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ON NORM ESTIMATION FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS IN GEOMETRIC FUNCTIONS THEORY

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Abstract. We investigate on some subclasses of analytic fuctions defined by subordination. Also, we give estimates of $\sup_{|z|<1}(1-|z|^2)|\frac{f^{''}(z)}{f^{'}(z)}|$, for functions belonging to extended class of starlike functions. For a locally univalent analytic function f defined on $\Delta=\{z\in\mathbb{C}:|Z|<1\}$, we consider the pre-Schwarzian norm by $\|T\|=\sup_{|z|<1}(1-|z|^2)|\frac{f^{''}(z)}{f^{'}(z)}|$. In this work, we find the sharp norm estimate for the functions f in the extended classes of starlike functions.

Keywords: Analytic Functions, Starlike functions, Pre-schwarzian derivatives, Subordination
2020 MSC: 30C45.

1. Introduction

The class of all analytic functions was denoted by $\mathcal{A}$. Functions belonging to this class can be displayed in the form of the following power series

$$f(z)=z+\sum_{n=1}^{\infty}a_nz^n. \tag{1}$$

The class of univalent functions in $\mathcal{A}$ which normalized with the conditions $f(0)=f^{'}(0)-1=0$ was represented by $\mathcal{S}$.
Also, let S^* denote the class of starlike functions that is defined as

$$S^*=\left\{f\in\mathcal{S};\ Re\left(\frac{zf^{'}(z)}{f(z)}\right)>0,\ z\in\Delta\right\}.$$

The study of subclasses of analytic functions has always been the focus of many researchers [7, 9–13].
One of the important tools used to identify the subclasses of univalent analytical functions are pre-Schwarzian derivative [15, 16].

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For a locally univalent analytic function f, the pre-Schwarzian derivative of f is defined by

$$T = \frac{f^{''}(z)}{f^{'}(z)}$$

and its norm is defined by

$$\|T\| = \sup_{|z|<1} \big(1 - |z|^2\big) \Big| \frac{f^{''}(z)}{f^{'}(z)} \Big|.$$

It is essential to note that $\|T\| < \infty$ if and only if f is uniformly locally univalent in Δ. It is also to be noted that if $f \in \mathcal{S}$, then $\|T\| \leq 6$. Conversely, it follows from Becker's theorem, if $f \in \mathcal{A}$ and $\|T\| \leq 1$, then $f \in \mathcal{S}$. The results are sharp [1, 2]. For functions belonging to the class of convex functions, $\|T\| \leq 4$. According to Yamashita [16], if $f \in S^*(\alpha)$, then $\|T\| \leq 6 - 4\alpha$. Bhowmik et al. [3] have obtained the estimate of the norm as $4 < \|T\| < 2\alpha + 2$ for functions in the class of concave univalent functions of order α.
The pre-Schwarzian derivative T and its norm, $\|T\|$, have important meanings in the theory of the Teichmuller space.

For example, other subclasses that have been studied by researchers using the pre-Schwarzian derivative can be mentioned as follows:
A function $f \in \mathcal{A}$ is called strongly starlike of order α if

$$\Big| arg \frac{zf^{'}(z)}{f(z)} \Big| < \frac{\alpha\pi}{2},$$

in Δ, where $0 \leq \alpha < 1$.
T. Sugawa in [15], has obtained the estimate of the pre-Schwarzian derivative norm for strongly starlike of order α, as

$$\|T\| \leq M(\alpha) + 2\alpha, \tag{2}$$

where $M(\alpha)$ is given by

$$M(\alpha) = \frac{4\alpha c(\alpha)^{\alpha+1}}{c(\alpha)^2 + 1},$$

and $c(\alpha)$ is the unique solution of the following equation with respect to x in the interval $(1, \infty)$:

$$\big(1 - \alpha\big)x^{\alpha+2} + (1 + \alpha)x^{\alpha} - x^2 - 1 = 0.$$

Moreover, the equality in (2) is sharp.
We say, f is subordinate to g and is written as $f(z) \prec g(z)$, if there exists a function $w(z)$ belonging to class of Schwartz functions which are satisfying $w(0) = 0$ and $|w(z)| < 1$, such that

$$f(z) = g(w(z))$$

where, f and g are analytic in Δ.
Let g be a univalent function, $f(z) \prec g(z)$ if and only if $f(0) = 0$ and $f(\Delta) \subset g(\Delta)$.
A function is called bi-univalent in open unit disk Δ, if f and f^{-1} are univalent in open unit disk Δ.
σ is considered a symbol of the class of bi-univalent functions Δ. For more information on the class σ, readers can refer to [14].
Also, for study the class of bi-univalent functions, we can use pre-Schwarzian derivative, for example S. Rana et al. [8] have obtained the estimate of the norm as

$$\|T\| \leq \min\left\{\frac{2(A-B)(A+2)}{A+1}, \frac{2(A-B)|A|}{(A+1)}\right\},$$

for functions in the class $S^*_\sigma[A,B]$ which is defined as follows:

