and $k=2$, separately. The case $k=1$ of Theorem 1.2 is proved in the following proposition.

Proposition 3.2. *Let $\psi : M \longrightarrow \mathbb{L}^4$ be an orientable L_1-biharmonic hypersurface immersed into the Lorentz-Minkowski space $\mathbb{L}^4$ with constant mean curvature. Then M is 1-maximal.*

Proof. If the shape operator S has the canonical form of type I, the proposition has been proved in [21]. So, we will consider cases II, III, IV of the canonical form of the shape operator S, with respect to an appropriate basis $\{e_1, e_2, e_3\}$ of T_pM, separately. In each case, first we prove that H_2 is constant by showing that the open set $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$ is empty.

S **is of type II.** In this case, P_1 and P_2 take the following form with respect to the local orthonormal frame $\{e_1, e_2, e_3\}$

(13)

$$P_1 = \begin{pmatrix} -(\mu+\lambda_3) & -\nu & 0 \\ \nu & -(\mu+\lambda_3) & 0 \\ 0 & 0 & -2\mu \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \mu\lambda_3 & \nu\lambda_3 & 0 \\ -\nu\lambda_3 & \mu\lambda_3 & 0 \\ 0 & 0 & \mu^2+\nu^2 \end{pmatrix}.$$

Using (5) and (8), we find

$$H_2 = \frac{1}{3}(\mu^2 + \nu^2 + 2\mu\lambda_3). \tag{14}$$

By using the first equation (11) and the inductive definition of P_2, we get

$$P_2(\nabla H_2) = \frac{9}{2}H_2\nabla H_2. \tag{15}$$

The gradient of H_2 can be expressed as

$$\nabla H_2 = -e_1(H_2)e_1 + e_2(H_2)e_2 + e_3(H_2)e_3. \tag{16}$$

By combining (13), (15) and (16), we obtain

$$\begin{cases} -\mu\lambda_3 e_1(H_2) + \nu\lambda_3 e_2(H_2) = -\frac{9}{2}H_2 e_1(H_2), \\ \nu\lambda_3 e_1(H_2) + \mu\lambda_3 e_2(H_2) = \frac{9}{2}H_2 e_2(H_2), \\ (\mu^2+\nu^2)e_3(H_2) = \frac{9}{2}H_2 e_3(H_2). \end{cases} \tag{17}$$

We show that ∇H_2 is in the direction of e_3. If $e_2(H_2) \neq 0$, the first two equations in (17) imply that $(\frac{9}{2}H_2 - \mu\lambda_3)^2 + (\nu\lambda_3)^2 = 0$. Thus, we find that $\nu\lambda_3 = 0$ and $\frac{9}{2}H_2 = \mu\lambda_3$, since $\nu \neq 0$, we get that H_2 vanishes identically on $\mathcal{U}$, this is a contradiction. Thus, $e_2(H_2) = 0$ holds identically on $\mathcal{U}$. Similarly, we conclude that $e_1(H_2) = 0$ holds identically on $\mathcal{U}$. Therefore, ∇H_2 is in the direction of e_3.

The third equation of (17) gives

$$H_2 = \frac{2}{9}(\nu^2 + \mu^2). \tag{18}$$

From the definition of H and (14), we also have

$$H_2 = \frac{4}{3}(2\mu - 3\varepsilon H)\mu. \tag{19}$$

The assumption of constancy of H, the relation $e_1(H_2) = e_2(H_2) = 0$ and (19) yield $e_1(\mu) = e_2(\mu) = 0$. Thus, from (18), we also obtain that $e_1(\nu) = e_2(\nu) = 0$.

Let us write

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k;\ i,j,k = 1,2,3.$$

We can easily obtain $\omega_{32}^1 = \omega_{31}^2$ (1), $\omega_{12}^3 = -\omega_{13}^2$ (2), $\omega_{21}^3 = \omega_{23}^1$ (3), $\omega_{11}^3 = \omega_{13}^1$ (4), $\omega_{22}^3 = \omega_{23}^2$ (5). From the relation $[e_1, e_2](H_2) = \nabla_{e_1} e_2(H_2) - \nabla_{e_2} e_1(H_2) = 0$, we easily get that $\omega_{12}^3 = \omega_{21}^3$ (6). Combining the relations (2), (3) and (6) gives $\omega_{13}^2 = -\omega_{23}^1$ (7).
Next, by substituting the set of triples

$$\{(1,2,1),(1,2,2),(1,3,1),(2,3,2),(1,3,3),(1,2,3),(1,3,2),(2,3,1)\}$$

into Codazzi equation ($\langle(\nabla_{e_i} S)e_j, e_k\rangle = \langle(\nabla_{e_j} S)e_i, e_k\rangle$), we obtain the following linear systems of equations.

