Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

THE COMPLEX-TYPE CYCLIC-FIBONACCI SEQUENCE AND ITS APPLICATIONS

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Article type: Research Article
(Received: 27 June 2022, Received in revised form 21 September 2022)
(Accepted: 30 September 2022, Published Online: 30 September 2022)

Abstract. In the present paper, we aim to generalize the notion of complex-type Fibonacci sequences to complex-type cyclic Fibonacci sequences. Firstly, we define the complex-type cyclic-Fibonacci sequence and then we give miscellaneous properties of this sequence by using the matrix method. Also, we study the complex-type cyclic-Fibonacci sequence modulo m. In addition, we describe the complex-type cyclic-Fibonacci sequence in a 2-generator group and investigate that in finite groups in details. Then, as our last result, we obtain the lengths of the periods of the complex-type cyclic-Fibonacci sequences in dihedral groups D_2, D_3, D_4, D_5, D_6 and D_8 with respect to their generating sets.

Keywords: The complex-type cyclic-Fibonacci sequence, Matrix, Group, Period.
2020 MSC: 11K31, 39B32, 11B50, 11C20, 20F05.

1. Introduction

As it is well-known, the following recurrence relation defines Fibonacci sequence:

$$F_k = F_{k-1} + F_{k-2}$$

for $k \geq 2$ and with initial conditions $F_0 = 0$ and $F_1 = 1$.

Also, further recurrence relation as below defines the Gaussian Fibonacci numbers $\{GF_n\}_{n=0}^{\infty}$

$$GF_{n+1} = GF_n + GF_{n-1}$$

for $n \geq 1$ and with initial conditions $GF_0 = i$ and $GF_1 = 1$, see [4].

In [14], the authors define the complex Fibonacci sequence $\{F_n^*\}$ as below for $n \geq 0$

$$F_n^* = F_n + iF_{n+1},$$

where F_n is the n^{th} Fibonacci number and $i = \sqrt{-1}$ is the imaginary unit [3,15].

Also, for any sequence $\{c_j\}_{j=0}^{k-1}$ of real numbers such that $c_{k-1} \neq 0$, $(k \geq 2)$, the k-generalized Fibonacci sequence $\{a_n\}_{n=0}^{+\infty}$ is given by the following relation:

$$a_{n+k} = c_{k-1}a_{n+k-1} + c_{k-2}a_{n+k-2} + \cdots + c_0 a_n$$

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DOI: 10.22103/jmmr.2022.19786.1292

Publisher: Shahid Bahonar University of Kerman
How to cite: Ö. Deveci, Ö. Erdağ, U. Güngöz, *The complex-type cyclic-Fibonacci sequence and its applications*, J. Mahani Math. Res. 2023; 12(2): 235-246.

for $n \geq 0$, where $a_0, a_1, \ldots, a_{k-1}$ are specified as initial conditions.

Using the companion matrix, the following closed-form formulas for the generalized sequence were given by Kalman [16].

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Also, he proved that

$$(A_k)^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

Various authors have studied many applications and interesting properties of the recurrence sequences related to those investigated in the present paper through the literature; see for example, [20, 23–25]. Especially, in [11] and [10], using the quaternions and complex numbers some new sequences were defined and many applications with miscellaneous properties of the sequences were obtained. In our paper, firstly a sequence so called the complex-type cyclic-Fibonacci sequence is defined and by the aid of the matrix method miscellaneous properties of the sequence are obtained.

We recall that when a sequence is composed only of repetitions of a fixed subsequence A sequence is periodic if after a certain points it consists only of repetitions of a fixed subsequence. We refer to the number of members in the shortest repeating subsequence as the period of the sequence. For instance, when a sequence with the terms $k, l, m, n, l, m, n, l, m, n, \ldots$ is considered, one would say it is periodic after the initial term k and it has period 3. Also, the first r terms in a sequence form a repeating subsequence, then it is said to be simply periodic with period r. For instance, when a sequence with the terms $k, l, m, n, k, l, m, n, k, l, m, n, \ldots$ is considered, one would say it is simply periodic with period 4.

It can be said that in the literature, Wall [26] started the research of the linear recurrence sequences modulo m by investigating the periods of the ordinary Fibonacci sequences modulo m. Recently, various authors such as in [13, 19, 21] have extended the theory to some special linear recurrence sequences.

For a finitely generated group $G = \langle B \rangle$, where $B = \{b_1, b_2, \ldots, b_n\}$, we refer to the sequence $x_u = b_{u+1}$, $0 \leq u \leq n-1$, $x_{n+u} = \prod_{v=1}^{n} x_{u+v-1}$, $u \geq 0$ as the Fibonacci orbit of G with respect to the generating set B, denoted as $F_B(G)$ in [6].