Definition 1.1. A function f given by (1) is said to be in the class $S^*_\sigma[A,B]$, if the following conditions are satisfied

$$\frac{zf^{'}(z)}{f(z)} \prec \frac{1+Az}{1+Bz},$$
$$\frac{wg^{'}(w)}{g(w)} \prec \frac{1+Aw}{1+Bw},$$

where $f \in \sigma$, $g = f^{-1}$, $w = f(z)$, $w \in \Delta$ and $-1 \leq A, B \leq 1$.

For the first time, Ma and Minda [5] extended the class of starlike functions by using of subordination method.
In fact, they introduced $S^*(\psi)$ as follow:

$$S^*(\psi) = \left\{f \in \mathcal{S};\ \frac{zf^{'}(z)}{f(z)} \prec \psi(z)\right\},$$

where, ψ is analytic and $Re\left\{\psi\right\} > 0,\ \psi(0) = 1,\ \psi^{'}(0) > 0$.
Recently, many researchers motivated by the work of Ma and Minda [5], introduced and studied some interesting extended class of starlike functions by choosing suitable ψ.
In this work, we find the sharp norm estimate for the functions f in the extended classes of starlike functions.
For our main purpose in this work, we need definitions of some extended classes of starlike functions. S^*_l, S^*_e, and S^*_c denote the classes of extended starlike

functions, respectively, Following [6, 9, 13] these are defined as:

$$S_l^* = \left\{ f \in \mathcal{S};\ \frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2},\ z \in \Delta \right\}, \tag{3}$$

$$S_e^* = \left\{ f \in \mathcal{S};\ \frac{zf'(z)}{f(z)} \prec e^z,\ z \in \Delta \right\}, \tag{4}$$

$$S_c^* = \left\{ f \in \mathcal{S};\ \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2,\ z \in \Delta \right\}. \tag{5}$$

2. Norm estimation for the classes S_l^*, S_e^*, and S_c^*

In this section, we obtain upper bound of $\|T\|$ for functions belonging to some extended class of starlike functions.

Theorem 2.1. *Let the function $f(z)$ given by* (1) *be in the class S_l^*. Then $\|T\| \le 2$.*

Proof. By (3), we have,

$$\frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2},\ z \in \Delta,$$

now, the definition of subordination yields that

$$\frac{zf'(z)}{f(z)} = w(z) + \sqrt{1+w(z)^2}, \tag{6}$$

where $w(z)$ is Schwartz function. Applying the Schwarz-Pick lemma, we get

$$\left| w^{'}(z) \right| \le \frac{1-|w(z)|^2}{1-|z|^2},\ z \in \Delta.$$

Logarithmic differentiation of (6) gives

$$\frac{1}{z} + \frac{f^{''}(z)}{f^{'}(z)} - \frac{f^{'}(z)}{f(z)} = \frac{w^{'}(z)}{\sqrt{1+w^2(z)}}. \tag{7}$$

From (6) and (7), we obtain the pre-Schwarzian derivative of f as follows:

$$T = \frac{f^{''}(z)}{f^{'}(z)} = \frac{1}{z}\left(w(z) + \sqrt{1+w^2(z)} \right) - \frac{1}{z} + \frac{w^{'}(z)}{\sqrt{1+w^2(z)}}.$$

By using the Schwarz-pick lemma and triangle inequality we conclude that,

$$\left| \frac{f^{''}(z)}{f^{'}(z)} \right| \le \frac{1}{|z|}\left(|w(z)| + \sqrt{1+w^2(z)} + 1 \right) + \frac{1-|w(z)|^2}{(1-|z|^2)\sqrt{1+w^2(z)}}. \tag{8}$$

Multiplying the inequality (8) by $(1-|z|^2)$, we get

$$(1-|z|^2)\left| \frac{f^{''}(z)}{f^{'}(z)} \right| \le \frac{1-|z|^2}{|z|}\left(|w(z)| + \sqrt{1+w^2(z)} + 1 \right) + \frac{1-|w(z)|^2}{\sqrt{1+w^2(z)}}. \tag{9}$$

