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EQUIVALENCE OF SEQUENTIAL HENSTOCK AND TOPOLOGICAL HENSTOCK INTEGRALS FOR INTERVAL VALUED FUNCTIONS

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Abstract. Suppose X is a locally compact Hausdorff space and $\Omega \in \triangle$. If F is an interval valued function defined in Ω with $F : \bar{\Omega} \to I_{\mathbb{R}}$. Suppose F is Topological Henstock integrable, is F Sequential Henstock integrable? Therefore, the purpose of this paper is to provide a positive response to this query.

Keywords: Sequential Henstock integral, Interval valued functions, Topological Henstock, guages.
2020 MSC: Primary 28C15, 28C20, 58C35, 46G12.

1. Introduction

A demonstrated generalization of the Riemann integral is the Henstock integral. It was introduced independently in the mid-1950s by R. Henstock and J. Kursweil to correct the deficiencies of the Riemann integral. Henstock integral is efficient in handling functions with high discontinuites and extreme oscillation as well as gives a simpler version of the link between integration and differentiation (see,[1,3,4,6,7,12,13,14,15,16,17]). Although the $\varepsilon - \delta$ concept is used in the Henstock integral's standard definition, the usage of sequences of gauge functions was then presented, along with the Sequential Henstock integral concept.(see [13]). The authors [5,6] discussed the equivalence results for certain Henstock integrals, Sequential Henstock integral and p-Henstock type integrals.

In 2018, Ray [14] introduced the notion of equivalence of Riemann integrals based on p-norm. Kim[8] established the equivalence of Perron, Henstock and variational Stieltjes Integral. Recently, the authors[5,6] and Paxton[13] proved the following, respectively for a real valued functions

(R_1). The Sequential Henstock integral and the Henstock integral are equal, i.e., $H_F[a,b] = SH_F[a,b]$.

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(R_2). The Topological Henstock and the Sequential Henstock integral are equal on $[a,b] \subset \mathbb{R}$, i.e., $DH_F[a,b] = SH_F[a,b]$.

In view of the above results, we raise the following common question: Is a function that is Topological Henstock integrable equivalent to a function that is the Sequential Henstock integrable when dealing with an interval-valued function? Therefore, this paper is aimed at answering this question and discuss an applicable example.

The following is an example of definition.

Definition 1.1 (13, 16)**.** A gauge function on $[a,b]$ is a positive real-valued function $\delta : [a,b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$

Definition 1.2 (13, 16)**.** A sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n}\, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, where $[u_{i-1}, u_i] \in [a,b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} < ... < u_{m_n} = b$.

Definition 1.3 (13)**.** A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable on $[a,b]$ if there exists a number $\alpha \in \mathbb{R}$ such that if $\varepsilon > 0$ there exists a function $\delta(x) > 0$ such that for $\delta(x)$-fine tagged partitions $P = \{(u_{i-1}\, u_i), t_i\}_{i=1}^{n}$, we have

$$|\sum_{i=1}^{n} f(t_i)[u_i - u_{(i-1)}] - \alpha| < \varepsilon,$$

where the number α is the Henstock integral of f on $[a,b]$. The family of all Henstock integrable functions on $[a,b]$ is denoted by $H[a,b]$ with $\alpha = (H)\int_{[a,b]} f(x)dx$.

The following terms have their definitions in a topological space.

Suppose that the Hausdorff space $X*$ is locally compact with subspace $\Omega \subset X^*$. We denote the closure of Ω as $\bar{\Omega}$ and the interior as $Int\Omega$. Let $\triangle$ be a family of subsets of X such that
i. If $\Omega \in \triangle$, then $\bar{\Omega}$ is compact.
ii. For each $x \in X$, the collection $\triangle(x) = \{\Omega \in \triangle | x \in Int\Omega\}$ is a neighbourhood base at x.
iii. If $\Omega, \omega \in \triangle$, then $\Omega \cap \omega \in \triangle$ and disjoint sets $C_1, ..., C_k \in \triangle$ such that $\Omega - \omega = \bigcup_{i=1}^{k} C_i$ do exist.

A gauge (topological) of $\Omega \in \triangle$ is a function U mapping to every $x \in \bar{\Omega}$ a neighbourhood $U(x)$ of x contained in X^*.

A division (topological) of $\Omega \in \triangle$ is a disjoint collection $\{\Omega_1, ..., \Omega_k\} \subset \triangle$ such that $\bigcup_{i=1}^{k} \Omega_i = \Omega$.