$$\begin{aligned} e_1(\nu) &= -e_2(\mu) - 2\nu\omega_{21}^2, \\ e_2(\nu) &= e_1(\mu) - 2\nu\omega_{12}^1, \end{aligned} \tag{20}$$

$$\begin{aligned} (\lambda_3 - \mu)\omega_{13}^1 + \nu\omega_{13}^2 &= e_3(\mu) + 2\nu\omega_{32}^1, \\ (\lambda_3 - \mu)\omega_{23}^2 - \nu\omega_{23}^1 &= e_3(\mu) - 2\nu\omega_{31}^2, \end{aligned} \tag{21}$$

$$\begin{aligned} e_1(\lambda_3) &= (\mu - \lambda_3)\omega_{31}^3 + \nu\omega_{32}^3, \\ e_2(\lambda_3) &= -\nu\omega_{31}^3 + (\mu - \lambda_3)\omega_{32}^3, \end{aligned} \tag{22}$$

$$\begin{aligned} e_3(\nu) &= (\lambda_3 - \mu)\omega_{13}^2 - \nu\omega_{13}^1, \\ e_3(\nu) &= -(\lambda_3 - \mu)\omega_{23}^1 - \nu\omega_{23}^2. \end{aligned} \tag{23}$$

Now, because the relation $e_1(\nu) = e_2(\nu) = e_1(\mu) = e_(\mu) = 0$ holds identically on $\mathcal{U}$, from (20) we get easily $\omega_{21}^2 = \omega_{12}^1 = 0$ (8). Also, because the relation $e_1(\lambda_3) = e_2(\lambda_3) = 0$ holds identically on $\mathcal{U}$, from the homogeneous linear system (22) with determinant $D = (\lambda - \mu)^2 + \nu^2 \neq 0$, we find that $\omega_{31}^3 = \omega_{32}^3 = 0$ (9). Moreover, by applying the equations of (23) and the equality (7), we have $\omega_{13}^1 = \omega_{23}^2$ (10). By using the relations (1), (7), (10) and the equations of (21), we conclude that $\omega_{32}^1 = \omega_{31}^2 = 0$ (11). Substituting (11) into the first equation of (21) gives

$$(3\varepsilon H - 3\mu)\omega_{13}^1 + \nu\omega_{13}^2 = e_3(\mu). \tag{24}$$

If $\omega_{13}^1 = \omega_{13}^2 = 0$ holds identically on $\mathcal{U}$, then it follows from (24) that $e_3(\mu) = 0$. Therefore, H_2 is constant on $\mathcal{U}$, which is a contradiction. Consequently, one of the following three cases occurs:
(a) $\omega_{13}^1 \neq 0$, $\omega_{13}^2 = 0$. The equation of Gauss as the following

$$\langle R(e_1, e_3)e_2, e_3\rangle = \langle S(e_3), e_2\rangle\langle S(e_1), e_3\rangle - \langle S(e_1), e_2\rangle\langle S(e_3), e_3\rangle,$$

and the relations (1)-(11) yield

$$e_3(\omega_{13}^2) = 2\omega_{13}^1\omega_{13}^2 + (2\mu - 3\varepsilon H)\nu.$$

Substituting $\omega_{13}^2 = 0$ into the above equation gives $(2\mu - 3\varepsilon H)\nu = 0$. Since μ is non-constant, we get $\nu = 0$, which is impossible.
(b) $\omega_{13}^1 = 0$, $\omega_{13}^2 \neq 0$. The equation of Gauss as the following

$$\langle R(e_1, e_3)e_1, e_3\rangle = \langle S(e_3), e_1\rangle\langle S(e_1), e_3\rangle - \langle S(e_1), e_1\rangle\langle S(e_3), e_3\rangle,$$

and the relations (1)-(11) imply that

$$e_3(\omega_{13}^1) = \frac{3}{4}H_2 - (\omega_{13}^1)^2 + (\omega_{13}^2)^2. \tag{25}$$

