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ON TIMELIKE HYPERSURFACES OF THE MINKOWSKI 4-SPACE WITH 1-PROPER SECOND MEAN CURVATURE VECTOR

Firooz Pashaie ✉, N. Tanoomand-khooshmehr, A. Rahimi, and L. Shahbaz

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Abstract. The mean curvature vector field of a submanifold in the Euclidean n-space is said to be *proper* if it is an eigenvector of the Laplace operator Δ. It is proven that every hypersurface with proper mean curvature vector field in the Euclidean 4-space $\mathbb{E}^4$ has constant mean curvature. In this paper, we study an extended version of the mentioned subject on timelike (i.e., Lorentz) hypersurfaces of Minkowski 4-space $\mathbb{E}^4_1$. Let $\mathbf{x} : M^3_1 \to \mathbb{E}^4_1$ be the isometric immersion of a timelike hypersurface M^3_1 in $\mathbb{E}^4_1$. The second mean curvature vector field $\mathbf{H}_2$ of M^3_1 is called *1-proper* if it is an eigenvector of the Cheng-Yau operator $\mathcal{C}$ (which is the natural extension of Δ). We show that each M^3_1 with 1-proper $\mathbf{H}_2$ has constant scalar curvature. By a classification theorem, we show that such a hypersurface is $\mathcal{C}$-biharmonic, $\mathcal{C}$-1-type or null-$\mathcal{C}$-2-type. Since the shape operator of M^3_1 has four possible matrix forms, the results will be considered in four different cases.

Keywords: Weak convex, Lorentz hypersurface, Biharmonic, $\mathcal{C}$-harmonic.
2020 MSC: Primary 53A30, 53B30, 53C40, 53C43.

1. Introduction

Surfaces of constant mean curvature play important roles in differential geometry and some physical theories. In this context, a geometric motivation is a well-known conjecture of Bang-Yen Chen (in 1987) which says that the only biharmonic submanifolds of Euclidean spaces are minimal ones. Some early improvements in the conjecture may be found (for instance) in [1,5,9,10]. Later on, many researchers have studied the hypersurfaces with proper mean curvature vector fields. The mean curvature vector field $\mathbf{H}$ of a hypersurface is said to be proper if it satisfies the equation $\Delta\mathbf{H} = \alpha\mathbf{H}$, where Δ is the Laplace operator and α is a real number. It is proved that the hypersurfaces of $\mathbb{E}^4$ with proper mean curvature vector field have constant mean curvature ([8]). In [4], the hypersurfaces of $\mathbb{E}^4_s$ with proper mean curvature vector field have been

✉ f_pashaie@maragheh.ac.ir, ORCID: 0000-0002-3020-7649
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studied. Also, some new results on (hyper-)surfaces with parallel mean curvature vector field in some (pseudo-)Riemannian 4-spaces can be found in [3,18]. We continue the matter using the Cheng-Yau operator $\mathcal{C}$, which is a routine extension of Laplace operator. We study the Lorentz hypersurfaces in pseudo-Euclidean 4-space $\mathbb{E}_1^4$ that whose second mean curvature vector field $\mathbf{H}_2$ is 1-proper (i.e., it satisfies the condition $\mathcal{C}\mathbf{H}_2 = \beta\mathbf{H}_2$ for a constant real number β). The shape operator of M_1^3 has four possible matrix forms, so we have to study four different cases, namely D_k-hypersurfaces of $\mathbb{E}_1^4$, where $k = 1, 2, 3, 4$. In D_1-hypersurfaces, we consider two cases depending on the number of distinct principal curvatures and we show that the D_1-hypersurfaces with 1-proper second mean curvature vector field and two or three distinct principal curvatures have constant scalar curvature. In cases D_2, D_3 and D_4, the shape operator is non-diagonal. In the non-diagonal cases, we show that if the hypersurface has 1-proper second mean curvature vector field and constant ordinary mean curvature, then it is 1-minimal.

2. Prerequisites

The preliminary concepts are recalled from [2,6,7,12–14,16,17]. The Minkowski 4-space $\mathbb{E}_1^4$ is the Euclidean space $\mathbb{E}^4$ endowed with the indefinite inner product defined by

$$< \mathbf{v}, \mathbf{w} >:= -v_1w_1 + \Sigma_{i=2}^4 v_iw_i,$$

for every $\mathbf{v}, \mathbf{w} \in \mathbb{E}^4$. Every timelike hypersurface M_1^3 in $\mathbb{E}_1^4$ is equipped with a Lorentz metric induced from $\mathbb{E}_1^4$. For each non-zero vector $\mathbf{v} \in \mathbb{E}_1^4$, the real number $< \mathbf{v}, \mathbf{v} >$ may be negative, zero or positive and then, $\mathbf{v}$ is said to be timelike, lightlike or spacelike, respectively. In general, for a Lorentz vector space V_1^3, a basis $\mathcal{B} := \{e_1, e_2, e_3\}$ is said to be *orthonormal* if it satisfies $< e_i, e_j >= \epsilon_i\delta_i^j$ (without Einstein convention) for $i, j = 1, 2, 3$, where $\epsilon_1 = -1$, $\epsilon_2 = 1$ and $\epsilon_3 = 1$. δ_i^j is the Kronecker delta. $\mathcal{B}$ is called *pseudo-orthonormal* if $\mathcal{B}$ satisfies $< e_2, e_2 >=< e_1, e_1 >= 0$, $< e_2, e_1 >= -1$ and $< e_3, e_i >= \delta_i^3$ for $i = 1, 2, 3$.

For a timelike hypersurface M_1^3 in $\mathbb{E}_1^4$ the shape operator S has four possible canonical matrix forms. When the metric on M_1^3 is of diagonal type $\mathcal{G}_1 := \mathrm{diag}[-1, 1, 1]$, its shape operator has matrix form

$$D_1 = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3] \text{ or } D_2 = \mathrm{diag}[\begin{bmatrix} \kappa & \lambda \\ -\lambda & \kappa \end{bmatrix}, \eta], \ (\lambda \neq 0).$$

Otherwise, the induced metric on M_1^3 is of non-diagonal type

$$\mathcal{G}_2 = \mathrm{diag}[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1]$$

and the shape operator is

$$D_3 = \mathrm{diag}[\begin{bmatrix} \kappa + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \kappa - \frac{1}{2} \end{bmatrix}, \lambda] \text{ or } D_4 = \begin{bmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{bmatrix}.$$

Now, by unified notations κ_i for $i = 1, 2, 3$, we define the principal curvatures of timelike hypersurfaces of Minkowski 4-space.