A partition (topological) of $\Omega \in \triangle$ is a set $P = \{(\Omega_1, t_1), ..., (\Omega_k, t_k)\}$ such that $\{\Omega_1, ..., \Omega_k\}$ is a division of Ω and $\{t_1, ..., t_k\} \subset \bar{\Omega}$. If U is a gauge on Ω, we say the partition P is U-fine if $\Omega_i \subset U(x)$ for $i = 1, 2, ..., k$.

A volume is a non-negative function such that $\nu(U) = \sum_{i=1}^{k} \nu(U_i)$ for every $\Omega \in \triangle$ and each division $\{\Omega_1, ..., \Omega_k\}$ of Ω.
Note: Volume here is intuitively defined as the "length" of the "intervals".

Definition 1.4 (13)**.** Suppose the Hausdorff space X^* is locally compact and $\Omega \in \triangle$ with $f : \bar{\Omega} \to \mathbb{R}$, then f is Topological Henstock integrable to $\alpha \in \mathbb{R}$ if for every $\varepsilon > 0$, neighbourhood $U(x) > 0$ exists such that $|\sum_{i=1}^{n} f(t_i)v(U_i) - \int_\Omega f| = |\sigma(f, P) - \int_\Omega f| < \varepsilon$ for every $U(x)$-fine partition P of Ω, where $\int_\Omega f = \alpha$ and $\sigma(f, P) = \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})$.

This Henstock integral uses the concept of neighbourhood system of a Topological space to define the integral value of the Topological space valued functions.

Definition 1.5 (13)**.** If there is a number α and a series of positive functions $\{\delta_n(x)\}_{n=1}^{\infty}$, then a function $f : [a, b] \to \mathbb{R}$ is Sequential Henstock integrable on $[a, b]$, if for any $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$S(f, P_n) = d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \to \text{as} \qquad n \to \infty,$$

we say that α is the Sequential Henstock integral of f on $[a, b]$ with $\alpha = \int_{[a,b]} f$.

Remark 1.6. If $\delta_n = \delta$, for all $n \in \mathbb{N}$, then we have a definition for Henstock integral.

Definition 1.7 (12)**.** Let $I_\mathbb{R} = \{I = [I^-, I^+]$: I be a closed bounded interval on the real line $\mathbb{R}\}$.
For $X^*, Y^* \in I_\mathbb{R}$, we define
i. $X^* \leq Y^*$ if and only if $Y^{*(-)} \leq X^{*(-)}$ and $X^{*(+)} \leq Y^{*(+)}$,
ii. $X^* + Y^* = Z^*$ if and only if $Z^{*(-)} = X^{*(-)} + Y^{*(-)}$ and $Z^{*(+)} = X^{*(+)} + Y^{*(+)}$,
iii. $X^*.Y^* = \{x.y : x \in X^*, y \in Y^*\}$, where

$$(X.Y)^{*(-)} = \min\{X^{*(-)}.Y^{*(-)}, X^{*(-)}.Y^{*(+)}, X^{*(+)}.Y^{*(-)}, X^{*(+)}.Y^{*(+)}\}$$

and

$$(X.Y)^{*(+)} = \max\{X^{*(-)}.Y^{*(-)}, X^{*(-)}.Y^{*(+)}, X^{*(+)}.Y^{*(-)}, X^{*(+)}.Y^{*(+)}\}.$$

Define $d(X^*, Y^*) = \max(|X^{*(-)} - Y^{*(-)}|, |X^{*(+)} - Y^{*(+)}|)$ as the distance between X^* and Y^*.

Now, we will define newly the interval Sequential Henstock (ISH) integral.

Definition 1.8. A function $F^* : [a,b] \to I_{\mathbb{R}}$ with interval values is Sequential Henstock integrable(ISH) to $I_0 \in I_{\mathbb{R}}$ on $[a,b]$ if for any $\varepsilon > 0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for every $\delta_n(x)-fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} F^*(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0) < \varepsilon.$$

We say that α is the Sequential Henstock integral of F^* on $[a,b]$ with $(IH)\int_{[a,b]} F^* = \alpha$ and $F^* \in ISH[a,b]$ where $I_0 = [I_0^-, I_0^+]$.

Definition 1.9. Suppose the Hausdorff space X^* is locally compact and let $\Omega \in \triangle$ with $F^* : \bar{\Omega} \to I_{\mathbb{R}}$, then F^* is Topological Henstock integrable(ITH) to $I_0 \in I_{\mathbb{R}}$ if for any $\varepsilon > 0$ there exists a neighbourhood $U_n(x) > 0$ such that

$$|\sum_{i=1}^{n} F^*(t_{i_n})\nu(U_{i_n}) - \int_{\Omega} F^*|$$

$$= \max|d(\sum_{i=1}^{n} F^{*(-)}(t_{i_n})\nu(U_{i_n})), I_0^-) < \varepsilon, d(\sum_{i=1}^{n} F^{*(+)}(t_{i_n})\nu(U_{i_n})), I_0^+) < \varepsilon|$$

for every $U_n(x)-$ fine partition P_n of Ω, where $(ITH)\int_{\Omega} F^* = I_0 = [I_0^-, I_0^+]$

Remark 1.10. It is obvious that if $F^* = F^{*(-)} = F^{*(+)}$ for all $x \in [a,b]$, the Definitions 1.8 and 1.9 implies the real-valued Sequential Henstock and Topological Henstock integrals.

