Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

TRUNCATED LOMAX-EXPONENTIAL DISTRIBUTION AND ITS FITTING TO FINANCIAL DATA

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Article type: Research Article
(Received: 09 May 2022, Received in revised form 28 August 2022)
(Accepted: 27 September 2022, Published Online: 27 September 2022)

Abstract. Nowadays, analyzing the losses data of the insurance and asset portfolios have special importance in risk analysis and economic problems. Therefore, having suitable distributions that are able to fit such data, is important. In this paper, a new distribution with decreasing failure rate function is introduced. Then, some important and applicable statistical indices in Insurance and Economics such as the moments and moment generating function, value at risk, tail value at risk, tail variance, and Shannon and Rényi entropies are obtained. One of the advantages of this distribution is that it has fewer parameters compared to other distributions that have been introduced so far. Finally, this distribution is utilized as a proper distribution to fit on a real data set.

Keywords: Exponential distribution, Mean residual life, Tail Value-at-Risk, Tail variance, Value-at-Risk.
2020 MSC: 91G80.

1. Introduction

Finding an appropriate distribution to describe the data is an important problem in statistics. Hence, the purpose of researchers is to introduce the distributions which are fitted to the characteristics of existing data. Many researchers proposed different generators to build new distributions by changing the baseline distributions. However, few of these distributions are based on truncated distributions. Barreto-Souza and Simas [3] introduced a new family of distributions with the truncated baseline distributions and called exp-$\boldsymbol{G}$ family. Rady et al. [13] proposed a new extension of the Lomax distribution by using power transformation named Power Lomax distribution (POLO). Golzar et al. [6] intruduced the Lomax-exponential distribution as an extension of exponential distribution which is a good model for skewed to right data. Meshkat et al. [12] and Mahdavi and Oliveira Silva [11] proposed a new generator to extend distributions by which Meshkat et al. introduced the Gamma-Weibull distribution and Mahdavi and Oliveira Silva provided the truncated exponential-exponential distribution. Ijaz and Asim [9] presented Lomax exponential distribution (LE) as a more flexible modification of the Lomax distribution to model

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DOI: 10.22103/jmmr.2022.19475.1257

Publisher: Shahid Bahonar University of Kerman
How to cite: Sh. Enami, *Truncated Lomax-exponential Distribution and its Fitting to Financial Data*, J. Mahani Math. Res. 2023; 12(2): 201-216.

the non-monotonic data sets in reliability theory. Hassan et al. [8] introduced a new truncated family as the right truncated power Lomax-G family and studied some structural properties of this family. The slashed Lomax distribution is proposed as an asymmetric distribution for fitting thick-tailed datasets by Li and Tian [10]. By using Weibull-G family, Hussain et al. [7] provided an extension of the Power Lomax distribution called the Weibull-Power Lomax distribution and discussed its statistical properties. A new modification of the inverse Lomax distribution is introduced by Ahmadini et al. [2] as the truncated Lomax inverse Lomax distribution which is more flexible to model the lifetime data sets.

Meshkat et al. [12] and Mahdavi and Oliveira Silva [11] idea is explained as follows. Let $F(.)$ and $G(.)$ be two cumulative distribution functions (CDF), then the truncated $F - G$ family CDF is given by

$$H(x) = \frac{F(G(x)) - F(0)}{F(1) - F(0)}, \tag{1}$$

with the corresponding probability density function (PDF) and hazard function, respectively, as follows

$$\begin{aligned} h(x) &= \frac{g(x)f(G(x))}{F(1) - F(0)}, \\ r(x) &= \frac{g(x)f(G(x))}{F(1) - F(G(x))}. \end{aligned}$$

The transformed distributions are used widely in economics, insurance and finance. Since the insurance and finance data are right-skewed or reverse "J"shaped, a new reverse "J"shaped distribution is introduced using Meshkat et al. [12] and Mahdavi and Oliveira Silva [11] idea in this paper. Then, some important statistical properties of this distribution like the moments, hazard function, moment generating function and Shannon and Rényi entropies and economics indices of this distribution like value at risk, tail value at risk, tail variance and mean residual lifetime are discussed. Finally, this distribution is utilized to fit the cost data set of power generation at various US power plants. Research shows that the proposed model has better fits compared to other models for this data.

2. Truncated Lomax-exponential distribution

In this section, a new distribution is introduced based on relation (1). Let $F(.)$ be the Lomax distribution function with α and λ parameters as follows

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \qquad x, \alpha, \lambda > 0,$$

and $G(.)$ be the exponential distribution function with mean σ as follows

$$G(x) = 1 - e^{-x/\sigma}, \qquad x, \sigma > 0.$$

By substituting $F(.)$ and $G(.)$ in relation (1), the truncated Lomax-exponential distribution is

$$H(x) = \frac{1-\lambda^{\alpha}\left(\lambda+1-e^{-x/\sigma}\right)^{-\alpha}}{1-\lambda^{\alpha}(\lambda+1)^{-\alpha}}, \qquad x,\alpha,\lambda,\sigma>0.$$