We put $\kappa_i := \lambda_i$ for $i = 1, 2, 3$ in the case $S = D_1$. In the case $S = D_2$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_3 := \eta$. In the case $S = D_3$, we take $\kappa_1 = \kappa_2 := \kappa$ and $\kappa_3 := \lambda$. Finally, when $S = D_4$, we take $\kappa_j := \kappa$ for $j = 1, 2, 3$.

M_1^3 is called a D_k*-hypersurface* if its shape operator is of form D_k for $k = 1, 2, 3, 4$.

Using the symmetric functions $s_0 := 1$, $s_1 = \sum_{j=1}^{3} \kappa_j$, $s_2 := \sum_{1\leq i_1<i_2\leq 3} \kappa_{i_1}\kappa_{i_2}$ and $s_3 := \kappa_1\kappa_2\kappa_3$, we define the *jth mean curvature* of M_1^3 (for $j = 0, 1, 2, 3$) by $H_j = \frac{1}{\binom{3}{j}} s_j$. When H_{j+1} is identically null, M_1^3 is called *j-minimal.* In the case D_1, we say that M_1^3 is *isoparametric* if its principal curvatures are constant. In other three cases, M_1^3 is called isoparametric if the minimal polynomial of its shape operator has constant coefficients. By Theorem 4.10 from [13], there is no isoparametric timelike hypersurface of E_1^4 with complex principal curvatures. The Newton operators on M_1^3 are given inductively by $P_0 = I$ and $P_j = s_j I - S \circ P_{j-1}$ for $j = 1, 2, 3$, where I is the identity map (see [2, 15]).

In four cases $S = D_i$ $(i = 1, 2, 3, 4)$, P_1 and P_1 have different forms. When $S = D_1$, we have $P_1 = \mathrm{diag}[\kappa_2 + \kappa_3, \kappa_1 + \kappa_3, \kappa_1 + \kappa_2]$ and $P_2 = \mathrm{diag}[\kappa_2\kappa_3, \kappa_1\kappa_3, \kappa_1\kappa_2]$.

In the case $S = D_2$, $P_1 = \mathrm{diag}[\begin{bmatrix} \kappa+\eta & -\lambda \\ \lambda & \kappa+\eta \end{bmatrix}, 2\kappa]$ and $P_2 = \mathrm{diag}[\begin{bmatrix} \kappa\eta & -\lambda\eta \\ \lambda\eta & \kappa\eta \end{bmatrix}, \kappa^2 + \lambda^2]$.

When $S = D_3$, $P_1 = \mathrm{diag}[\begin{bmatrix} \kappa+\lambda-\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \kappa+\lambda+\frac{1}{2} \end{bmatrix}, 2\kappa]$ and $P_2 = \mathrm{diag}[\begin{bmatrix} (\kappa-\frac{1}{2})\lambda & -\frac{1}{2}\lambda \\ \frac{1}{2}\lambda & (\kappa+\frac{1}{2})\lambda \end{bmatrix}, \kappa^2]$.

In the case $S = D_4$, $P_1 = \begin{bmatrix} 2\kappa & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 2\kappa & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\kappa \end{bmatrix}$, and $P_2 = \begin{bmatrix} \kappa^2-\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}\kappa \\ \frac{1}{2} & \kappa^2+\frac{1}{2} & \frac{\sqrt{2}}{2}\kappa \\ \frac{\sqrt{2}}{2}\kappa & \frac{\sqrt{2}}{2}\kappa & \kappa^2 \end{bmatrix}$.

The Cheng-Yau operator is given by $\mathcal{C}(f) := \mathrm{tr}(P_1 \circ \nabla^2 f)$ for every smooth real function f on M_1^3. The Hessian map ∇^2 is defined as $(\nabla^2 f)X = \nabla_X(\nabla f)$ for every smooth vector fields X on M_1^3, where $\nabla f = \sharp df$.

For a timelike hypersurface $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ we have (from [2, 15]) $\mathcal{C}\mathbf{x} = c_1 H_2 \mathbf{n}$ and then

$\mathcal{C}^2\mathbf{x} = -6[9H_2 I - 2P_2]\nabla H_2 - 6[9H_1 H_2^2 + 3H_2 H_3 - \mathcal{C}H_2]\mathbf{n}$.

$\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ is said to be $\mathcal{C}$*-biharmonic* if it satisfies the equation $\mathcal{C}^2\mathbf{x} = 0$. Also, the second mean curvature vector field $\mathbf{H}_2$ of $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ is called *1-proper* if $\mathbf{x}$ satisfies the equation $\mathcal{C}\mathbf{H}_2 = \beta\mathbf{H}_2$ for a real number β. Clearly, one can obtain simpler conditions on M_1^3 to have proper second mean curvature vector field as:

$$(1)\qquad \text{(i) } \mathcal{C}H_2 = H_2(\beta + 9H_1H_2 - 3H_3), \quad \text{(ii) } P_2\nabla H_2 = \frac{9}{2}H_2\nabla H_2.$$

The well-known structure equations on $\mathbb{E}_1^4$ are given by $d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j$, $\omega_{ij}+\omega_{ji} = 0$ and $d\omega_{ij} = \sum_{l=1}^{4} \omega_{il}\wedge\omega_{lj}$. Restricted on M, we have $\omega_4 = 0$ and then $0 = d\omega_4 = \sum_{i=1}^{3} \omega_{4,i} \wedge \omega_i$. So by Cartan's lemma, there exist functions h_{ij} such that $\omega_{4,i} = \sum_{j=1}^{3} h_{ij}\omega_j$ and $h_{ij} = h_{ji}$ which give the second fundamental form of M, as $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_4$. The mean curvature H is given by $H = \frac{1}{3}\sum_{i=1}^{3} h_{ii}$. Therefore, we obtain the structure equations on M as follow.

$$\text{(i) } \omega_{ij} + \omega_{ji} = 0, \text{ (ii) } d\omega_i = \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j,$$

$$\text{(iii) } d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{3} R_{ijkl}\omega_k \wedge \omega_l,$$

$$\text{(iv) } R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$$

for $i, j = 1, 2, 3$, where R_{ijkl} denotes the components of the Riemannian curvature tensor of M. Denoting the covariant derivative of h_{ij} by h_{ijk}, we have

$$\text{(v) } dh_{ij} = \sum_{k=1}^{3} h_{ijk}\omega_k + \sum_{k=1}^{3} h_{kj}\omega_{ik} + \sum_{k=1}^{3} h_{ik}\omega_{jk},$$

$$(2)\qquad \text{(vi) } h_{ijk} = h_{ikj}.$$

Now we recall the definition of an $\mathcal{C}$-finite type hypersurface from [11].