Also recall that a k-nacci (k-step Fibonacci) sequence in a finite group is defined via a sequence of group terms x_0, x_1, x_2, …, x_n, … which is obtained by the following recurrence formula when initial (seed) terms x_0, x_1, x_2, …, x_{j-1} are given:

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \leq n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \geq k. \end{cases}$$

Here, noting that once the terms of the seed set x_0, x_1, x_2, …, x_{j-1} generate a group, the k-nacci sequence will be reflecting the structure of the group. Thus, as in [18], $F_k\left(G; x_0, x_1, x_2, \ldots, x_{j-1}\right)$ denotes k-nacci sequence of a group G generated by x_0, x_1, x_2, …, x_{j-1}.

Note also that the orbit of a k-generated group is a k-nacci sequence.

From [10], we use the following definition as our preliminary information.

Definition 1.1. Let G be a k-generated group. For a generating k-tuple $(x_1, x_2, \ldots, x_k)$, the complex-type k-Finonacci orbit is defined by $a_i = x_{i+1}$, $(0 \leq i \leq k-1)$,

$$a_{n+k} = (a_n)^{i^k} (a_{n+1})^{i^{k-1}} \ldots (a_{n+k-1})^{i}, \ n \geq 0$$

where the following conditions are achieved for any $x, y \in G$ and any integer u:

(i). Let e be the identity of G and consider $z = a + ib$, where a, b are integers, then

$*\ x^z \equiv x^{a(\text{mod } |x|)+ib(\text{mod } |x|)} = x^{a(\text{mod } |x|)} x^{ib(\text{mod } |x|)} = x^{ib(\text{mod } |x|)} x^{a(\text{mod } |x|)} = x^{ib(\text{mod } |x|)+a(\text{mod } |x|)}$,

$*\ x^{ia} = \left(x^i\right)^a = (x^a)^i$,

$*\ e^u = e$,

$*\ x^{0+i0} = e$.

(ii). Given $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, b_1, a_2 and b_2 are integers, $y^{-z_2} x^{-z_1} = (x^{z_1} y^{z_2})^{-1}$.

(iii). If $yx \neq xy$, then $y^i x^i \neq x^i y^i$.

(iv). $y^i x^i = (xy)^i$ and $x^{-1} y^{-1} = \left(x^i y^i\right)^i$,

(v). $y^i x = x y^i$ and so $x^i y^{-1} = \left(x y^i\right)^i$ and $x^{-1} y^i = \left(x^i y\right)^i$.

As we earlier stated, Wall [26] started the study of the recurrence sequences in groups. In [27], the Fibonacci sequences in abelian groups was studied by Wilcox. Then, Campbell et al. [7] expanded the theory to some finite simple groups. In their study, the basic Fibonacci length of the basic Fibonacci orbit in a 2-generator group and the Fibonacci length of the Fibonacci orbit were defined. Moreover, some researchers have also considered the concept of the Fibonacci length for more than two generators, see for example [5, 6]. In [18], Knox indicated that a k-nacci sequence in a finite group is periodic. Recently, several authors have extended the theory to some special linear recurrence sequences, see for example [1, 2, 8, 9, 12, 17, 22]. Deveci and Shannon [10] defined the complex-type k-Fibonacci orbit of a k-generator group. They showed that when a k-generator group is finite, its complex-type k-Fibonacci orbit is periodic. In our paper, using the terms of 2-generator groups which is called the complex-type cyclic-Fibonacci orbit, the complex-type cyclic-Fibonacci sequence is redefined. Then the sequence in finite groups is examined in detail. Finally for some $n \geq 2$, the lengths of the periods of the complex-type cyclic-Fibonacci orbits of the dihedral group D_n are obtained as applications of the results obtained.

2. The Complex-type Cyclic-Fibonacci Sequence

Now for every integer $n \geq 1$, define the complex-type cyclic-Fibonacci sequence by the below homogeneous linear recurrence relation

$$f_{n+2}^{(c,i)} = \begin{cases} f_{n+1}^{(c,i)} + f_n^{(c,i)} & n \equiv 0 \, (\mathrm{mod}\ 4) \\ i\left(f_{n+1}^{(c,i)} + f_n^{(c,i)}\right) & n \equiv 1 \, (\mathrm{mod}\ 4) \\ -f_{n+1}^{(c,i)} - f_n^{(c,i)} & n \equiv 2 \, (\mathrm{mod}\ 4) \\ -i\left(f_{n+1}^{(c,i)} + f_n^{(c,i)}\right) & n \equiv 3 \, (\mathrm{mod}\ 4) \end{cases},$$

where $f_1^{(c,i)} = 0$, $f_2^{(c,i)} = 1$ and $i = \sqrt{-1}$. From the relations in the definitions of the complex-type cyclic-Fibonacci numbers, Fibonacci numbers and Gaussian Fibonacci numbers, we derive the following relation:

$$GF_n + F_n = (-1)^{n+1} \cdot \left(f_{4n-2}^{(c,i)} + f_{4n-4}^{(c,i)}\right)$$

for $n \geq 2$.