If X has the $H(.)$ distribution function, then X follows the truncated Lomax-exponential distribution with parameters α, λ and σ denoted by $X \sim TLE(\alpha,\lambda,\sigma)$. The PDF and hazard function of this distribution is, respectively,

$$\begin{aligned} h(x) &= \frac{\alpha\lambda^{\alpha}e^{-x/\sigma}\left(\lambda+1-e^{-x/\sigma}\right)^{-(\alpha+1)}}{\sigma\left(1-\lambda^{\alpha}(\lambda+1)^{-\alpha}\right)}, \qquad x,\alpha,\lambda,\sigma>0,\\ r(x) &= \frac{\alpha e^{-x/\sigma}\left(\lambda+1-e^{-x/\sigma}\right)^{-(\alpha+1)}}{\sigma\left[\left(\lambda+1-e^{-x/\sigma}\right)^{-\alpha}-(\lambda+1)^{-\alpha}\right]}, \qquad x,\alpha,\lambda,\sigma>0. \end{aligned}$$

3. Properties and Indices

In this section, some important and applicable statistical indices in Insurance and Economics such as the moments and moment generating function, value at risk, tail value at risk, tail variance, and Shannon and Rényi entropies are discussed.

3.1. Shape of PDF and hazard function.

In this subsection, the shape of PDF $TLE(\alpha,\lambda,\sigma)$ is studied. Obviously, the function

$$k(x) = x\left(\lambda+1-x\right)^{-(\alpha+1)},$$

is increasing in $0<x<1$ for positive values of α and λ. Therefore, by combining $k(x)$ and $e^{-x/\sigma}$ and multiplying to an appropriate positive coefficient, the function $h(x)$ become decreasing in $x>0$. Consequently, the Mode is equal to 0.

To discuss the shape of hazard function of $TLE(\alpha,\lambda,\sigma)$, it can be easily shown that the function

$$k(x) = \frac{x(\lambda+1-x)^{-(\alpha+1)}}{(\lambda+1-x)^{-\alpha}-(\lambda+1)^{-\alpha}},$$

is increasing in $0<x<1$ for positive values of α and λ. Therefore, by combining $k(x)$ and $e^{-x/\sigma}$ and multiplying to an appropriate positive coefficient, the function $r(x)$ become decreasing in $x>0$. Consequently, the TLE distribution has the DFR property.

Now, the hazard function is studied for large values of x as follows,

$$\begin{aligned}
\lim_{x\to\infty} r(x) &= \lim_{x\to\infty} \frac{\alpha e^{-x/\sigma}\left(\lambda+1-e^{-x/\sigma}\right)^{-(\alpha+1)}}{\sigma\left[\left(\lambda+1-e^{-x/\sigma}\right)^{-\alpha}-(\lambda+1)^{-\alpha}\right]} \\
&= \frac{\alpha}{\sigma}\lim_{x\to\infty}\left(\lambda+1-e^{-x/\sigma}\right)^{-(\alpha+1)} \\
&\qquad \times \lim_{x\to\infty}\frac{e^{-x/\sigma}}{\left[\left(\lambda+1-e^{-x/\sigma}\right)^{-\alpha}-(\lambda+1)^{-\alpha}\right]} \\
&= \frac{\alpha}{\sigma}(\lambda+1)^{-(\alpha+1)}\lim_{x\to\infty}\frac{\frac{-1}{\sigma}e^{-x/\sigma}}{\alpha\left(\lambda+1-e^{-x/\sigma}\right)^{-(\alpha+1)}\frac{1}{\sigma}e^{-x/\sigma}} \\
&= \frac{(\lambda+1)^{-(\alpha+1)}}{\sigma}\lim_{x\to\infty}\frac{1}{\left(\lambda+1-e^{-x/\sigma}\right)^{-(\alpha+1)}} \\
&= \frac{(\lambda+1)^{-(\alpha+1)}}{\sigma}\frac{1}{(\lambda+1)^{-(\alpha+1)}} = \frac{1}{\sigma}.
\end{aligned}$$

Thus for large values of x, the function $r(x)$ does not depend on α and λ.

For some values of the parameters, the plots of the PDF and the hazard function are shown in Figure 1.

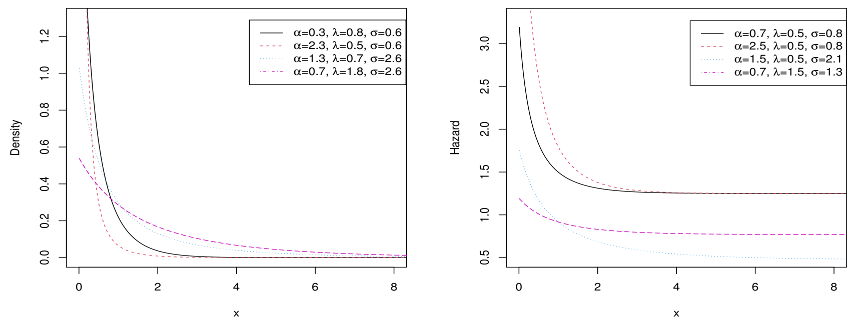

FIGURE 1. Plots of the PDF and hazard function of the $TLE(\alpha,\lambda,\sigma)$ distribution for various values of parameters.

3.2. **Moments and Moment generating function.** In this subsection, the k-th non-central moments and moment generating function are investigated.