Definition 2.1. A timelike hypersurface $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ is called *of* $\mathcal{C}$*-finite type* if $\mathbf{x} = \sum_{i=0}^{m} \mathbf{x}_i$ for a natural number m, satisfying the condition $\mathcal{C}\mathbf{x}_i = \beta_i\mathbf{x}_i$, where $\mathbf{x}_0$ is a constant vector and for $i = 1, 2, \cdots, m$, each β_i is a real number and each $\mathbf{x}_i : M_1^3 \to \mathbb{E}_1^4$ is a smooth map. If β_i's are mutually distinct then M_1^3 is called *of* $\mathcal{C}$*-m-type*. In addition, it is said to be *of null-*$\mathcal{C}$*-m-type* if for at least one i $(1 \leq i \leq m)$ we have $\beta_i = 0$.

3. Results

First, we emphasize that a timelike hypersurface in the Minkowski 4-space cannot be both $\mathcal{C}$-biharmonic and of $\mathcal{C}$-finite type.

Proposition 3.1. There is no $\mathcal{C}$-biharmonic timelike hypersurface of $\mathcal{C}$-finite type in the Minkowski 4-space.

Proof. Let $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ be both $\mathcal{C}$-biharmonic and of $\mathcal{C}$-finite type. So, we have $\mathbf{x} = \sum_{i=0}^{m} \mathbf{x}_i$, where $\mathcal{C}\mathbf{x}_0 = 0$ and $\mathcal{C}\mathbf{x}_i = \beta_i \mathbf{x}_i$ for mutually distinct real numbers β_i $(i = 1, \ldots, m)$. Then, we obtain $0 = \mathcal{C}^s\mathbf{x} = \lambda_1^s \mathbf{x}_1 + \cdots + \lambda_k^s \mathbf{x}_k$, for $s = 1, 2, 3, \ldots$, which gives $\mathbf{x}_1 = \cdots = \mathbf{x}_m = 0$. This is a contradiction. □

In the rest results, we consider different cases based on the matrix from of the shape operator of timelike hypersurfaces in $\mathbb{E}_1^4$.

3.1. **Diagonal shape operator.** In this subsection, we assume that $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ is a D_1-hypersurface with 1-proper second mean curvature vector field.

Theorem 3.2. A D_1-hypersurface $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ has 1-proper second mean curvature vector field if and only if it is $\mathcal{C}$-biharmonic, $\mathcal{C}$-1-type or $\mathcal{C}$-null-2-type.

Proof. First, assume that $\mathcal{C}\mathbf{H}_2 = \beta\mathbf{H}_2$ for a constant β. If $\beta = 0$, then M_1^3 is $\mathcal{C}$-biharmonic. Otherwise, taking $\mathbf{x}_1 = \frac{1}{\beta}\mathcal{C}\mathbf{x}$ and $\mathbf{x}_0 = \mathbf{x} - \mathbf{x}_1$, we have $\mathcal{C}\mathbf{x}_1 = \frac{1}{c}\mathcal{C}^2\mathbf{x} = \frac{6}{c}\mathcal{C}\mathbf{H}_2 = 6\mathbf{H}_2 = \mathcal{C}\mathbf{x}$.

According to $\mathbf{x}_0$ is constant or non-constant, M_1^3 is $\mathcal{C}$-1-type or $\mathcal{C}$-null-2-type. The converse is straightforward. □

The D_1-hypersurfaces with 1-proper second mean curvature vector field are studied in two different cases (with different results and different methods of proof) according to the number of distinct principal curvatures. In Theorems 3.3 and 3.4 the hypersurface is assumed to have three distinct principal curvatures.

Theorem 3.3. Every D_1-hypersurface $\mathbf{x} : M_1^3 \to \mathrm{E}_1^4$ with three distinct principal curvatures and 1-proper second mean curvature vector field has constant second mean curvature.

Proof. Assume that H_2 is non-constant. By showing that $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$ is empty, we get a contradiction. With respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M_1^3, its shape operator S is diagonal such that $Se_i = \lambda_i e_i$ for $i = 1, 2, 3$, where λ_i's are mutually distinct. Also, we have $P_2 e_i = \mu_{i,2} e_i$ for $i = 1, 2, 3$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (**??**)(ii) we get

$$e_i(H_2)(\mu_{i,2} - \frac{9}{2}H_2) = 0 \tag{3}$$

for $i = 1, 2, 3$. Each point of $\mathcal{U}$ has an open neighborhood on which we have $e_i(H_2) \neq 0$ for at least one i. So, without loss of generality, we can assume that $e_1(H_2) \neq 0$ and then we have $\mu_{1,2} = \frac{9}{2}H_2$, (locally) on $\mathcal{U}$, which gives $\lambda_2\lambda_3 = \frac{9}{2}H_2$. Now, we prove three simple claims.

Claim 1: $e_2(H_2) = e_3(H_2) = 0$.
If either $e_2(H_2) \neq 0$ or $e_3(H_2) \neq 0$, then by (3) we get $\mu_{1,2} = \mu_{2,2} = \frac{9}{2}H_2$ or $\mu_{1,2} = \mu_{3,2} = \frac{9}{2}H_2$, which give $\lambda_3(\lambda_2 - \lambda_1) = 0$ or $\lambda_2(\lambda_1 - \lambda_3) = 0$. But, since λ_i's are assumed to be mutually distinct, we get $\lambda_3 = 0$ or $\lambda_2 = 0$, which gives $H_2 = 0$ on $\mathcal{U}$. The result is in contradiction with the definition of $\mathcal{U}$.

Claim 2: $e_2(\lambda_1) = e_3(\lambda_1) = 0$.
Since H is assumed to be constant on M, we have $e_2(\lambda_1) = e_2(3H - \lambda_1 - \lambda_2) = -e_2(\lambda_1) - e_2(\lambda_2)$. On the other hand, from two recent results $e_2(H_2) = 0$ and $\lambda_2\lambda_3 = \frac{9}{2}H_2$ we get

$$e_2(\lambda_1\lambda_3) + e_2(\lambda_1\lambda_2) = e_2(3H_2 - \frac{9}{2}H_2) = 0,$$

which gives $\lambda_1 e_2(\lambda_2 + \lambda_3) + (\lambda_2 + \lambda_3)e_2\lambda_1 = 0$, and then we have

$$\lambda_1 e_2(3H - \lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1 = \lambda_1 e_2(-\lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1 = (-\lambda_1 + \lambda_2 + \lambda_3)e_2\lambda_1 = 0.$$

Therefore, assuming $e_2(\lambda_1) \neq 0$, we get $\lambda_1 = \lambda_2 + \lambda_3$ which gives contradiction

$$e_2(\lambda_1) = e_2(\lambda_2 + \lambda_3) = e_2(3H - \lambda_1) = -e_2(\lambda_1).$$

Consequently, $e_2(\lambda_1) = 0$.