By setting

$$M = \begin{bmatrix} -1 & -i \\ i & 0 \end{bmatrix}$$

and by using an induction method on n, we find the relationship between the terms of the sequence $\left\{f_n^{(c,i)}\right\}$ and the matrix M as follows:

$$(M)^n = \begin{bmatrix} f_{4n+2}^{(c,i)} & \overline{f_{4n+1}^{(c,i)}} \\ f_{4n+1}^{(c,i)} & f_{4n+2}^{(c,i)} - i \cdot f_{4n+1}^{(c,i)} \end{bmatrix}.$$

We use the above definitions and define the matrices:

$$A_1 = \begin{bmatrix} i & i \\ 1 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -i & -i \\ 1 & 0 \end{bmatrix}$$

and

$$A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $M = A_4A_3A_2A_1$. Using the above identities, we define the following matrix:

$$B^n = A_uA_{u-1}\cdots A_1M^k,$$

where $n = 4k + u$ such that $u, k \in \mathbb{N}$. So we get

$$B^n\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}f_{n+1}^{(c,i)}\\ f_n^{(c,i)}\end{bmatrix} \tag{1}$$

for $n = 4k + u$ such that $u, k \in \mathbb{N}$.

Now we investigate the Simpson formulas of the complex-type cyclic-Fibonacci sequence.

If $n = 4k + 1$ $(k \in \mathbb{N})$, then

$$B^n = A_1M^k = \begin{bmatrix} f_{n+2}^{(c,i)} & -\overline{\left(f_{n+2}^{(c,i)} + f_{n+1}^{(c,i)}\right)} \\ f_{n+1}^{(c,i)} & -f_n^{(c,i)}\end{bmatrix}.$$

So we get

$$\left(f_{n+2}^{(c,i)}\right)\left(-f_n^{(c,i)}\right) - \left(f_{n+1}^{(c,i)}\right)\left(-\overline{\left(f_{n+2}^{(c,i)} + f_{n+1}^{(c,i)}\right)}\right) = (-1)^{k+1}\cdot i.$$

If $n = 4k + 2$ $(k \in \mathbb{N})$, then

$$B^n = A_2A_1M^k = \begin{bmatrix} f_{n+2}^{(c,i)} & \overline{f_{n+1}^{(c,i)}} \\ f_{n+1}^{(c,i)} & -\mathrm{Re}\left(f_{n+1}^{(c,i)}\right) + i\cdot\left[\mathrm{Im}\left(f_{n+1}^{(c,i)}\right) - \mathrm{Re}\left(f_{n+1}^{(c,i)}\right)\right]\end{bmatrix}.$$

So we get

$$\left(f_{n+2}^{(c,i)}\right)\left(-\mathrm{Re}\left(f_{n+1}^{(c,i)}\right) + i\cdot\left[\mathrm{Im}\left(f_{n+1}^{(c,i)}\right) - \mathrm{Re}\left(f_{n+1}^{(c,i)}\right)\right]\right) - \left(f_{n+1}^{(c,i)}\right)\left(\overline{f_{n+1}^{(c,i)}}\right) = (-1)^{k+1}\cdot i.$$

If $n = 4k + 3$ $(k \in \mathbb{N})$, then

$$B^n = A_3A_2A_1M^k = \begin{bmatrix} f_{n+2}^{(c,i)} & \mathrm{Re}\left(f_{n+1}^{(c,i)}\right) - \mathrm{Im}\left(f_{n+1}^{(c,i)}\right) \\ f_{n+1}^{(c,i)} & \overline{f_n^{(c,i)}}\end{bmatrix}.$$

So we get

$$\left(f_{n+2}^{(c,i)}\right)\left(\overline{f_n^{(c,i)}}\right) - \left(f_{n+1}^{(c,i)}\right)\left(\mathrm{Re}\left(f_{n+1}^{(c,i)}\right) - \mathrm{Im}\left(f_{n+1}^{(c,i)}\right)\right) = (-1)^k.$$

If $n = 4k + 4$ $(k \in \mathbb{N})$, then

$$B^n = M^{k+1} = \begin{bmatrix} f_{n+2}^{(c,i)} & -f_{n+1}^{(c,i)} \\ f_{n+1}^{(c,i)} & f_{n+2}^{(c,i)} - i\cdot f_{n+1}^{(c,i)}\end{bmatrix}.$$