Combining this with $\omega_{13}^1 = 0$ gives $\frac{3}{4}H_2 + (\omega_{13}^2)^2 = 0$. But this is a contradiction since H_2 is non-constant on $\mathcal{U}$.
(c) $\omega_{13}^1 \neq 0,\ \omega_{13}^2 \neq 0$. By differentiating of (19) with respect to e_3, applying the first equation of (23) and (24), we find

$$e_3(H_2) = \frac{4}{3}\nu(\varepsilon H - \mu)\omega_{13}^2 - \frac{7}{3}H_2\omega_{13}^1. \tag{26}$$

Comparing (18) and (19) gives

$$\frac{2}{9}\nu^2 - \frac{22}{9}\mu^2 + 4\varepsilon H\mu = 0. \tag{27}$$

Consequently, we have

$$\nu^2 = 11\mu^2 - 18\varepsilon H\mu. \tag{28}$$

By differentiating of (27) with respect to e_3, applying the first equation of (23) and (24), we find

$$\omega_{13}^2 = \frac{22\mu^2 - 42\varepsilon H\mu + 27H^2}{\nu(14\mu - 12\varepsilon H)}\omega_{13}^1. \tag{29}$$

Substituting (29) into the equation (26) gives

$$e_3(H_2) = \frac{Q_3(\mu)}{7\mu - 6\varepsilon H}, \tag{30}$$

where $Q_3(\mu) = -\frac{160}{3}\mu^3 + 136\varepsilon H\mu^2 - 96H^2\mu + 18\varepsilon H^3$, is a polynomial in terms of μ of degree 3.
Differentiating (30) with respect to e_3 and using (24), (25) and (29), we obtain

$$e_3(e_3(H_2)) = \frac{Q_7(\mu)(w_{13}^1)^2 + Q_9(\mu)}{(7\mu - 6\varepsilon H)^3(44\mu^2 - 72\varepsilon H\mu)}, \tag{31}$$

where

$$\begin{aligned} Q_7(\mu) = {} & (1408\mu^4 - 5208\varepsilon H\mu^3 + 5778H^2\mu^2 - 1944\varepsilon H^3\mu + 729H^4)Q_3(\mu) \\ & + (-3080\mu^5 + 13224\varepsilon H\mu^4 - 19530H^2\mu^3 + 11232\varepsilon H^3\mu^2 - 1944H^4\mu)\frac{\partial(Q_3(\mu))}{\partial\mu}, \end{aligned}$$

and $Q_9(\mu) = (4312\mu^6 - 20916\varepsilon H\mu^5 + 36936H^2\mu^4 - 28080\varepsilon H^3\mu^3 + 7776H^4\mu^2)Q_3(\mu)$ are polynomials in terms of μ of degree 7 and 9, respectively.
Next, by the definition of the L_1-operator from (9), L_1H_2 is locally given by

$$L_1(H_2) = -\langle P_1\nabla_{e_1}(\nabla H_2), e_1\rangle + \langle P_1\nabla_{e_2}(\nabla H_2), e_2\rangle + \langle P_1\nabla_{e_3}(\nabla H_2), e_3\rangle.$$

Since $\nabla H_2 = e_3(H_2)e_3$, from (13) and the relations (1)-(11), we find

$$L_1H_2 = 2(\mu - \lambda_3)\omega_{13}^1 e_3(H_2) - 2\nu\omega_{13}^2 e_3(H_2) - 2\mu e_3e_3(H_2).$$

Comparing this with the second equation of (11) gives

$$(3H\varepsilon-\mu)\omega_{13}^1 e_3(H_2)+\nu\omega_{13}^2 e_3(H_2)+\mu e_3e_3(H_2) + 96\varepsilon\mu^5-464H\mu^4+744\varepsilon H^2\mu^3-396H^3\mu^2=0. \tag{32}$$