Similarly, one can show $e_3(\lambda_1) = 0$. So, Claim 2 is proved.

Claim 3: $e_2(\lambda_3) = e_3(\lambda_2) = 0$.
Using the notations

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k, \ (i, j = 1, 2, 3), \tag{4}$$

and the compatibility condition $\nabla_{e_k} < e_i, e_j >= 0$, we have

$$\omega_{ki}^i = 0, \ \omega_{ki}^j + \omega_{kj}^i = 0, \ (i, j, k = 1, 2, 3) \tag{5}$$

and applying the Codazzi equation (see [14], page 115, Corollary 34(2))

$$(\nabla_V S)W = \nabla_W S)V, \ (\forall V, W \in \chi(M)) \tag{6}$$

on the basis $\{e_1, e_2, e_3\}$, we get for distinct $i, j, k = 1, 2, 3$

$$(a) \ e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \ (b) \ (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j. \tag{7}$$

Also, by a straightforward computation of components of the identity $(\nabla_{e_i} S)e_j - (\nabla_{e_j} S)e_i \equiv 0$ for distinct $i, j = 1, 2, 3$, we get $[e_2, \ e_3](H_2) = 0$, $\omega_{12}^1 = \omega_{13}^1 = \omega_{13}^2 = \omega_{21}^3 = \omega_{32}^1 = 0$ and

$$\omega_{21}^2 = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}, \ \omega_{31}^3 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}, \omega_{23}^2 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}, \ \omega_{32}^3 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}. \tag{8}$$

Therefore, the covariant derivatives $\nabla_{e_i}e_j$ simplify to $\nabla_{e_1}e_k = 0$ for $k = 1, 2, 3$, and

(9)
$$\nabla_{e_2}e_1 = \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}e_2,\ \nabla_{e_3}e_1 = \frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}e_3, \nabla_{e_2}e_2 = \frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}e_1,$$
$$\nabla_{e_3}e_2 = \frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}e_3, \nabla_{e_2}e_3 = \frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}e_2,\ \nabla_{e_3}e_3 = \frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_1 + \frac{e_2(\lambda_3)}{\lambda_3-\lambda_2}e_2.$$

Now, the Gauss equation for $< R(e_2, e_3)e_1,\ e_2 >$ and $< R(e_2, e_3)e_1,\ e_3 >$ show that

$$e_3\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right) = \frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{10}$$

$$e_2\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right) = \frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right). \tag{11}$$

We also have the Gauss equation for $< R(e_1, e_2)e_1,\ e_2 >$ and $< R(e_3, e_1)e_1,\ e_3 >$, which give the following relations

(12)
$$e_1\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right) + \left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)^2 = \lambda_1\lambda_2,\ e_1\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right) + \left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)^2 = \lambda_1\lambda_3.$$

Finally, we obtain from the Gauss equation for $< R(e_3, e_1)e_2, e_3 >$ that

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\right) = \frac{e_1(\lambda_3)e_2(\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)}. \tag{13}$$

On the other hand, it follows from Claim 1 that

(14)
$$-\mu_{1,1}e_1e_1(H_2) + \left(\mu_{2,1}\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \mu_{3,1}\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)e_1(H_2) - 9H_2^2(H - \frac{3}{2}\lambda_1) = 0.$$

By differentiating (14) along e_2 and e_3 (and using respectively (10) and (11)) we obtain

$$e_2\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}\right) = \frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{15}$$

$$e_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right) = \frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2} - \frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right). \tag{16}$$

Using (9), we find that

$$[e_1,\ e_2] = \frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}e_2. \tag{17}$$

Applying both sides of the equality (17) on $\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}$, using (15), (12), and (13), we deduce that

$$\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{18}$$

(18) shows that $e_2(\lambda_3)=0$ or

$$\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}. \tag{19}$$

From equation (19), by differentiating both sides along e_1 and applying (12), we get $\lambda_2=\lambda_3$, which is a contradiction, so we have to confirm the result $e_2(\lambda_3)=0$.

Analogously, using (9), we find that $[e_1,\ e_3]=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_3$. By a similar manner, we deduce that

$$\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=0, \tag{20}$$

and one can show that $e_3(\lambda_2)$ necessarily has to be vanished.

Hence, we have obtained $e_2(\lambda_3)=e_3(\lambda_2)=0$, which by applying the Gauss equation for $< R(e_2,e_3)e_1,e_3 >$, gives the following equality

$$\frac{e_1(\lambda_3)e_1(\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}=\lambda_2\lambda_3. \tag{21}$$

Finally, using (12), differentiating (21) along e_1 gives

$$\lambda_2\lambda_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0, \tag{22}$$

which implies $\lambda_2\lambda_3=0$ (since we have seen above that $\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)\neq 0$). Therefore, we obtain $H_2=0$ on $\mathcal{U}$, which is a contradiction. Hence H_2 is constant on M_1^3. □

In Theorem 3.4, we follow Defever's techniques to prove our result (see [8]).

Theorem 3.4. Let $\mathbf{x}: M_1^3 \to \mathbb{E}_1^4$ be a D_1-hypersurface with three distinct principal curvatures and 1-proper second mean curvature vector field. If M_1^3 has constant ordinary mean curvature, then it is 1-minimal.

Proof. By Theorem 3.3 and by assumption, H_2 and H_1 both are constant. We prove that H_2 is null. If $H_2\neq 0$, by (**??**)(i) we obtain that H_3 is constant. Therefore, all of mean curvatures H_i (for $i=1,2,3$) are constant, which means that M^3 is isoparametric. By Corollary 2.7 in [13], an isoparametric timelike D_1-hypersurface has at most one non-zero principal curvature, which contradicts the assumption that, three principal curvatures of M are mutually distinct. So $H_2\equiv 0$. □

Now, in the next two theorems we suppose that the shaper operator has exactly two distinct principal curvatures.