So we get

$$\left(f_{n+2}^{(c,i)}\right)\left(f_{n+2}^{(c,i)} - i\cdot f_{n+1}^{(c,i)}\right) + \left(f_{n+1}^{(c,i)}\right)^2 = (-1)^{k+1}.$$

3. The Complex-type Cyclic-Fibonacci Sequence in Groups

If we reduce the sequence $\left\{f_n^{(c,i)}\right\}$ modulo m, taking the least nonnegative residues, we get the below recurrence sequence:

$$\left\{f_n^{(c,i)}(m)\right\} = \left\{f_1^{(c,i)}(m),\ f_2^{(c,i)}(m),\ \ldots,\ f_j^{(c,i)}(m),\ \ldots\right\},$$

where $f_j^{(c,i)}(m)$ refers to the n^{th} term of the complex-type cyclic-Fibonacci sequence when read modulo m. Here we may note that in the sequences $\left\{f_n^{(c,i)}\right\}$ and $\left\{f_n^{(c,i)}(m)\right\}$, the recurrence relations are the same.

Theorem 3.1. *The sequence $\left\{f_n^{(c,i)}(m)\right\}$ is periodic and the length of its period is divisible by 4.*

Proof. Consider the set

$$\begin{aligned} Q = & \{(z_1, z_2) \mid z_k\text{'s are complex numbers } a_k + ib_k \text{ where} \\ & a_k \text{ and } b_k \text{ are integers such that } 0 \leq a_k, b_k \leq m-1 \text{ and } k \in \{1,2\}\}. \end{aligned}$$

Assume $|Q|$ is the cardinality of the set Q. Because the set Q is finite, one can easily see that there are $|Q|$ distinct 2-tuples of the complex-type cyclic-Fibonacci sequence modulo m. Hence, it is easily seen that at least one of these 2-tuples appears twice in the sequence $\left\{f_n^{(c,i)}(m)\right\}$. Let $f_u^{(c,i)}(m) \equiv f_v^{(c,i)}(m)$ and $f_{u+1}^{(c,i)}(m) \equiv f_{v+1}^{(c,i)}(m)$. If $v - u \equiv 0 \pmod 4$, then we get $f_{u+2}^{(c,i)}(m) \equiv f_{v+2}^{(c,i)}(m)$, $f_{u+3}^{(c,i)}(m) \equiv f_{v+3}^{(c,i)}(m)$, ... Thus, it is clear that the subsequence following this 2-tuple repeats, that is, $\left\{f_n^{(c,i)}(m)\right\}$ is a periodic sequence and the length of its period must be divided by 4. □

Here $h_{f_n^{(c,i)}}(m)$ stands for the lengths of periods of the sequence $\left\{f_n^{(c,i)}(m)\right\}$. It can be clearly seen from the equation (1), $h_{f_n^{(c,i)}}(m)$ is the smallest positive integer α such that $B^\alpha \equiv I \pmod m$.

For any integer matrix $A = [a_{ij}]$, $A \pmod m$ refers that all entries of A are modulo m, that is, $(a_{ij} \pmod m) = A \pmod m$. Now we take the set $\langle A \rangle_m = \{(A)^n \pmod m \mid n \geq 0\}$ into consideration. If $(\det A, m) \neq 1$, then the set $\langle A \rangle_m$ is a semigroup. If $(\det A, m) = 1$, then the set $\langle A \rangle_m$ is a cyclic group. Also, since $\det M = -1$ for any positive integer $m \geq 2$, the set $\langle M \rangle_m$ is a cyclic group. From (1), it is clear that $h_{f_n^{(c,i)}}(m) = 4\left|\langle M \rangle_m\right|$.

Theorem 3.2. *Assume that p is a prime number. If t is the smallest positive integer such that $\left|\langle M \rangle_{p^{t+1}}\right| \neq \left|\langle M \rangle_{p^t}\right|$, then $\left|\langle M \rangle_{p^{t+1}}\right| = p\left|\langle M \rangle_{p^t}\right|$.*

Proof. Assume that α is a positive integer and $|\langle M\rangle_m|$ is denoted by $l_{f_n^{(c,i)}}(m)$. Let I be 2×2 identity matrix and $(M)^{l_{f_n^{(c,i)}}(p^{\alpha+1})} \equiv I\left(\bmod p^{\alpha+1}\right)$. Then we can derive $(M)^{l_{f_n^{(c,i)}}(p^{\alpha+1})} \equiv I\,(\bmod p^{\alpha})$, which means that $l_{f_n^{(c,i)}}(p^{\alpha})$ divides $l_{f_n^{(c,i)}}\left(p^{\alpha+1}\right)$. Moreover, we may write $(M)^{l_{f_n^{(c,i)}}(p^{\alpha})} = I + \left(m_{i,j}^{(\alpha)}\cdot p^{\alpha}\right)$, by the binomial theorem. Hence, we obtain:

$$(M)^{l_{f_n^{(c,i)}}(p^{\alpha})\cdot p} = \left(I + \left(m_{i,j}^{(\alpha)}\cdot p^{\alpha}\right)\right)^p = \sum_{n=0}^{p}\binom{p}{i}\left(m_{i,j}^{(\alpha)}\cdot p^{\alpha}\right)^n \equiv I\left(\bmod\ p^{\alpha+1}\right).$$