The k-th non-central moment can be written as

$$
\begin{aligned}
E(X^k) &= \int_0^\infty x^k h(x)dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma\left(1-\lambda^\alpha(\lambda+1)^{-\alpha}\right)}\int_0^\infty x^k e^{-x/\sigma}\left(\lambda+1-e^{-x/\sigma}\right)^{-(\alpha+1)}dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma\left(1-\lambda^\alpha(\lambda+1)^{-\alpha}\right)(\lambda+1)^{\alpha+1}}\int_0^\infty x^k e^{-x/\sigma}\left(1-\frac{e^{-x/\sigma}}{\lambda+1}\right)^{-(\alpha+1)}dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)}\int_0^\infty x^k e^{-x/\sigma}\left(1-\frac{e^{-x/\sigma}}{\lambda+1}\right)^{-(\alpha+1)}dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)} \\
&\quad\times\int_0^\infty x^k e^{-x/\sigma}\left[1+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(-\alpha-1-i+1)}{s!}\left(-\frac{e^{-x/\sigma}}{\lambda+1}\right)^s\right]dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)} \\
&\quad\times\left[\int_0^\infty x^k e^{-x/\sigma}dx+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^s}\int_0^\infty x^k e^{-(s+1)x/\sigma}dx\right] \\
&= \frac{\alpha\lambda^\alpha k!\sigma^{k+1}}{\sigma(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)}\left[1+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^s(s+1)^{k+1}}\right] \\
&= \frac{\alpha\lambda^\alpha k!\sigma^{k}}{(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)}\left[1+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^s(s+1)^{k+1}}\right],
\end{aligned}
$$

where the fifth equality is obtained by substituting $t = -\dfrac{e^{-x/\sigma}}{\lambda+1}$ and $a = -(\alpha+1)$ in the expansion

$$
(2)\qquad (1+t)^a = 1+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(a-i+1)}{s!}t^s,\qquad |t|<1, a\in(-\infty,\infty).
$$

Similarly, the moment generating function is obtained as

$$
M_X(t) = \frac{\alpha\lambda^\alpha}{(1+\lambda)\left((\lambda+1)^\alpha-\lambda^\alpha\right)}\left[\frac{1}{1-\sigma t}+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^s(s+1-\sigma t)}\right],\ t<\frac{1}{\sigma}.
$$

3.3. **Value at risk.** Value at risk represents the maximum loss on the asset portfolio or the maximum amount of damage to an insurance company over a specified period of time and in normal conditions under a certain level of confidence. If the level of confidence is p $(0 < p < 1)$, the corresponding value at risk to the random loss X with CDF $H_X(.)$ is defined as

$$VaR_X(p) = H_X^{-1}(p) = \inf\{x : H_X(x) \geq p\}.$$

That is, the maximum financial loss is equal to $VaR_X(p)$ with probability p. It can be easily investigated that if $X \sim TLE(\alpha, \lambda, \sigma)$, then

$$VaR_X(p) = -\sigma \log\left(1 - \lambda((1 - p + p\lambda^\alpha(\lambda+1)^{-\alpha})^{-1/\alpha} - 1)\right).$$

If m denotes the median of random variable X, then

$$m = VaR_X(0.5) = -\sigma \log\left(1 - \lambda((0.5 + 0.5\lambda^\alpha(\lambda+1)^{-\alpha})^{-1/\alpha} - 1)\right).$$

In addition, $X \sim TLE(\alpha, \lambda, \sigma)$ if U is a random variable from a uniform distribution $(0, 1)$ and random variable X is

$$X = -\sigma \log\left[1 - \lambda((1 - U + U\lambda^\alpha(\lambda+1)^{-\alpha})^{-1/\alpha} - 1)\right].$$

This follows because

$$\begin{aligned}
& U = \frac{1 - \lambda^\alpha\left(\lambda + 1 - e^{-X/\sigma}\right)^{-\alpha}}{1 - \lambda^\alpha(\lambda+1)^{-\alpha}} \\
\Rightarrow\quad & \frac{[1 - U(1 - \lambda^\alpha(\lambda+1)^{-\alpha})]^{-\frac{1}{\alpha}}}{(\lambda^\alpha)^{-\frac{1}{\alpha}}} = \lambda + 1 - e^{-X/\sigma} \\
\Rightarrow\quad & e^{-X/\sigma} = \lambda + 1 - \lambda\left[1 - U(1 - \lambda^\alpha(\lambda+1)^{-\alpha})\right]^{-\frac{1}{\alpha}} \\
\Rightarrow\quad & X = -\sigma \log\left(1 - \lambda((1 - U + U\lambda^\alpha(\lambda+1)^{-\alpha})^{-1/\alpha} - 1)\right).
\end{aligned}$$