Theorem 3.5. Let $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ be a D_1-hypersurfaces with 1-proper second mean curvature vector field. If M_1^3 has exactly two distinct principal curvatures, then its second mean curvature is constat.

Proof. Let λ_1 and λ_2 be the principal curvatures of M_1^3 of multiplicities 1 and 2, respectively. Taking $\mathcal{U} := \{p \in M_1^3 : \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{U}$ is empty. We suppose that $\{e_1, e_2, e_3\}$ is a local orthonormal basis of tangent bundle on $\mathcal{U}$ such that $Se_j = \lambda_j e_j$ for $j = 1, 2, 3$. In the assumed case, we have

$$\lambda_1 = \lambda_2 = \lambda, \ \lambda_3 = \mu.$$

Therefore, we obtain

$$\mu_{1,2} = \mu_{2,2} = \lambda\mu, \ \mu_{3,2} = \lambda^2, \ 3H_2 = \lambda^2 + 2\lambda\mu. \tag{23}$$

By condition (**??**)(ii), we have

$$P_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2. \tag{24}$$

Then, using the polar decomposition

$$\nabla H_2 = \sum_{i=1}^{3} \epsilon_i < \nabla H_2, e_i > e_i, \tag{25}$$

we see that (24) is equivalent to

$$\epsilon_i < \nabla H_2, e_i > (\mu_{i,2} - \frac{9}{2} H_2) = 0$$

for $i = 1, 2, 3$.

For each i, from $< \nabla H_2, e_i > \neq 0$ on $\mathcal{U}$ we get

$$\mu_{i,2} = \frac{9}{2} H_2. \tag{26}$$

By definition, we have $\nabla H_2 \neq 0$ on $\mathcal{U}$, which gives one or both of the following states.

State 1. $< \nabla H_2, e_i > \neq 0$, for $i = 1$ or $i = 2$. By equalities (23) and (26), we obtain

$$\lambda\mu = \frac{9}{2}(\frac{2}{3}\lambda\mu + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(2\mu + \frac{3}{2}\lambda) = 0. \tag{27}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\mu = -\frac{3}{4}\lambda$, $H_2 = -\frac{1}{6}\lambda^2$.

State 2. $< \nabla H_2, e_3 > \neq 0$. By equalities (23) and (26), we obtain

$$\lambda^2 = \frac{9}{2}(\frac{2}{3}\lambda\mu + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(3\mu + \frac{1}{2}\lambda) = 0. \tag{28}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we have $\mu = -\frac{1}{6}\lambda$, $H_2 = \frac{2}{9}\lambda^2$.

Both states requires the same calculation, so we consider for instance State 2. Let us denote the maximal integral submanifold through $x \in \mathcal{U}$, corresponding to λ by $\mathcal{U}_1^{n-1}(x)$. We write

$$d\lambda = \sum_{i=1}^{3} \lambda_{,i}\omega_i \ d\mu = \sum_{j=1}^{3} \mu_{,j}\omega_j. \tag{29}$$

Then, we have $\lambda_{,1} = \lambda_{,2} = 0$. We can assume that $\lambda > 0$ on $\mathcal{U}$, then we have (in State 2)

$$\mu = \frac{-1}{6}\lambda < 0. \tag{30}$$

By means of (2), we obtain

$$\sum_{k=1}^{3} h_{ijk}\omega_k = \delta_{ij} d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}, \tag{31}$$

for $i, j, k = 1, 2, 3$. Here, we adopt the notational convention that $a, b, c = 1, 2$. From (29) and (31), we have

$$\begin{aligned} &h_{12k} = h_{21k} = 0, \\ &h_{aab} = 0,\ h_{aa3} = \lambda_{,3}, \\ &h_{33a} = 0,\ h_{333} = \mu_{,3}. \end{aligned} \tag{32}$$

Combining this with (2) and the formula

$$\sum_{i=1}^{3} h_{a3i}\omega_i = dh_{a3} + \sum_{i=1}^{3} h_{i3}\omega_{ia} + \sum_{i=1}^{3} h_{ai}\omega_{i3} = (\lambda - \mu)\omega_{a3},$$

we obtain from (30)

$$\omega_{a3} = \frac{\lambda_{,3}}{\lambda - \mu}\omega_a = \frac{6\lambda_{,3}}{7\lambda}\omega_a. \tag{33}$$

Therefore we have

$$d\omega_3 = \sum_{a=1}^{2} \omega_{3a} \wedge \omega_a = 0.$$

Notice that we may consider λ to be locally a function of the parameter s, where s is the arc length of an orthogonal trajectory of the family of the integral submanifolds corresponding to λ. We may put $\omega_3 = ds$.
Thus, for $\lambda = \lambda(s)$, we have

$$d\lambda = \lambda_{,3} ds,\ \lambda_{,3} = \lambda'(s),$$

so from (33), we get

$$\omega_{a3} = \frac{\lambda_{,3}}{\lambda - \mu}\omega_a = \frac{6\lambda'(s)}{7\lambda}\omega_a. \tag{34}$$

According to the structure equations of $\mathbb{E}^4_1$ and (34), we may compute

(35)

$$(i):\ d\omega_{a3} = \sum_{b=1}^{2} \omega_{ab} \wedge \omega_{b3} + \omega_{a4} \wedge \omega_{43} = \left(\frac{6\lambda'}{7\lambda}\right) \sum_{b=1}^{2} \omega_{ab} \wedge \omega_b - \lambda\mu\omega_a \wedge ds,$$

$$(ii):\ d\omega_{a3} = d\left\{\frac{6\lambda'}{7\lambda}\omega_a\right\} = \left(\frac{6\lambda'}{7\lambda}\right)' ds \wedge \omega_a + \left(\frac{6\lambda'}{7\lambda}\right) d\omega_a$$

$$= \left\{-\left(\frac{6\lambda'}{7\lambda}\right)' + \left(\frac{6\lambda'}{7\lambda}\right)^2\right\} \omega_a \wedge ds + \left(\frac{6\lambda'}{7\lambda}\right) \sum_{b=1}^{2} \omega_{ab} \wedge \omega_b.$$

Comparing equalities (35)(i) and (35)(ii), we get $\left(\frac{6\lambda'}{7\lambda}\right)' - \left(\frac{6\lambda'}{7\lambda}\right)^2 - \lambda\mu = 0$, which, by combining with (30), gives

$$\left(\frac{6\lambda'}{7\lambda}\right)' - \left(\frac{6\lambda'}{7\lambda}\right)^2 - \left(\frac{-1}{6}\right)\lambda^2 = 0. \tag{36}$$