Then we have $(M)^{l_{f_n^{(c,i)}}(p^{\alpha})\cdot p} \equiv I\left(\bmod p^{\alpha+1}\right)$, which implies that $l_{f_n^{(c,i)}}\left(p^{\alpha+1}\right)$ divides $l_{f_n^{(c,i)}}(p^{\alpha})\cdot p$. Therefore, it can be clearly seen that $l_{f_n^{(c,i)}}\left(p^{\alpha+1}\right) = l_{f_n^{(c,i)}}(p^{\alpha})$ or $l_{f_n^{(c,i)}}\left(p^{\alpha+1}\right) = l_{f_n^{(c,i)}}(p^{\alpha})\cdot p$, and the latter is true if and only if there is an $m_{i,j}^{(\alpha)}$ which is not divisible by p. Since we suppose that t is the smallest positive integer such that $l_{f_n^{(c,i)}}\left(p^{t+1}\right) \neq l_{f_n^{(c,i)}}(p^{t})$, there is an $m_{i,j}^{(t)}$ which cannot be divided by p. This yields that $l_{f_n^{(c,i)}}\left(p^{t+1}\right) = l_{f_n^{(c,i)}}(p^{t})\cdot p$. So we have the conclusion. □

Theorem 3.3. *Let m_1 and m_2 be positive integers with $m_1, m_2 \geq 2$, then* $\left|\langle M\rangle_{\mathrm{lcm}[m_1,m_2]}\right| = \mathrm{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right]$.

Proof. Let $|\langle M\rangle_m|$ be denoted by $l_{f_n^{(c,i)}}(m)$ and let $\mathrm{lcm}[m_1,m_2] = m$. Clearly, $(M)^{l_{f_n^{(c,i)}}(m_1)} \equiv I\,(\bmod m_1)$ and $(M)^{l_{f_n^{(c,i)}}(m_2)} \equiv I\,(\bmod m_2)$. By means of the least common multiple operation, this implies that $(M)^{l_{f_n^{(c,i)}}(m)} \equiv I\,(\bmod m_1)$ and $(M)^{l_{f_n^{(c,i)}}(m)} \equiv I\,(\bmod m_2)$. So we get $\left|\langle M\rangle_{m_1}\right| \mid |\langle M\rangle_m|$ and $\left|\langle M\rangle_{m_2}\right| \mid |\langle M\rangle_m|$, which means that $\mathrm{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right]$ divides $\left|\langle M\rangle_{\mathrm{lcm}[m_1,m_2]}\right|$. Now we consider $\mathrm{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right] = \delta$. Then we can write $M^{\delta} \equiv I\,(\bmod m_1)$ and $M^{\delta} \equiv I\,(\bmod m_2)$, which yields that $M^{\delta} \equiv I\,(\bmod m)$. Thus, it is seen that $\mathrm{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right]$ is divisible by $\left|\langle M\rangle_{\mathrm{lcm}[m_1,m_2]}\right|$. So we have the conclusion. □

Assume G is a finite j-generator group and suppose that X is a subset of $\underbrace{G\times G\times\cdots\times G}_{j\text{ times}}$ such that $(x_1, x_2, \ldots, x_j) \in X$ if and only if G is generated by $x_1, x_2, \ldots, x_j$. Here, $(x_1, x_2, \ldots, x_j)$ is said to be a generating j-tuple for G.

Definition 3.4. Let G be a 2-generator group and (x_1, x_2) is a generating pair of G. Then, we define the complex-type cyclic-Fibonacci orbit by

$$c_1 = x_1,\ c_2 = x_2, c_n = \begin{cases} (c_{n-2})(c_{n-1}) & \text{for } n \equiv 0\,(\bmod 4) \\ (c_{n-2})^{i}(c_{n-1})^{i} & \text{for } n \equiv 1\,(\bmod 4) \\ (c_{n-2})^{-1}(c_{n-1})^{-1} & \text{for } n \equiv 2\,(\bmod 4) \\ (c_{n-2})^{-i}(c_{n-1})^{-i} & \text{for } n \equiv 3\,(\bmod 4) \end{cases},\quad (n > 2).$$

Let the notation $F^{(i,c)}_{(x_1,x_2)}(G)$ denote the complex-type cyclic-Fibonacci orbit of G for generating pair (x_1, x_2).