3.4. **Tail value at risk and tail variance.** In the previous subsection, it is observed that an investor or an insurance company is sure with a $(100 \times p)\%$ confidence that it will not incur a loss greater than $VaR_X(p)$. Therefore, an investor or insurance company may encounter more losses than expected $VaR_X(p)$. So to prevent bankruptcy, it is important to know the amount of loss that exceeds the expectation. To find the average loss that is more than $VaR_X(p)$, the criterion of value at risk is used, which is defined as follows

$$TVaR_X(p) = E[X | X > VaR_X(p)].$$

If $X \sim TLE(\alpha, \lambda, \sigma)$, then

$$\begin{aligned}
TVaR_X(p) &= \frac{1}{1-p}\int_{VaR_X(p)}^{\infty} x h(x) dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma(1-p)\left(1 - \lambda^\alpha(\lambda+1)^{-\alpha}\right)} \\
&\quad \times \int_{VaR_X(p)}^{\infty} x e^{-x/\sigma}\left(\lambda + 1 - e^{-x/\sigma}\right)^{-(\alpha+1)} dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha\lambda^{\alpha}}{\sigma(1-p)\left(1-\lambda^{\alpha}(\lambda+1)^{-\alpha}\right)(\lambda+1)^{\alpha+1}} \\
&\quad \times \int_{VaR_X(p)}^{\infty} xe^{-x/\sigma}\left(1-\frac{e^{-x/\sigma}}{\lambda+1}\right)^{-(\alpha+1)} dx \\
&= \frac{\alpha\lambda^{\alpha}}{\sigma(1-p)(\lambda+1)\left((\lambda+1)^{\alpha}-\lambda^{\alpha}\right)} \\
&\quad \times \int_{VaR_X(p)}^{\infty} xe^{-x/\sigma}\left(1-\frac{e^{-x/\sigma}}{\lambda+1}\right)^{-(\alpha+1)} dx \\
&= \frac{\alpha\lambda^{\alpha}}{\sigma(1-p)(\lambda+1)\left((\lambda+1)^{\alpha}-\lambda^{\alpha}\right)} \\
&\quad \times \int_{VaR_X(p)}^{\infty} xe^{-x/\sigma}\left[1+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(-\alpha-1-i+1)}{s!}\left(-\frac{e^{-x/\sigma}}{\lambda+1}\right)^{s}\right] dx \\
&= \frac{\alpha\lambda^{\alpha}}{\sigma(1-p)(\lambda+1)\left((\lambda+1)^{\alpha}-\lambda^{\alpha}\right)}\left[\int_{VaR_X(p)}^{\infty} xe^{-x/\sigma}dx+\right. \\
&\quad \left.\times\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^{s}}\int_{VaR_X(p)}^{\infty} xe^{-(s+1)x/\sigma}dx\right] \\
&= \frac{\alpha\lambda^{\alpha}\sigma^{2}}{\sigma(1-p)(\lambda+1)\left((\lambda+1)^{\alpha}-\lambda^{\alpha}\right)}\left[\int_{\frac{1}{\sigma}VaR_X(p)}^{\infty} xe^{-x}dx+\right. \\
&\quad \left.\times\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^{s}(s+1)^{2}}\int_{\frac{s+1}{\sigma}VaR_X(p)}^{\infty} xe^{-x}dx\right] \\
&= \frac{\alpha\lambda^{\alpha}\sigma}{(1-p)(\lambda+1)\left((\lambda+1)^{\alpha}-\lambda^{\alpha}\right)}\left[\Gamma(2,\frac{1}{\sigma}VaR_X(p))+\right. \\
&\quad \left.\times\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^{s}(s+1)^{2}}\Gamma(2,\frac{s+1}{\sigma}VaR_X(p))\right], \qquad (3)
\end{aligned}
$$

where $\Gamma(a,t)=\int_t^{\infty} x^{a-1}e^{-x}dx$ and the fifth equality is obtained by substituting $t=-\dfrac{e^{-x/\sigma}}{\lambda+1}$ and $a=-(\alpha+1)$ in relation (2).

As mentioned, knowing the amount of loss that exceeds $VaR_X(p)$ is important to prevent bankruptcy. So, the variance of this loss is known as tail variance and is as follows

$$TV_X(p)=E[X^2|X>VaR_X(p)]-(TVaR_X(p))^2, \qquad (4)$$

If $X \sim TLE(\alpha, \lambda, \sigma)$, then

$$
\begin{aligned}
E[X^2 \mid X > VaR_X(p)] &= \frac{1}{1-p}\int_{VaR_X(p)}^{\infty} x^2 h(x)dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma(1-p)\left(1-\lambda^\alpha(\lambda+1)^{-\alpha}\right)}\int_{VaR_X(p)}^{\infty} x^2 e^{-x/\sigma}\left(\lambda+1-e^{-x/\sigma}\right)^{-(\alpha+1)}dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma(1-p)\left(1-\lambda^\alpha(\lambda+1)^{-\alpha}\right)(\lambda+1)^{\alpha+1}} \\
&\quad \times \int_{VaR_X(p)}^{\infty} x^2 e^{-x/\sigma}\left(1-\frac{e^{-x/\sigma}}{\lambda+1}\right)^{-(\alpha+1)}dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma(1-p)(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)}\int_{VaR_X(p)}^{\infty} x^2 e^{-x/\sigma}\left(1-\frac{e^{-x/\sigma}}{\lambda+1}\right)^{-(\alpha+1)}dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma(1-p)(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)} \\
&\quad \times \int_{VaR_X(p)}^{\infty} x^2 e^{-x/\sigma}\left[1+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(-\alpha-1-i+1)}{s!}\left(-\frac{e^{-x/\sigma}}{\lambda+1}\right)^s\right]dx \\
&= \frac{\alpha\lambda^\alpha}{\sigma(1-p)(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)} \\
&\quad \times \left[\int_{VaR_X(p)}^{\infty} x^2 e^{-x/\sigma}dx+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^s}\int_{VaR_X(p)}^{\infty} x^2 e^{-(s+1)x/\sigma}dx\right] \\
&= \frac{\alpha\lambda^\alpha\sigma^3}{\sigma(1-p)(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)} \\
&\quad \times \left[\int_{\frac{1}{\sigma}VaR_X(p)}^{\infty} x^2 e^{-x}dx+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^s(s+1)^2}\int_{\frac{s+1}{\sigma}VaR_X(p)}^{\infty} x^2 e^{-x}dx\right] \\
&= \frac{\alpha\lambda^\alpha\sigma^2}{(1-p)(\lambda+1)\left((\lambda+1)^\alpha-\lambda^\alpha\right)} \\
&\quad \times \left[\Gamma(3,\frac{1}{\sigma}VaR_X(p))+\sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\alpha+i)}{s!(\lambda+1)^s(s+1)^2}\Gamma(3,\frac{s+1}{\sigma}VaR_X(p))\right]. \qquad (5)
\end{aligned}
$$