Defining function $\beta(s) := \left(\frac{1}{\lambda(s)}\right)^{\frac{6}{7}}$ for $s \in (-\infty, +\infty)$, from (36) we get $\beta'' = \left(\frac{1}{6}\right)\beta^{\frac{-8}{6}}$, which by integrating, gives $(\beta')^2 = -\beta^{\frac{-2}{6}} + c$, where c is the constant of integration. The last equation is equivalent to

$$(\lambda')^2 = -\left(\frac{7}{6}\right)^2 \lambda^4 + c\left(\frac{7}{6}\right)^2 \lambda^{\frac{26}{7}}. \tag{37}$$

Now, in order to compare two sides of condition (**??**)(i), we need to compute $\nabla_{e_i}\nabla H_2$ and $P_1(e_i)$ for $i = 1, 2, 3$. From (27) we have $\nabla H_2 = \frac{4}{9}\lambda\lambda' e_3$, which by using (34), gives

(38)

$$\nabla_{e_a}\nabla H_2 = \frac{4}{9}\lambda\lambda'\nabla_{e_a} e_3 = \frac{4}{9}\lambda^r\lambda' \sum_b \omega_{3b}(e_a)e_b = -\frac{8}{21}\lambda'^2 e_a,$$

$$\nabla_{e_3}\nabla H_2 = \frac{4}{9}\nabla_{e_3}(\lambda\lambda' e_3) = \frac{4}{9}\lambda'^2 e_3 + \frac{4}{9}\lambda\lambda'' e_3.$$

By using (23) and (30), we compute $P_1(e_a)$ and $P_1(e_3)$.

$$P_1(e_1) = \frac{5}{6}\lambda e_1,\ P_1(e_2) = \frac{5}{6}\lambda e_2\ P_1(e_3) = 2\lambda e_3. \tag{39}$$

From (38) and (39), we get

$$\mathcal{C}(H_2) = 6H_2\left(\frac{-10(\lambda')^2}{21\lambda} + \frac{2(\lambda')^2}{3\lambda} + \frac{2}{3}\lambda''\right). \tag{40}$$

From (**??**(i)), we have $\mathcal{C}(H_2) = H_2 tr(S^2 \circ P_1) = 2H_2\frac{11}{6}\lambda^3$, which Combining with (40), gives

$$\lambda\lambda'' + \left(1 + \frac{-5}{7}\right)\lambda'^2 - 2\frac{33}{12}\lambda^4 = 0. \tag{41}$$

On the other hand, the equality (36) is equivalent to

$$\lambda\lambda'' = \frac{13}{7}\lambda'^2 + \frac{-7}{36}\lambda^4. \tag{42}$$

Now, substituting (42) and (41), we obtain

$$\frac{15}{7}\lambda'^2 + \frac{191}{36}\lambda^4 = 0. \tag{43}$$

From equations (37), (43) and (27), we get that H_2 is locally constant on $\mathcal{U}$, which is a contradiction with the definition of $\mathcal{U}$. Hence H_2 is constant on M.

By a similar discussion, one can get the same result in State 1. □

Theorem 3.6. Every D_1-hypersurface $\mathbf{x} : M_1^3 \to \mathrm{E}_1^4$ with at most two distinct principal curvatures, 1-proper second mean curvature vector field and constant ordinary mean curvature is 1-minimal.

Proof. H_1 is constant and by Theorem 3.5 we know that H_2 is also constant. If $H_2 \neq 0$, by (**??**)(i) we obtain that H_3 is constant. Therefore, all of mean curvatures H_i (for $i = 1, 2, 3$) are constant, which means that M_1^3 is isoparametric. By Corollary 2.7 in [13], an isoparametric timelike D_1-hypersurface has at most one nonzero principal curvature, which is a contradiction. So $H_2 \equiv 0$.. □

3.2. **Non-diagonal case.** In this subsection, we assume that $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ has non-diagonal shape operator of type D_2, D_3 or D_4. We show that if such a hypersurface has 1-proper second mean curvature vector field, then its second mean curvature has to be constant. In addition, if it has constant ordinary mean curvature, then it has to be 1-minimal.

Proposition 3.7. Let $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ be a D_2-hypersurface with 1-proper second mean curvature vector field. If M_1^3 has constant ordinary mean curvature and a constant real principal curvature, then its second and third mean curvatures are constant.

Proof. Suppose that H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption M_1^3 has three distinct principal curvatures. Then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form B_4, such that $Se_1 = \kappa e_1 - \lambda e_2$, $Se_2 = \lambda e_1 + \kappa e_2$, $Se_3 = \eta e_3$ and then we have $P_2 e_1 = \kappa\eta e_1 + \lambda\eta e_2$, $P_2 e_2 = -\lambda\eta e_1 + \kappa\eta e_2$ and $P_2 e_3 = (\kappa^2 + \lambda^2) e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (**??**)(ii) we get

$$(44) \qquad \begin{aligned} &(i)\ \epsilon_1 e_1(H_2)(\kappa\eta - \frac{9}{2}H_2) = \epsilon_2 e_2(H_2)\lambda\eta, \\ &(ii)\ \epsilon_2 e_2(H_2)(\kappa\eta - \frac{9}{2}H_2) = -\epsilon_1 e_1(H_2)\lambda\eta, \\ &(iii)\ \epsilon_3 e_3(H_2)(\kappa^2 + \lambda^2 - \frac{9}{2}H_2) = 0. \end{aligned}$$

Now, we prove three simple claims.

Claim 1: $e_1(H_2) = e_2(H_2) = 0$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (44)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$(45) \qquad \begin{aligned} &(i)\ \kappa\eta - \frac{9}{2}H_2 = \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\lambda\eta, \\ &(ii)\ \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}(\kappa\eta - \frac{9}{2}H_2) = -\lambda\eta, \end{aligned}$$

which by substituting (i) in (ii), gives $\lambda\eta(1 + (\frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)})^2) = 0$, then $\lambda\eta = 0$. Since by assumption $\lambda \neq 0$, we get $\eta = 0$. So, by $(45)(i)$, we have $H_2 = 0$.

Similarly, if $e_2(H_2) \neq 0$, then by dividing both sides of equalities $(44)(i, ii)$ by $\epsilon_2 e_2(H_2)$ we get

$$(46) \qquad \begin{aligned} &(i)\ \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}(\kappa\eta - \frac{9}{2}H_2) = \lambda\eta, \\ &(ii)\ \kappa\eta - \frac{9}{2}H_2 = -\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}\lambda\eta, \end{aligned}$$

which, by substituting (i) in (ii), gives $\lambda\eta(1 + (\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)})^2) = 0$, then $\lambda\eta = 0$. Since by assumption $\lambda \neq 0$, we get $\eta = 0$. So, by $(46)(ii)$, we have $H_2 = 0$.