Theorem 3.5. *If G is a finite group, then the complex-type cyclic-Fibonacci orbit of G is a periodic sequence and the length of its period is divisible by 4.*

Proof. Consider the set

$$\begin{aligned} S &= \Big\{ \Big((s_1)^{a_1 (\bmod |s_1|) + i b_1 (\bmod |s_1|)}, (s_2)^{a_2 (\bmod |s_2|) + i b_2 (\bmod |s_2|)} : \\ &\quad \sqrt{-1} = i,\ s_1, s_2 \in G \text{ and } a_1, a_2, b_1, b_2 \in Z \Big\}. \end{aligned}$$

Since the group G is finite, S is a finite set. Then for any $u \geq 0$, there exists $v > u$ such that $c_u = c_v$ and $c_{u+1} = c_{v+1}$. If $v - u \equiv 0 \pmod 4$, then we get $c_{u+2} = c_{v+2}$, $c_{u+3} = c_{v+3}$, ... Then due to the repeating, for all generating pairs, the sequence $F^{(i,c)}_{(x_1,x_2)}(G)$ is periodic and the length of its period must be divided by 4. □

Now, consider that the length of the period of the orbit $F^{(i,c)}_{(x_1,x_2)}(G)$ is denoted by $LF^{(i,c)}_{(x_1,x_2)}(G)$. By the definition of the orbit $F^{(i,c)}_{(x_1,x_2)}$, it is easy to see that the chosen generating set and the order in which the assignments of x_1, x_2 are made determine the length of the period of this sequence in a finite group.

Now, we mark the lengths of the periods of the orbits $F^{(i,c)}_{(x,y)}(D_2)$, $F^{(i,c)}_{(x,y)}(D_3)$, $F^{(i,c)}_{(x,y)}(D_4)$, $F^{(i,c)}_{(x,y)}(D_5)$, $F^{(i,c)}_{(x,y)}(D_6)$ and $F^{(i,c)}_{(x,y)}(D_8)$. The dihedral group D_n of order $2n$ is defined as follows:

$$D_n = \langle x, y \mid x^n = y^2 = (xy)^2 = e \rangle$$

for every $n \geq 2$. Note that $|x| = n$, $|y| = 2$, $xy = yx^{-1}$ and $yx = x^{-1}y$. By direct calculation, we obtain the orbit $F^{(i,c)}_{(x,y)}(D_n)$ as follows:

$$\begin{aligned}
c_1 &= x,\ c_2 = y,\ c_3 = x^i y^i, \\
c_4 &= x^i y^{i+1},\ c_5 = y^i,\ c_6 = x^i y, \\
c_7 &= x y^{i+1},\ c_8 = x^{-1-i} y^i,\ c_9 = x^{-1-2i} y^i, \\
c_{10} &= x^{-2+i},\ c_{11} = x^{3-3i} y,\ c_{12} = x^{5+2i} y, \\
c_{13} &= x^{-1+2i},\ c_{14} = x^{6+4i} y,\ c_{15} = x^{-6-7i} y^i, \\
c_{16} &= x^{-11i} y^{i+1},\ c_{17} = x^{-4+6i} y^i,\ c_{18} = x^{4-17i} y, \\
c_{19} &= x^{-23-8i} y^{i+1},\ c_{20} = x^{27+9i} y^i,\ c_{21} = x^{17+50i} y^i, \\
c_{22} &= x^{44-41i},\ c_{23} = x^{-91+61i} y,\ c_{24} = x^{-135-20i} y, \\
c_{25} &= x^{41-44i},\ c_{26} = x^{-176-64i} y,\ c_{27} = x^{108+217i} y^i, \\
c_{28} &= x^{-68+281i} y^{i+1},\ c_{29} = x^{64-40i} y^i,\ c_{30} = x^{-132+321i} y, \\
c_{31} &= x^{361+196i} y^{i+1},\ c_{32} = x^{-493-125i} y^i,\ c_{33} = x^{-321-854i} y^i, \\
c_{34} &= x^{-814+729i},\ c_{35} = x^{1583-1135i} y,\ c_{36} = x^{2397+406i} y, \\
c_{37} &= x^{-729+814i},\ c_{38} = x^{3126+1220i} y,\ c_{39} = x^{-2034-3855i} y^i,
\end{aligned}$$