Thus, tail variance is obtained by substituting the relations (3) and (5) in the relation (4).

3.5. **Mean residual lifetime.** In reliability, mean residual lifetime attracts wide attention in practice. Let X be the lifetime of a system that works at time x. So, the mean residual lifetime of the system at time x is defined as $\mu(x) = E[X - x|X > x]$. This concept is also of especial importance in insurance science. To prevent bankruptcy, the insurance companies insure themselves with larger companies. This type of insurance and the larger company are called reinsurance and a reinsurer, respectively. Let X be the amount of random loss to an insurance company. In this type of insurance, the reinsurer undertakes to pay the difference, $X - x$, if the amount of damage to the insurance company exceeds a certain threshold of x. Therefore, the average loss to the reinsurer will be $E[X - x|X > x]$, which is the mean residual lifetime. If $X \sim TLE(\alpha, \lambda, \sigma)$, then the mean residual lifetime is given as

$$\begin{aligned}
\mu(x) &= E[X - x|X > x] \\
&= \frac{1}{1 - H(x)} \int_x^\infty (1 - H(t))dt \\
&= \frac{(\lambda+1)^{-\alpha}}{(\lambda+1)^{-\alpha} - \left(\lambda+1-e^{-\frac{x}{\sigma}}\right)^{-\alpha}} \int_x^\infty \left(1 - \left(1 - \frac{e^{-\frac{t}{\sigma}}}{\lambda+1}\right)^{-\alpha}\right) dt \\
&= \frac{(\lambda+1)^{-\alpha}}{\left(\lambda+1-e^{-\frac{x}{\sigma}}\right)^{-\alpha} - (\lambda+1)^{-\alpha}} \int_x^\infty \sum_{s=1}^{\infty} \frac{\prod_{i=1}^{s}(-\alpha - i + 1)}{s!} \left(-\frac{e^{-t/\sigma}}{\lambda+1}\right)^s dt \\
&= \frac{(\lambda+1)^{-\alpha}\sigma}{\left(\lambda+1-e^{-\frac{x}{\sigma}}\right)^{-\alpha} - (\lambda+1)^{-\alpha}} \sum_{s=1}^{\infty} \frac{\prod_{i=1}^{s}(\alpha + i - 1)}{s!} \int_x^\infty \left(\frac{e^{-t/\sigma}}{\lambda+1}\right)^s dt \\
&= \frac{(\lambda+1)^{-\alpha}\sigma}{\left(\lambda+1-e^{-\frac{x}{\sigma}}\right)^{-\alpha} - (\lambda+1)^{-\alpha}} \sum_{s=1}^{\infty} \frac{\prod_{i=1}^{s}(\alpha + i - 1)}{s^2(s-1)!(\lambda+1)^s} e^{-sx/\sigma},
\end{aligned}$$

where the fourth equality is resulted by using the relation (2). Since the truncated Lomax-exponential distribution has the DFR property (Ghai and Mi [5]), $\mu(x)$ is an increasing function of x.

3.6. **Entropy.** Entropy is applied to measure the variability and uncertainty of a random variable, which is widely used in reliability theory. Shannon entropy (Shannon [15]) and Rényi entropy (Rényi [14]) are the most important types of entropy obtained in this paper for the truncated Lomax-exponential distribution. If X is a random variable with the density function $h(.)$, then Shannon entropy for X is defined as $\eta_X = E[\log h(X)]$, and Rényi entropy for X is defined as $R_X(\rho) = \frac{1}{1-\rho} \log(\int h(x)^\rho dx)$ for $\rho > 0$ and $\rho \neq 1$. If