Claim 2: $e_3(H_2) = 0$.

If $e_3(H_2) \neq 0$, then from equality $(44)(iii)$ we have $\kappa^2 + \lambda^2 = \frac{9}{2}H_2$, which gives $\kappa^2 + \lambda^2 = -6\kappa\eta$, where $\eta = 3H_1 - 2\kappa$ and η and H_1 are assumed to be constant on $\mathcal{U}$. So κ is also constant on $\mathcal{U}$, and then $H_2 = \frac{-4}{3}\kappa\eta = \frac{8}{3}\kappa^2 - 4H_1\kappa$ is constant on $\mathcal{U}$. □

Theorem 3.8. Let $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ be a D_2-hypersurface with proper second mean curvature vector field. If M_1^3 has constant ordinary mean curvature and a constant real principal curvature, then it is 1-minimal.

Proof. By Proposition 3.7, the second mean curvature of M_1^3 is constant, which gives $\mathcal{C}(H_2) = 0$. Then, by (**??**)(i), we have $9H_1H_2^2 - 3H_2H_3 = 0$, which gives $(7\eta - 4\kappa)\kappa^2\eta^2 = 0$.

Now, if $7\eta = 4\kappa$, then from $\kappa^2 + \lambda^2 = -6\kappa\eta$ we get $\frac{31}{7}\kappa^2 + \lambda^2 = 0$, and then $\kappa = \lambda = 0$, which gives $H_2 = H_3 = 0$. Also, if $\kappa^2\eta^2 = 0$, then we have $H_2 = H_3 = 0$. □

Theorem 3.9. Let $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ be a D_3-hypersurface with 1-proper second mean curvature vector field. If M_1^3 has constant ordinary mean curvature, then it is 1-minimal.

Proof. Suppose that H_2 is not constant. Taking $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we show that $\mathcal{U} = \emptyset$. With respect to an orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_2$, such that $Se_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Se_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Se_3 = \lambda e_3$ and then we have $P_2e_1 = (\kappa - \frac{1}{2})\lambda e_1 + \frac{1}{2}\lambda e_2$, $P_2e_2 = -\frac{1}{2}\lambda e_1 + (\kappa + \frac{1}{2})\lambda e_2$ and $P_2e_3 = \kappa^2 e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2)e_i$, from condition (**??**)(ii) we have

$$\text{(47)} \qquad \begin{aligned} &(i)\ \epsilon_1 e_1(H_2)[(\kappa - \frac{1}{2})\lambda - \frac{9}{2}H_2] = \epsilon_2 e_2(H_2)\frac{\lambda}{2}, \\ &(ii)\ \epsilon_2 e_2(H_2)[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2] = -\epsilon_1 e_1(H_2)\frac{\lambda}{2}, \\ &(iii)\ \epsilon_3 e_3(H_2)(\kappa^2 - \frac{9}{2}H_2) = 0. \end{aligned}$$

It remains to prove the following claim.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$.

If $e_1(H_2) \neq 0$, by dividing both sides of (47)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$\text{(48)} \qquad \begin{aligned} &(i)\ (\kappa - \frac{1}{2})\lambda - \frac{9}{2}H_2 = \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\frac{\lambda}{2}, \\ &(ii)\ \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2] = -\frac{\lambda}{2}, \end{aligned}$$

which gives $\frac{\lambda}{2}(1+u)^2 = 0$, where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. Then, either $\lambda = 0$ or $u = -1$. If $\lambda = 0$, then we get $H_2 = 0$ from (48)(i). Also, by assumption $\lambda \neq 0$ we get $u = -1$ which gives $\kappa\lambda = \frac{9}{2}H_2$, then $\kappa(3\kappa + 4\lambda) = 0$ and finally $\kappa = -\frac{4}{3}\lambda$ (since $\kappa = 0$ gives $H_2 = 0$ again). Hence, we have $H_2 = \frac{2}{9}\kappa\lambda = -\frac{8}{27}\lambda^2$ and $H_1 = -\frac{5}{9}\lambda$. H_1 is constant, then H_2 is constant and $e_1(H_2) = 0$. This is a contradiction. Hence, $e_1(H_2) = 0$. The claim $e_2(H_2) = 0$ can be proven by a similar manner.

Now, if $e_3(H_2) \neq 0$, then by (47)(iii) we get $\kappa^2 = \frac{9}{2}H_2$, then $\kappa(\kappa + 6\lambda) = 0$, which gives $\kappa = 0$ or $\kappa = -6\lambda$. If $\kappa = 0$, then $H_2 = 0$, and if $\kappa = -6\lambda$ then since $H_1 = -\frac{11}{3}\lambda$ is assumed to be constant, we get that H_2 is constant and then $e_3(H_2) = 0$. Which is a contradiction, so we have $e_3(H_2) = 0$.

Now, we prove that $H_2 \equiv 0$. Since the shape operator is of type II, there exist two possible cases as:

Case 1: M_1^3 has a principal curvature κ of multiplicity 3;

Case 2: The only distinct principal curvatures of M_1^3 are κ and λ of multiplicities 2 and 1, respectively.

In Case 1, we have $H_1 = \kappa$, $H_2 = \kappa^2$ and $H_3 = \kappa^3$. By (**??**)(i), we have $3H_1H_2^2 = H_2H_3$, which gives $\kappa^5 = 0$, and then $H_2 = 0$.

In Case 2, we have $H_1 = \frac{1}{3}(2\kappa + \lambda)$, $H_2 = \frac{1}{3}(\kappa^2 + 2\kappa\lambda)$ and $H_3 = \kappa^2\lambda$. We assume that $H_2 \neq 0$ and continue in two subcases as follow. Since $H_2 \neq 0$, then $\kappa \neq 0$ and by using (**??**)(i) we obtain that H_3 is constant. Therefore, all of mean curvatures H_i (for $i = 1, 2, 3$) are constant, which means that M_1^3 is isoparametric. By Corollary 2.7 in [13], an isoparametric D_3-hypersurface of type D_3 in the Einstein space has at most one non-zero principal curvature, so we get $\lambda = 0$. Then $H_1 = \frac{2}{3}\kappa$, $H_2 = \frac{1}{3}\kappa^2$ and $H_3 = 0$, hence, by (**??**)(i), we get $\kappa = 0$, which contradicts with the assumption of this case. Therefore $H_2 = 0$. □

Proposition 3.10. Let $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ be a D_4-hypersurface with 1-proper second mean curvature vector field. Then H_2 is constant.