Substituting (29), (30) and (31) into equation (32) yields

$$(\omega_{13}^1)^2=\frac{R_9(\mu)}{R_7(\mu)}, \tag{33}$$

where

$$R_9(\mu)=-2Q_9(\mu)-724416\varepsilon\mu^9+6549536H\mu^8-24992016\varepsilon H^2\mu^7+52130808H^3\mu^6 - 64151136\varepsilon H^4\mu^5+46552320H^5\mu^4-18444672\varepsilon H^6\mu^3+3079296H^7\mu^2,$$

and $R_7(\mu)=2Q_7(\mu)+(1540\mu^4-6612\varepsilon H\mu^3+9765H^2\mu^2-5616\varepsilon H^3\mu+972H^4)Q_3(\mu)$, are polynomials in terms of μ of degree 9 and 7, respectively.
Acting with e_3 on (33), we get

$$e_3((\omega_{13}^1)^2)=\frac{Q_{18}(\mu)}{(7\mu-6\varepsilon H)(R_7(\mu))^2}\omega_{13}^1, \tag{34}$$

where $Q_{18}(\mu)=\frac{1}{2}(20\mu^2-36H\varepsilon\mu+9H^2)(\frac{\partial R_7(\mu)}{\partial\mu}Q_9(\mu)-\frac{\partial R_9(\mu)}{\partial\mu}R_7(\mu))$, is a polynomial in terms of μ of degree 18.
On the other hand, by substituting (29) and (33) into (25), we have

$$e_3((\omega_{13}^1)^2)=\frac{Q_{13}(\mu)}{Q_{11}(\mu)}\omega_{13}^1, \tag{35}$$

where

$$Q_{13}(\mu)=(-2156\mu^6+10458\varepsilon H\mu^5-18468H^2\mu^4+14040\varepsilon H^3\mu^3-3888H^4\mu^2)R_7(\mu) + \frac{1}{2}(1672\mu^4-5376\varepsilon H\mu^3+4680H^2\mu^2-324\varepsilon H^3\mu-729H^4)R_9(\mu),$$

and $Q_{11}(\mu)=(11\mu^2-18H\varepsilon\mu)(7\mu-6\varepsilon H)^2R_7(\mu)$, are polynomials in terms of μ of degree 13 and 11, respectively.
Now, comparing (34) and (35) yields

$$Q_{18}(\mu)Q_{11}(\mu)-(7\mu-6\varepsilon H)Q_{13}(\mu)(R_7(\mu))^2=0,$$

or equivalently, we obtain a polynomial in terms of μ of degree 29 with constant coefficients. Therefore, we conclude that μ is constant, which leads to a contradiction.

S **is of type III.** In this case, P_2 takes the following form with respect to the local pseudo-orthonormal frame $\{e_1,e_2,e_3\}$

$$P_2=\begin{pmatrix}\lambda\lambda_3 & 0 & 0\\ -\lambda_3 & \lambda\lambda_3 & 0\\ 0 & 0 & \lambda^2\end{pmatrix}. \tag{36}$$

Using (6) and (8), we find

$$H_2 = \frac{\lambda^2 + 2\lambda\lambda_3}{3}. \tag{37}$$

By using the first equation (11) and the inductive definition of P_2, we get

$$P_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2. \tag{38}$$

On the other hand, since the frame $\{e_1, e_2, e_3\}$ is pseudo-orthonormal, the gradient of H_2 can be expressed as

$$\nabla H_2 = -e_2(H_2)e_1 - e_1(H_2)e_2 + e_3(H_2)e_3. \tag{39}$$

Combining (36), (38) and (39), we obtain

$$\begin{cases} -\lambda\lambda_3 e_2(H_2) = \frac{9}{2} H_2 e_2(H_2), \\ \lambda_3 e_2(H_2) - \lambda\lambda_3 e_1(H_2) = \frac{9}{2} H_2 e_1(H_2), \\ \lambda^2 e_3(H_2) = \frac{9}{2} H_2 e_3(H_2). \end{cases} \tag{40}$$