$X \sim TLE(\alpha, \lambda, \sigma)$, then Shannon entropy for X is given as

$$\begin{aligned}
\eta_X &= E[\log h(X)] \\
&= \sigma - \log\alpha - \alpha\log\lambda + \log\left(1 - \lambda^{\alpha}(\lambda+1)^{-\alpha}\right) \\
&\quad + \frac{1}{\sigma}E[X] + (\alpha-1)E\left[\log\left(\lambda + 1 - e^{-X/\sigma}\right)\right] \\
&= \sigma - \log\alpha - \alpha\log\lambda + \log\left(1 - \lambda^{\alpha}(\lambda+1)^{-\alpha}\right) + \frac{1}{\sigma}E[X] \\
&\quad \times + (\alpha-1)\log(\lambda+1) + (\alpha-1)E\left[\log\left(1 - \frac{e^{-X/\sigma}}{\lambda+1}\right)\right] \\
&= \sigma - \log\alpha - \alpha\log\lambda + \log\left(1 - \lambda^{\alpha}(\lambda+1)^{-\alpha}\right) + \frac{1}{\sigma}E[X] \\
&\quad \times + (\alpha-1)\log(\lambda+1) - (\alpha-1)E\left[\sum_{s=1}^{\infty}\frac{e^{-sX/\sigma}}{s(\lambda+1)^s}\right] \\
&= \sigma - \log\alpha - \alpha\log\lambda + \log\left(1 - \lambda^{\alpha}(\lambda+1)^{-\alpha}\right) + \frac{1}{\sigma}E[X] \\
&\quad \times + (\alpha-1)\log(\lambda+1) - (\alpha-1)E\left[\sum_{s=1}^{\infty}\frac{M_X\left(-\frac{s}{\sigma}\right)}{s(\lambda+1)^s}\right],
\end{aligned}$$

where the fourth equality is obtained by substituting $t = \dfrac{e^{-\frac{X}{\sigma}}}{\lambda+1}$ in the expansion

$$\log(1-t) = -\sum_{s=1}^{\infty}\frac{t^s}{s}, \qquad |t| < 1.$$

Similarly, by using the relation (2) can be shown that Rényi entropy for X is as follows

$$\begin{aligned}
R_X(\rho) &= \frac{1}{1-\rho}\Bigg[\log\alpha + \rho\alpha\log\lambda - (\rho-1)\log\sigma - \rho(\alpha+1)\log(\lambda+1) \\
&\quad -\rho\log\left(1 - \lambda^{\alpha}(\lambda+1)^{-\alpha}\right) + \log\left(\frac{1}{\rho} + \sum_{s=1}^{\infty}\frac{\prod_{i=1}^{s}(\rho\alpha+\rho+i-1)}{(\lambda+1)^s s!(s+\rho)}\right)\Bigg].
\end{aligned}$$

4. **Parameter estimation**

In this section, the maximum likelihood estimators (MLEs) of three parameters LTE distribution are discussed. Let $x_1, \ldots, x_n$ be the observed values from the truncated Lomax-exponential distribution, then the log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (\alpha, \lambda, \sigma)^T$ is given by

$$\begin{aligned}
\ell(\boldsymbol{\theta}) &= n\log\alpha - n\log\sigma - n\log\left(\lambda^{-\alpha} - (\lambda+1)^{-\alpha}\right) \\
&\quad -\frac{1}{\sigma}\sum_{i=1}^{n}x_i - (\alpha+1)\sum_{i=1}^{n}\log\left(\lambda + 1 - e^{-x_i/\sigma}\right).
\end{aligned}$$

Also, the components of the score vector $s(\boldsymbol{\theta}) = \left(\frac{\partial \ell(\boldsymbol{\theta})}{\partial \alpha}, \frac{\partial \ell(\boldsymbol{\theta})}{\partial \lambda}, \frac{\partial \ell(\boldsymbol{\theta})}{\partial \sigma}\right)^T$ are obtained as

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \alpha} &= \frac{n}{\alpha} + n\frac{\lambda^{-\alpha}\log\lambda - (\lambda+1)^{-\alpha}\log(\lambda+1)}{\lambda^{-\alpha} - (\lambda+1)^{-\alpha}} - \sum_{i=1}^{n} \log\left(\lambda + 1 - e^{-x_i/\sigma}\right), \\
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \lambda} &= n\alpha\frac{\lambda^{-(\alpha+1)} - (\lambda+1)^{-(\alpha+1)}}{\lambda^{-\alpha} - (\lambda+1)^{-\alpha}} - (\alpha+1)\sum_{i=1}^{n} \frac{1}{\lambda + 1 - e^{-x_i/\sigma}}, \\
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i + \frac{\alpha+1}{\sigma^2}\sum_{i=1}^{n} \frac{e^{-x_i/\sigma}}{\lambda + 1 - e^{-x_i/\sigma}}.
\end{aligned}$$

By solving the nonlinear system $s(\boldsymbol{\theta}) = \mathbf{0}$, the maximum likelihood estimation (MLE) of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}$, is obtained.

The components of Fisher information matrix is provided by computed the negative second partial derivatives of $\ell(\boldsymbol{\theta})$ as follows

$$I(\boldsymbol{\theta}) = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}, \tag{6}$$

where

$$\begin{aligned}
I_{11} &= -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha^2} = -\frac{n\left(\alpha^2\lambda^\alpha(\lambda+1)^\alpha(\log(\lambda) - \log(\lambda+1))^2 - (\lambda^\alpha - (\lambda+1)^\alpha)^2\right)}{\alpha^2\left(\lambda^\alpha - (\lambda+1)^\alpha\right)^2}, \\
I_{22} &= -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda^2} = -\frac{\alpha^2 n\left(\lambda^{\alpha+1} - (\lambda+1)^{\alpha+1}\right)^2}{\left(\lambda^2+\lambda\right)^2\left(\lambda^\alpha - (\lambda+1)^\alpha\right)^2} - \frac{\alpha(\alpha+1)n\left((\lambda+1)^{\alpha+2} - \lambda^{\alpha+2}\right)}{\left(\lambda^2+\lambda\right)^2\left(\lambda^\alpha - (\lambda+1)^\alpha\right)} \\
&\quad - \frac{\alpha+1}{\left(\lambda - e^{-\frac{x}{\sigma}} + 1\right)^2},
\end{aligned}$$