Proof. Taking $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S is of form $\tilde{B}_3$, then $Se_1 = \kappa e_1 + \frac{\sqrt{2}}{2}e_3$, $Se_2 = \kappa e_2 - \frac{\sqrt{2}}{2}e_3$, $Se_3 = -\frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_2 + \kappa e_3$ and then we have $P_2e_1 = (\kappa^2 - \frac{1}{2})e_1 - \frac{1}{2}e_2 - \frac{\sqrt{2}}{2}\kappa e_3$, $P_2e_2 = \frac{1}{2}e_1 + (\kappa^2 + \frac{1}{2})e_2 + \frac{\sqrt{2}}{2}\kappa e_3$ and $P_2e_3 = \frac{\sqrt{2}}{2}\kappa e_1 + \frac{\sqrt{2}}{2}\kappa e_2 + \kappa^2 e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2)e_i$, from condition (**??**)(ii) we have

$$
(49)\qquad
\begin{aligned}
&(i)\ \epsilon_1e_1(H_2)[(\kappa^2 - \frac{1}{2}) - \frac{9}{2}H_2] + \frac{1}{2}\epsilon_2e_2(H_2) + \frac{\sqrt{2}}{2}\epsilon_3e_3(H_2)\kappa = 0,\\
&(ii)\ \frac{-1}{2}\epsilon_1e_1(H_2) + \epsilon_2e_2(H_2)[(\kappa^2 + \frac{1}{2}) - \frac{9}{2}H_2] + \frac{\sqrt{2}}{2}\epsilon_3e_3(H_2)\kappa = 0,\\
&(iii)\ \epsilon_1e_1(H_2)\frac{-\sqrt{2}}{2}\kappa + \epsilon_2e_2(H_2)\frac{\sqrt{2}}{2}\kappa + \epsilon_3e_3(H_2)(\kappa^2 - \frac{9}{2}H_2) = 0.
\end{aligned}
$$

Now, we prove the following claim:

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$.

If $e_1(H_2) \neq 0$, then by dividing by $\epsilon_1e_1(H_2)$, and using the identity $H_2 = \kappa^2$ we get

$$
(50)\qquad
\begin{aligned}
&(i)\ -\frac{1}{2} - \frac{7}{2}\kappa^2 + \frac{1}{2}u_1 + \frac{\sqrt{2}}{2}u_2\kappa = 0,\\
&(ii)\ \frac{-1}{2} + u_1(\frac{1}{2} - \frac{7}{2}\kappa^2) + \frac{\sqrt{2}}{2}u_2\kappa = 0,\\
&(iii)\ \frac{-\sqrt{2}}{2}\kappa + \frac{\sqrt{2}}{2}u_1\kappa - \frac{7}{2}\kappa^2)u_2 = 0,
\end{aligned}
$$

where $u_1 := \frac{\epsilon_2e_2(H_2)}{\epsilon_1e_1(H_2)}$ and $u_2 := \frac{\epsilon_3e_3(H_2)}{\epsilon_1e_1(H_2)}$, which gives $\kappa^2(u_1 - 1) = 0$. If $\kappa = 0$, then $H_2 = 0$. Assuming $\kappa \neq 0$, we get $u_1 = 1$, which using (50)(iii), gives

$u_2 = 0$. Substituting $u_1 = 1$ and $u_2 = 0$ in (50)(i), we obtain again $\kappa = 0$, which is impossible. Hence $e_1(H_2) \equiv 0$.

Therefore, using the result $e_1(H_2) \equiv 0$, the system of equations (49) gives

$$(51) \quad \begin{aligned} &(i)\ \frac{1}{2}\epsilon_2 e_2(H_2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0,\\ &(ii)\ \epsilon_2 e_2(H_2)(\frac{1}{2} - \frac{7}{2}\kappa^2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0,\\ &(iii)\ \epsilon_2 e_2(H_2)\frac{\sqrt{2}}{2}\kappa - \epsilon_3 e_3(H_2)\frac{7}{2}\kappa^2 = 0. \end{aligned}$$

Comparing (i) and (ii), we get $\kappa e_2(H_2) = 0$, which using (iii) gives $\kappa e_3(H_2) = 0$, and then, using (i), gives $e_2(H_2) = 0$. Then, the second claim (i.e., $e_2(H_2) = 0$) is proved.

Now, using the results $e_1(H_2) = e_2(H_2) = 0$, we get $\kappa e_3(H_2) = 0$, which, using $H_2 = \kappa^2$, implies $\kappa e_3(\kappa^2) = 0$ and then $e_3(\kappa^3) = 0$, and finally $e_3(H_2) = 0$. □

Theorem 3.11. Let $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ be a D_4-hypersurface with 1-proper second mean curvature vector field. If the ordinary mean curvature of M_1^3 is constant, then it is 1-minimal. Furthermore, all of mean curvatures of M_1^3 are null.

Proof. By Proposition 3.10, the 2th mean curvature of M_1^3 is constant, which by (**??**)(i), gives $L_1 H_2 = 9H_1H_2^2 - 3H_2H_3 = 0$, and then $3H_1H_2^2 = H_2H_3$, which using $H_1 = \kappa$, $H_2 = \kappa^2$ and $H_3 = \kappa^3$, gives $\kappa^5 = 0$, and then $H_1 = H_2 = H_3 = 0$. □

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Firooz Pashaie
Orcid number: 0000-0002-3020-7649
Department of Mathematics
University of Maragheh, P.O.Box 55181-83111
Maragheh, Iran
Email address: f_pashaie@maragheh.ac.ir

Naser Tanoomand-khooshmehr
Orcid number:0000-0003-0448-3030
Department of Mathematics
University of Maragheh, P.O.Box 55181-83111
Maragheh, Iran
Email address: nasertanoumand@gmail.com

Asghar Rahimi
Orcid number: 0000-0003-2095-6811
Department of Mathematics
University of Maragheh, P.O.Box 55181-83111
Maragheh, Iran
Email address: rahimi@maragheh.ac.ir

Leila Shahbaz
Orcid number: 0000-0001-6312-6231
Department of Mathematics
University of Maragheh, P.O.Box 55181-83111
Maragheh, Iran
Email address: l_shahbaz@maragheh.ac.ir