$$\begin{aligned}
c_{40} &= x^{1092-5075i}y^{i+1},\ c_{41} = x^{-1220+942i}y^{i},\ c_{42} = x^{2312-6017i}y,\\
c_{43} &= x^{-6959-3532i}y^{i+1},\ c_{44} = x^{9271+2485i}y^{i},\ c_{45} = x^{6017+16230i}y^{i},\\
c_{46} &= x^{15288-13745i},\ c_{47} = x^{-29975+21305i}y,\ c_{48} = x^{-45263-7560i}y,\\
c_{49} &= x^{13745-15288i},\ c_{50} = x^{-59008-22848i}y,\ c_{51} = x^{38136+72753i}y^{i},\\
c_{52} &= x^{-20872+95601i}y^{i+1},\ c_{53} = x^{22848-17264i}y^{i},\ c_{54} = x^{-43720+112865i}y,\\
c_{55} &= x^{130129+66568i}y^{i+1},\ c_{56} = x^{-173849-46297i}y^{i},\ c_{57} = x^{-112865-303978i}y^{i},\\
c_{58} &= x^{-286714+257681i},\ c_{59} = x^{561659-399579i}y,\ c_{60} = x^{848373+141898i}y,\\
c_{61} &= x^{-257681+286714i},\ c_{62} = x^{1106054+428612i}y,\ c_{63} = x^{-715326-1363735i}y^{i},\\
c_{64} &= x^{390728-1792347i}y^{i+1},\ c_{65} = x^{-428612+324598i}y^{i},\ c_{66} = x^{819340-2116945i}y,\\
c_{67} &= x^{-2441543-1247952i}y^{i+1},\ c_{68} = x^{3260883+868993i}y^{i},\ c_{69} = x^{2116945+5702426i}y^{i},\\
c_{70} &= x^{5377828-4833433i},\ c_{71} = x^{-10535859+7494773i}y,\ c_{72} = x^{-15913687-2661340i}y,\\
c_{73} &= x^{4833433-5377828i},\ c_{74} = x^{-20747120-8039168i}y,\ c_{75} = x^{13416996+25580553i}y^{i},\\
c_{76} &= x^{-7330124+33619721i}y^{i+1},\ c_{77} = x^{8039168-6086872i}y^{i},\ c_{78} = x^{-15369292+39706593i}y,\\
c_{79} &= x^{45793465+23408460i}y^{i+1},\ c_{80} = x^{-61162757-16298133i}y^{i},\ c_{81} = x^{-39706593-106956222i}y^{i},\\
c_{82} &= x^{-100869350+90658089i},\ c_{83} = x^{197614311-140575943i}y,\ c_{84} = x^{298483661+49917854i}y,\\
c_{85} &= x^{-90658089+100869350i},\ c_{86} = x^{389141750+150787204i}y,\ c_{87} = x^{-251656554-479799839i}y^{i},\\
c_{88} &= x^{137485196-630587043i}y^{i+1},\ c_{89} = x^{-150787204+114171358i}y^{i},\ c_{90} = x^{288272400-744758401i}y,\\
c_{91} &= x^{-858929759-439059604i}y^{i+1},\ c_{92} = x^{1147202159+305698787i}y^{i},\ c_{93} = x^{744758401+2006131918i}y^{i},\\
c_{94} &= x^{1891960560-1700433121i},\ c_{95} = x^{-3706565039+2636718961i}y,\ c_{96} = x^{-5598525599-936285840i}y,\\
c_{97} &= x^{1700433121-1891960560i},\ c_{98} = x^{-7298958720-2828246400i}y,\ c_{99} = x^{4720206960+8999391841i}y^{i},\\
c_{100} &= x^{-2578751760+11827638241i}y^{i+1},\ldots
\end{aligned}$$

Using the above information, the orbits $F_{(x,y)}^{(i,c)}(D_2)$, $F_{(x,y)}^{(i,c)}(D_3)$, $F_{(x,y)}^{(i,c)}(D_4)$, $F_{(x,y)}^{(i,c)}(D_5)$, $F_{(x,y)}^{(i,c)}(D_6)$ and $F_{(x,y)}^{(i,c)}(D_8)$ become, respectively:

$$\begin{aligned}
c_{13} &= x^{-1+2i} = x = c_1,\ c_{14} = x^{6+4i}y = y = c_2,\\
c_{15} &= x^{-6-7i}y^{i} = x^{i}y^{i} = c_3,\ c_{16} = x^{-11i}y^{1+i} = x^{i}y^{1+i} = c_4,\ldots,
\end{aligned}$$

$$\begin{aligned}
c_{97} &= x^{1700433121-1891960560i} = x = c_1,\\
c_{98} &= x^{-7298958720-2828246400i}y = y = c_2,\\
c_{99} &= x^{4720206960+8999391841i}y^{i} = x^{i}y^{i} = c_3,\\
c_{100} &= x^{-2578751760+11827638241i}y^{i+1} = x^{i}y^{1+i} = c_4,\ldots,
\end{aligned}$$