$$\begin{aligned}
I_{33} &= -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha^2} = -\frac{n\sigma - 2x}{\sigma^3} + \frac{(\alpha+1)(x-2\sigma)}{\sigma^4(\lambda+1)e^{x/\sigma} - 1} + \frac{(\alpha+1)x}{\sigma^4\left((\lambda+1)e^{x/\sigma} - 1\right)^2}, \\
I_{12} &= I_{21} = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda \partial \lambda_0} = -\frac{n\lambda^{\alpha-1}}{(\lambda+1)\left((\lambda+1)^\alpha - \lambda^\alpha\right)} - \frac{n}{\lambda} + \frac{1}{\lambda - e^{-\frac{x}{\sigma}} + 1} \\
&\quad - \frac{\alpha n\lambda^{\alpha-1}(\lambda+1)^{\alpha-1}(\log(\lambda) - \log(\lambda+1))}{\left(\lambda^\alpha - (\lambda+1)^\alpha\right)^2}, \\
I_{13} &= I_{31} = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda \partial \beta} = -\frac{x}{\sigma^2\left((\lambda+1)e^{x/\sigma} - 1\right)}, \\
I_{23} &= I_{32} = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda_0 \partial \beta} = \frac{(\alpha+1)xe^{x/\sigma}}{\sigma^2\left((\lambda+1)e^{x/\sigma} - 1\right)^2}.
\end{aligned}$$

Under conditions that are existed for parameters in the interior of the parameter space but not on the boundary,

$$\sqrt{n}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \xrightarrow{d} N_3(\mathbf{0}, I^{-1}(\boldsymbol{\theta})),$$

where $I^{-1}(\boldsymbol{\theta})$ is inverse of the Fisher information matrix. In addition, a $100(1-\alpha)\%$ asymptotic confidence interval (ACI) for each parameter $\boldsymbol{\theta}_i$ is as follows

$$\left(\hat{\boldsymbol{\theta}}_i - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{I}^{-1}(\boldsymbol{\theta})_i}{n}}, \hat{\boldsymbol{\theta}}_i - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{I}^{-1}(\boldsymbol{\theta})_i}{n}}\right),$$

where $\hat{I}^{-1}(\boldsymbol{\theta})_i$ is the i-th diagonal element of $\hat{I}^{-1}(\boldsymbol{\theta})$.

5. Real data analysis

As an application of the truncated Lomax-exponential distribution, data on the cost of electricity generation by various US power plants in 1970 are analyzed. This data set consists of 158 observations and is available in the **Ecdat** package as **Electricity** in R software. The scaled total time on test plot (TTT plot) is used to determine the behavior of the failure rate function. Let $F(.)$ be a continuous distribution function with inverse $F^{-1}(.)$. Brunk et al. [4] defined the concept of total time on test as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} (1 - F(u))du,$$

and the scaled total time on test as

$$\phi_F(t) = \frac{H_F^{-1}(t)}{H_F^{-1}(1)}.$$

Aarset [1] showed that if $\phi_F(t)$ is concave at interval $[0, 1]$, then the failure rate function is increasing and if $\phi_F(t)$ is convex at interval $[0, 1]$, then the failure rate function is decreasing. The empirical version of the function $\phi_F(t)$ is applied to determine the behavior of the failure rate function using the data. If $x_1, \ldots, x_n$ are the observations of a random variable, then the scaled total time on test transform is defined as follows

$$T\left(\frac{i}{n}\right) = \frac{\sum_{k=1}^{i} x_{k:n} - (n-i)x_{i:n}}{\sum_{k=1}^{i} x_{k:n}}, \qquad i = 0, 1, \ldots, n,$$

where $x_{k:n}$ is the observed value of k-th ordered statistic. The plot obtained from the connection of points $(\frac{i}{n}, T(\frac{i}{n}))$ for $i = 0, 1, \ldots, n$, is called the empirical scaled total time on test plot (empirical TTT plot).

Figure 2 displays the empirical TTT plot of the cost data set of power generation at various US power plants in 1970. Since this plot is convex, it is found that the distribution with the DFR property is appropriate to fit the data set. Thus, the truncated Lomax-exponential (TLE) distribution is an appropriate distribution for this data set. The truncated Lomax-exponential distribution