Therefore, by using the inequality $|w(z)| \leq |z|$ for all $z \in \Delta$, we see that

$$(1-|z|^2)\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(|z|+\sqrt{1+|z|^2}+1\right)+1+|z|. \tag{10}$$

Taking the supremum value from both sides in the unit disc, the inequality (10) becomes

$$\sup_{z\in\Delta}(1-|z|^2)\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \sup_{z\in\Delta}(1+|z|).$$

This completes the proof. □

Theorem 2.2. *Let the function* $f(z)$ *given by* (1) *be in the class* S_e^*. *Then* $\|T\| \leq 2$.

Proof. Since $f \in S_e^*$, we have

$$\frac{zf^{'}(z)}{f(z)} \prec e^z.$$

Using the definition of subordination, there exists a Schwarz function $w(z)$ with $w(0)=0$ and $|w(z)|<1$ such that

$$\frac{zf^{'}(z)}{f(z)} = e^{w(z)}. \tag{11}$$

By logarithmic differentiation of (11), we ge

$$\frac{1}{z}+\frac{f^{''}(z)}{f^{'}(z)}-\frac{f^{'}(z)}{f(z)} = w^{'}(z). \tag{12}$$

Also, equation (11) gives us,

$$\frac{f^{'}(z)}{f(z)} = \frac{1}{z}e^{w(z)}. \tag{13}$$

Equations (12) and (13) give us the pre-Schwarzian derivative of f as follows

$$T = \frac{f^{''}(z)}{f^{'}(z)} = \frac{1}{z}\left(e^{w(z)}-1\right)+w^{'}(z).$$

By applying triangle inequality and the Schwarz-Pick lemma, we obtain

$$\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \frac{1}{|z|}\left(e^{|w(z)|}-1\right)+\frac{1-|w(z)|^2}{1-|z|^2},\ z\in\Delta.$$

For all $z \in \Delta$, we have

$$(1-|z|^2)\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(e^{|w(z)|}-1\right)+1-|w(z)|^2.$$

Since $w(z)$ is Schwarz function and $|w(z)| \leq |z|$, we have

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(e^{|z|}-1\right)+1+|z|.$$

We take the lower limit as $z \to 1^-$ to obtain the following inequality

$$\sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq \sup_{z\in\Delta}\left\{\frac{1-|z|^2}{|z|}\left(e^{|z|}-1\right)+1+|z|\right\}.$$

This completes the proof. □

Theorem 2.3. *Let the function* $f(z)$ *given by* (1) *be in the class* S_c^*. *Then* $\|T\| \leq \frac{8}{3}$.

Proof. In order to prove this theorem, we use a similar procedure. Since $f \in S_c^*$, we have

$$\frac{zf'(z)}{f(z)} \prec 1+\frac{4}{3}z+\frac{2}{3}z^2.$$

By using subordination method we see that,

$$\frac{zf'(z)}{f(z)} = 1+\frac{4}{3}w(z)+\frac{2}{3}w(z)^2, \tag{14}$$

where $w(z)$ is the Schwarz function. By logarithmic differentiation of (14), we get

$$\frac{1}{z}+\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)} = \frac{\frac{4}{3}w'(z)\left(1+w(z)\right)}{1+\frac{4}{3}w(z)+\frac{2}{3}w(z)^2}. \tag{15}$$

From (14), we have

$$\frac{f'(z)}{f(z)} = \frac{1}{z}\left(1+\frac{4}{3}w(z)+\frac{2}{3}w(z)^2\right). \tag{16}$$

By the help of the Schwarz-Pick lemma and triangle inequality, from (15) and (16), we obtain

(17)

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{|z|}\left(\frac{4}{3}|w(z)|+\frac{2}{3}|w(z)|^2\right)+\frac{4}{3}\left(\frac{1-|w(z)|^2}{1-|z|^2}\right)\left|\frac{1+w(z)}{1+\frac{4}{3}w(z)+\frac{2}{3}w(z)^2}\right|.$$

Consequently, for all $z \in \Delta$, we conclude that,

(18)

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(\frac{4}{3}|w(z)|+\frac{2}{3}|w(z)|^2\right)+\frac{4}{3}\left(1-|w(z)|^2\right)\left|\frac{1+w(z)}{1+\frac{4}{3}w(z)+\frac{2}{3}w(z)^2}\right|.$$