If $e_2(H_2) \neq 0$, the first equation in (40) gives $\lambda\lambda_3 = -\frac{9}{2} H_2$, then by the second equation of (40), we conclude that $\lambda_3 = 0$, which yields $H_2 = 0$ and this is a contradiction. So, $e_2(H_2) = 0$. Easily, we can verify that ∇H_2 is either in the direction of e_3 ($e_1(H_2) = 0$, and $e_3(H_2) \neq 0$), or in the direction of e_2 ($e_1(H_2) \neq 0$, and $e_3(H_2) = 0$). In the first case, the third equation in (40) gives $\lambda^2 = \frac{9}{2} H_2$. Combining this with (37) implies that $\lambda(\lambda + 6\lambda_3) = 0$, on $\mathcal{U}$. Hence, $\lambda = -6\lambda_3$ on $\mathcal{U}$. From this and H being constant, we conclude that λ is also a constant, therefore, H_2 is constant on $\mathcal{U}$ and this is a contradiction. In the second case, the second equation in (40) gives $\lambda\lambda_3 = -\frac{9}{2} H_2$. Combining this with (37) implies that $\lambda(8\lambda_3 + 3\lambda) = 0$, on $\mathcal{U}$, which leads to a contradiction.

S **is of type IV.** Since H is constant, (7) and (8) imply that H_2 is also constant on $\mathcal{U}$, which is a contradiction.

Therefore, $\mathcal{U}$ is empty, and we conclude that H_2 is constant. If H_2 is nonzero, the second equation in (11) implies that $3HH_2^2 = H_3$, since by assumption, H is constant, thus, we get H_3 is also constant. Hence, all of the mean curvatures H_is are constant functions, which means that M is isoparametric. Corollary 3.1 shows that it does not occur. So, $H_2 \equiv 0$. This finishes the proof. □

3.2. **L_2-biharmonic hypersurfaces.** The case $k = 2$ of Theorem 1.2 is proved in the following proposition. Note that the Cayley-Hamilton theorem makes the proof of Proposition 3.3 very much easier than that of Proposition 3.2.

Proposition 3.3. *Let $\psi : M \longrightarrow \mathbb{L}^4$ be an orientable L_2-biharmonic hypersurface immersed into Lorentz-Minkowski space $\mathbb{L}^4$ with constant 2-th mean curvature. Then M is 2-maximal.*

Proof. First, we prove that H_3 is constant by showing that the open set $\mathcal{U} = \{p \in M : \nabla H_3^2(p) \neq 0\}$ is empty. By the Cayley-Hamilton theorem, we have $P_3 = 0$, so

$$(S \circ P_2)\nabla H_3 = \varepsilon H_3 \nabla H_3,$$

which jointly with the first equation in (12) yields ∇H_3^2 vanishes identically on $\mathcal{U}$, which is a contradiction. Thus, H_3 is constant. If H_3 is nonzero, then the second equation in (12) implies that $H \equiv 0$. Therefore, all of the mean curvatures H_i are constant functions, which is equivalent to say that M is isoparametric. Corollary 3.1 shows that this is impossible. So, $H_3 = 0$. This finishes the proof. □

Now, Propositions 3.2 and 3.3 finished the proof of Theorem 1.2 stated in the introduction.

Remark 3.4. In the proof of Theorem 1.2, some calculations have been done with Maple.

References

[1] M. Aminian, S. M. B. Kashani, *L_k-Biharmonic Hypersurfaces in Euclidean Space*, Taiwan J Math., 19., no 3 (2015) 861-874.

[2] M. Aminian, S. M. B. Kashani, *L_k-Biharmonic Hypersurfaces in Space Form*, Acta Math. Vietnam., 42., no 3 (2017) 471-490.

[3] A. Arvanitoyeorgos, F. Defever, G. Kaimakamis and V. J. Papantoniou, *Biharmonic Lorentz Hypersurfaces in E_1^4*, Pacific J. Math., 229., no 2 (2007) 293-305.

[4] B. Y. Chen, *Some Open Problems and Conjetures on Submanifolds of Finite Type*, Soochow J. Math., 17., (1991) 169-188.

[5] B. Y. Chen, S. Ishikawa, *Biharmonic Surfaces in Pseudo-Euclidean Spaces*, Mem. Fac. Sci. Kyushu Univ. ser A., 45., (1991) 323-347.

[6] Deepika., R. S. Gupta, *On Biharmonic Lorentz Hypersurfaces with non-Diagonal Shape Operator*, Int. Electron. J. Geom., 10., no 1, (2017) 96-111.

[7] Deepika., R. S. Gupta and A. Sharfuddin, *Biharmonic Hypersurfaces with Constant Scalar Curvature in E_s^5*, Kyungpook Math. J., 56., no 1 (2016) 273-293.