$$\begin{aligned}
c_{25} &= x^{41-44i} = x = c_1,\ c_{26} = x^{-176-64i}y = y = c_2,\\
c_{27} &= x^{108+217i}y^{i} = x^{i}y^{i} = c_3,\ c_{28} = x^{-68+281i}y^{i+1} = x^{i}y^{1+i} = c_4,\ldots,
\end{aligned}$$

$$\begin{aligned}
c_{97} &= x^{1700433121-1891960560i} = x = c_1,\\
c_{98} &= x^{-7298958720-2828246400i}y = y = c_2,\\
c_{99} &= x^{4720206960+8999391841i}y^{i} = x^{i}y^{i} = c_3,\\
c_{100} &= x^{-2578751760+11827638241i}y^{i+1} = x^{i}y^{1+i} = c_4,\ldots,
\end{aligned}$$

$$\begin{aligned} c_{97} &= x^{1700433121-1891960560i} = x = c_1, \\ c_{98} &= x^{-7298958720-2828246400i}y = y = c_2, \\ c_{99} &= x^{4720206960+8999391841i}y^i = x^i y^i = c_3, \\ c_{100} &= x^{-2578751760+11827638241i}y^{i+1} = x^i y^{1+i} = c_4, \ldots, \end{aligned}$$

and

$$\begin{aligned} c_{49} &= x^{13745-15288i} = x = c_1,\ c_{50} = x^{-59008-22848i}y = y = c_2, \\ c_{51} &= x^{38136+72753i}y^i = x^i y^i = c_3,\ c_{52} = x^{-20872+95601i}y^{i+1} = x^i y^{1+i} = c_4, \ldots. \end{aligned}$$

So we get $LF_{(x,y)}^{(i,c)}(D_2) = 12$, $LF_{(x,y)}^{(i,c)}(D_3) = 96$, $LF_{(x,y)}^{(i,c)}(D_4) = 24$, $LF_{(x,y)}^{(i,c)}(D_5) = 96$, $LF_{(x,y)}^{(i,c)}(D_6) = 96$ and $LF_{(x,y)}^{(i,c)}(D_8) = 48$.

Corollary 3.6. *For $n = 2^k$ such that $k \geq 2$, the length of the period of the complex-type cyclic-Fibonacci orbit $LF_{(x,y)}^{(i,c)}(D_n)$ is $6n$.*

Proof. From the orbit $F_{(x,y)}^{(i,c)}(D_n)$, we can deduce the following:

$$\begin{aligned} c_1 &= x,\ c_2 = y, \ldots, \\ c_{25} &= x^{41-44i},\ c_{26} = x^{-176-64i}y, \ldots, \\ c_{49} &= x^{13745-15288i},\ c_{50} = x^{-59008-22848i}y, \ldots, \\ c_{24u+1} &= x^{4u\lambda_1+1-4u\lambda_2 i},\ c_{24u+2} = x^{-4u\lambda_3-4u\lambda_4 i}y, \ldots, \end{aligned}$$

where $\gcd(\lambda_1, \lambda_2) = 1$. So we need an $u \in \mathbb{N}$ such that $4u = \tau n$ for $\tau \in \mathbb{N}$. If $n = 2^k$ such that $k \geq 2$, then $u = \frac{n}{4}$, and we obtain $LF_{(x,y)}^{(i,c)}(D_n) = 24\frac{n}{4} = 6n$. □

4. Conclusion and Discussion

In Section 2, we defined the complex-type cyclic-Fibonacci sequence and then we obtained the relationships among the elements of the sequence and the generating matrix of the sequence. Also, we gave the Simpson formula of the complex-type cyclic-Fibonacci sequence.

In Section 3, we studied the complex-type cyclic-Fibonacci sequence modulo m. Furthermore, we got the cyclic groups generated by reducing the multiplicative orders of the generating matrices and the auxiliary equations of these sequences modulo m and then, we investigated the orders of these cyclic groups. Moreover, using the terms of 2-generator groups which is called the complex-type cyclic-Fibonacci orbit, we redefined the complex-type cyclic-Fibonacci sequence. Also, the sequence in finite groups was examined in detail. Finally, for some $n \geq 2$ as applications of the results obtained, we got the lengths of the periods of the complex-type cyclic-Fibonacci orbits of the dihedral group D_n and we reached the length of the period of the complex-type cyclic-Fibonacci orbit $LF_{(x,y)}^{(i,c)}(D_n)$ for $n = 2^k$ when $k \geq 2$. One may consider the following open question whether or not it is possible to obtain some numerical results for other values of n and write general formulas for these results.

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