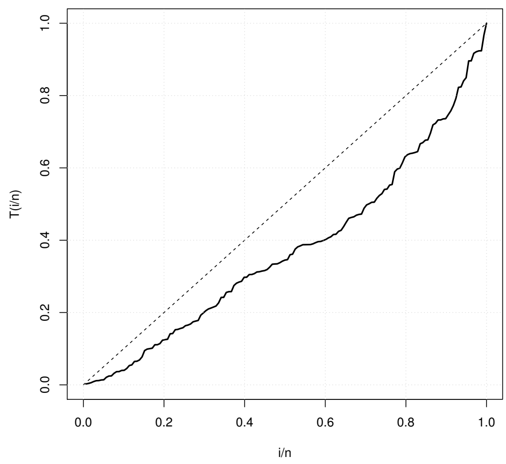

FIGURE 2. The empirical TTT plot of the cost data set of power generation at US power plants

is compared to the truncated exponential-exponential (TEE) distribution introduced by Mahdavi and Oliveira Silva [11] with PDF

$$f(x,\alpha,\lambda) = \frac{\alpha\lambda e^{-\lambda x} e^{\alpha e^{-\lambda x}}}{e^{\alpha} - 1}, \qquad x > 0, \alpha > 0, \lambda > 0,$$

and the gamma (G) distribution with PDF

$$f(x,\alpha,\lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \qquad x > 0, \alpha > 0, \lambda > 0,$$

and the weibull (W) distribution with PDF

$$f(x,\alpha,\lambda) = \alpha\lambda^{\alpha} x^{\alpha-1} e^{-(\lambda x)^{\alpha}}, \qquad x > 0, \alpha > 0, \lambda > 0,$$

that have the DFR property. In order to compare the models for this data set, the Kolmogorov-Smirnov test statistic (KS), the Anderson-Darling test statistic (AD), the Cramer-Von Misestest statistic (CVM) are used and the MLEs of the parameters and this statistics are provided in Table 1. The probability value of each statistic is also given in front of the statistic value in parentheses.

As observed in Table 1, the truncated Lomax-exponential distribution provides a better fit than the other models based on these three goodness-of-fit statistics.

Figure 3 displays how to fit the densities and distribution functions of different models versus the histogram and the empirical distribution for the data set. Also, the probability-probability plot fitted to this data set corresponding to the truncated Lomax-exponential distribution is presented in Figure 4

Table 1. The MLEs of the parameters and KS, AD, the CVM statistics with their corresponding probability value.

Model	Estimation	KS statistic	AD statistic	CVM statistic
TLE	$\hat{\alpha} = 3.212, \hat{\lambda} = 0.479, \hat{\sigma} = 216.784$	0.050(0.827)	0.395(0.853)	0.059(0.820)
TEE	$\hat{\alpha} = 3.817, \hat{\lambda} = 0.007$	0.058(0.664)	0.687(0.569)	0.082(0.680)
G	$\hat{\alpha} = 0.673, \hat{\lambda} = 0.013$	0.096(0.107)	1.471(0.184)	0.287(0.147)
W	$\hat{\alpha} = 0.757, \hat{\lambda} = 0.023$	0.069(0.431)	0.856(0.442)	0.161(0.359)

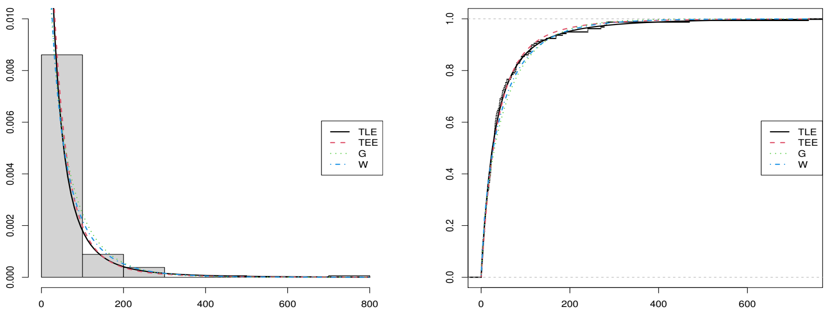

Figure 3. The fitted densities and distribution functions to the cost data set of power generation at US power plants

which the empirical distribution function of each data versus the theoretical distribution function is plotted. These plots also confirm the appropriate fit of this distribution to the data.

6. **Conclusion**

Regarding the importance of analyzing the insurance and finance data, and since they are usually right-skewed or reverse "J" shaped, this study was performed to provide a suitable distribution to fit such data. Important financial and insurance indicators were studied by introducing the truncated Lomax-exponential distribution with the characteristics of financial data. Finally, by considering the cost data set of power generation at various US power plants as a real data set, the proposed distribution, and gamma, Weibull, and truncated exponential-exponential distributions were fitted to the data set. By using the Kolmogorov-Smirnov test statistic, the Anderson-Darling test statistic, the Cramer-Von Misestest statistic, it is observed that the proposed distribution provides a better fit to this real data set than the other models. Therefore, this distribution can provide an appropriate fit to reverse "J" shaped finance data.

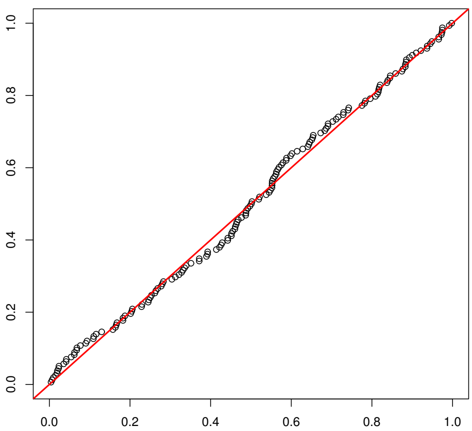

FIGURE 4. The probability-probability plot of TLE distribution to the cost data set of power generation at US power plants

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