By the inequality $|w(z)| \leq |z|$ for all $z \in \Delta$, we obtain

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(\frac{4}{3}|z|+\frac{2}{3}|z|^2\right)+\frac{4}{3}\left(1+|z|\right). \tag{19}$$

Hence, the limit of $(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|$ and $\frac{1-|z|^2}{|z|}\left(\frac{4}{3}|z|+\frac{2}{3}|z|^2\right)+\frac{4}{3}\left(1+|z|\right)$ exist when z tends to 1 from the left and it equals the supremum of $(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|$ and $\frac{1-|z|^2}{|z|}\left(\frac{4}{3}|z|+\frac{2}{3}|z|^2\right)+\frac{4}{3}\left(1+|z|\right)$. Therefore

$$\sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq \sup_{z\in\Delta}\left\{\frac{1-|z|^2}{|z|}\left(\frac{4}{3}|z|+\frac{2}{3}|z|^2\right)+\frac{4}{3}\left(1+|z|\right)\right\}.$$

This completes the proof. □

3. Conclusion

The norm of pre-Schwarzian derivatives has always been the main interest for researchers to study the Univalent and bi-Univalent classes. Many studies related to this problem are around analytic normalized functions. Here, we find the sharp norm estimate for the functions f in the extended classes of starlike functions. In future research, we may obtain the bounds of the norm of pre-Schwarzian derivatives for classes of functions which are defined in terms of the quantum derivative operators.

References

[1] Becker. J,*Lownersche differentialgleichung and quasikonform fortsetzbare schlichte funktionen*, J. Reine Angew. Math., 255 (1972), 23–43.

[2] Becker. J, Pommerenke. C, *Schlichtheitskriterien und Jordangebiete*, J. Reine Angew. Math., 354 (1984), 74–94.

[3] Bhowmik. B, Ponnusamy. S, Wirths. K. J, *Schlichtheitskriterien und Jordangebiete*, J. Reine Angew. Math., 354 (1984), 74–94.

[4] Duren. P, *A Univalent Functions*, in: Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, (1983).

[5] Ma. W. C, Minda. D, *A unified treatment of some special classes of univalent function*, in: Proceedings of the Conference on Complex Analysis, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA. (Tianjin, 1992), 157-169.

[6] Mendiratta. R, Nagpal. S, Ravichandran. V, *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc. 38 (2015), no. 1, 365–386.

[7] Najafzadeh. Sh, Rahmatan. H, Haji. H, *Application of subordination for estimating the Hankel determinant for subclass of univalent functions*, Creat. Math. Inform, **30**(2021), 69–74.

[8] Rana. S, Goswami. P, Dubey. R. S, *The norm of pre-Schwarzian derivative on subclasses of bi-univalent functions*, AIMS Mathematics, 3(4)(2018), 600–607

[9] Raina. R. K, Sokol. J, *Some properties related to a certain class of starlike functions*, C. R. Math. Acad. Sci. Paris 353 (2015), no. 11, 973–978.
[10] Rahmatan. H, Shokri. A, Ahmad. H, Botmart. T *Subordination Method for the Estimation of Certain Subclass of Analytic Functions Defined by the q-Derivative Operator*, Journal of Function Spaces, Volume 2022, Article ID 5078060, 9 pages
[11] Rahmatan. H, Haji. H, Najafzadeh. SH, *Coefficient estimates and Fekete-Szegö coefficient inequality for new subclasses of Bi-univalent functions*, Caspian Journal of Mathematical Sciences, **10**(1)(2021), 39–50.
[12] Rahmatan. H, Najafzadeh. SH, Haji. H, *Initial coefficient bounds for interesting subclasses of meromorphic and bi-univalent functions*, J. Mahani Math. Res. Cent. 11(2)(2022), 9–17.
[13] Sharma. K, Jain. N. K, Ravichandran. V,*Starlike functions associated with a cardioid*, Afr. Mat. 27 (2016), no. 5-6, 923–939.
[14] Srivastava. H. M, Mishra. A. K and Gochhayat. P., *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. vol. 23(2010), pp. 1188–1192.
[15] Sugawa. T,*On the norm of pre-Schwarzian derivatives of strongly starlike functions*, Kyoto University. 9(1997), 178–185.
[16] Yamashita. S, *Norm estimates for function starlike or convex of order alpha*, Hokkaido Math. J., 28 (1999), 217–230.

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