[8] F. Defever, G. Kaimakamis, and V. Papantoniou, *Biharmonic Hypersurfaces of the 4-Dimensional Semi-Euclidean Space E_s^4*, J. Math. Anal. Appl., 315., 1 (2006) 276-286.

[9] R. S. Gupta, *Biharmonic Hypersurfaces in E_s^5*, An. St. Univ. Al. I. Cuza., 2., no 2 (2016) 585-593.

[10] J. Hahn, *Isoparametric Hypersurfaces in the Pseudo-Riemannian Space Forms*, Math. Z., 187., (1984) 195-208.

[11] G. Y. Jiang, *2-Harmonic Maps and Their First and Second Variational Formulas*, Chin. Ann. Math. Ser A., 7., (1986) 389-402.

[12] J. Liu, L. Du, *Classification of Proper Biharmonic Hypersurfaces in Pseudo-Riemannian Space Forms*, Differ. Geom. Appl., 41,. (2015) 110-122.

[13] P. Lucas, H.F. Ramı́rez-Ospina, *Hypersurfaces in Lorentz-Minkowski Space Satisfying $L_k\psi = A\psi + b$*, Geom. Dedicata., 153., no 1 (2011) 151-175.

[14] P. Lucas, H.F. Ramírez-Ospina, *Hypersurfaces in Pseudo-Euclidean Spaces Satisfying a Linear Condition on the Linearized Operator of a Higher Order Mean Curvature*, Differ. Geom. Appl., 31.,no 2 (2013) 175-1.89.

[15] M. A. Magid, *Isometric Immersions of Lorentz Space with Parallel Second Fundamental Forms*, Tsukuba J. Math., 8., no 1 (1984) 31-54.

[16] A. Mohammadpouri, *Hypersurfaces with L_r-Pointwise* 1*-Type Gauss Map*, Math. Phys. Anal. Geom., 14., no 1 (2018) 67-77.

[17] A. Mohammadpouri, F. Pashaiei, *On the Classification of Hypersurfaces in Euclidean Spaces Satisfying $L_r \overrightarrow{\mathbb{H}}_{r+1} = \lambda \overrightarrow{\mathbb{H}}_{r+1}$*, Proyecciones J. Math., 35., no 1 (2016) 1-10.

[18] A. Mohammadpouri, F. Pashaie, *L_r-Biharmonic Hypersurfaces in $\mathbb{E}^4$*, Bol. Soc. Paran. Mat., 38., no 5 (2020) 9-18.

[19] A. Mohammadpouri, F. Pashaie, and S. Tajbakhs, *L_1-Biharmonic Hypersurfaces in Euclidean Spaces with Three Distinct Principal Curvatures*, Iran. J. Math. Sci. Inform., 13., no 2 (2018) 49-70.

[20] V. J. Papantoniou, K. Petoumenos, *Biharmonic Hypersurfaces of Type M_2^3 in E_2^4*, Houston J. Math., 38., no 1 (2012) 93-114.

[21] F. Pashaie, A. Mohammadpouri, *L_k-Biharmonic Spacelike Hypersurfaces in Minkowski* 4*-Space $\mathbb{E}_1^4$*, Sahand commun. math. anal., 5., no 1 (2017) 21-30.

[22] R. C. Reilly, *Variational Properties of Functions of the Mean Curvatures for Hypersurfaces in Space Forms*, J. Differential Geom., 8., no 3 (1973) 465-477.

[23] M. Shams. Solary, *Eigenvalues for Tridiagonal 3-Toeplitz Matrices*, J. Mahani math. res. cent., 10.,no 2 (2021) 63-72.

[24] H. Urakawa, *Geometry of Biharmonic Mappings: Differential Geometry of Variational Methods*, World Scientific pub Co Inc., Singapore, 2018.

Rahim Hosseinoughli
Orcid number: 0000-0003-1829-2411
Faculty of Mathematics, Statistics and Computer Sciences
University of Tabriz
Tabriz, Iran
Email address: r.hoseinoghli@tabrizu.ac.ir

Akram Mohammadpouri
Orcid number: 0000-0002-8546-6445
Faculty of Mathematics, Statistics and Computer Sciences
University of Tabriz
Tabriz, Iran
Email address: pouri@tabrizu.ac.ir