Proposition 1.11. *$F^* \in ITH$ if and only if $F^{*(-)}, F^{*(+)} \in SH$.*

Proof. Suppose $F^* \in ITH$. Then $F^{*(-)}, F^{*(+)} \in TH$. Then by Theorems 2.3 and 2.4 of [8] it has been shown that TH is equivalent to SH, so we have that if $F^* \in ITH$, then $F^{*(-)}, F^{*(+)} \in SH$ which in-turns implies that $F^* \in ISH$. Hence, $F^* \in ITH \implies F^{*(-)}, F^{*(+)} \in SH$.
Conversely, Suppose $F^{*(-)}, F^{*(+)} \in SH$, then $F^* \in ISH$. We have that SH is equivalent to TH(Theorem 1 of [10]). If $F^{*(-)}, F^{*(+)} \in SH$, then $F^{*(-)}, F^{*(+)} \in TH$. Since by Theorem 3.2 of [10] $F^{*(-)}, F^{*(+)} \in SH$ we have $F^* \in ISH$. Then it follows that $F^{*(-)}, F^{*(+)} \in SH \implies F^* \in ITH$.

□

2. Main Results

We state and prove the following theorems:

Theorem 2.1. *Let X^* be a locally compact Hausdorff space. Then The interval Topological Henstock is equivalent to interval Sequential Henstock integral, i.e., $(ITH)\int_{\Omega} F^* = (ISH)\int_a^b F^*$ on $I = [a,b]$.*

Proof. Since the Hausdorff space X^* is locally compact then by Heine - Borel's theorem, each $[u_{i-1}, u_i] \subset [u_{(i-1)_n}, u_{i_n}] \subset [a, b] \subset \mathbb{R}$ is compact. Hence, any point $I_0 \in I_{\mathbb{R}}$ can be found in the open interval $[a, b]$, which in turn is contained in the neighbourhood $[U_{(i-1)_n}, U_{i_n}] \subset X^*$, so that $I_{\mathbb{R}}$ becomes a locally compact Hausdorff space with intervals. Therefore, we demonstrate the outcome for $X = \max[I^-_{\mathbb{R}}, I^+_{\mathbb{R}}]$ and the topological partition $P_n = \{(\Omega_{n_1}, t_{n_1}), ..., (\Omega_{n_k}, t_{n_k}) \subset [a, b] : t \in [a, b], a, b \in \mathbb{R}, a < b\}$ of Ω is proved. The interval Sequential Topological Henstock integral so reduces to Sequential Henstock integral under this condition. Hence by proposition 1.12, Definition 1.8 implies Definition 1.9. □

Theorem 2.2. *Suppose X^* is a locally compact Hausdorff space. Then The interval Sequential Henstock is equivalent to interval Henstock integral, i.e., then, $(ISH)\int_\Omega F^* = (ITH)\int_a^b F^*$ on $I = [a, b] \subset \mathbb{R}$.*

Proof. For $\Omega \in \triangle$, let $v(U_n) = u_{i_n} - u_{(i-1)_n}$ and sequence of positive gauges $U_{\delta_n}(t_{i_n})$ on Ω such that $U_{\delta_n}(t_{i_n}) = (t_{i_n} - \delta_n(t_{i_n}), t_{i_n} + \delta_n(t_{i_n})), (i = 1, 2, ...k)$ is a U_n- fine partitions P_n on Ω with positive sequence $\{\delta_n(x)\}_{n=1}^{\infty}$ for each $x \in [a, b]$. Thus, $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ is $\delta_n(x) - fine$ partitions. Since $[u_{(i-1)_n}, u_{i_n}] \subset (t_{i_n} - \delta_n(t_{i_n}), t_{i_n} + \delta_n(t_{i_n})), i = 1, 2, ...k$. Therefore, for any closed intervals $[a, b] \subset \mathbb{R}$ Definition 1.10 holds. For every $\varepsilon > 0$, there exists a positive sequence $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for all $\delta_n(x) - fine$ tagged partitions, we have

$$|\sum_{i=1}^{m_n} F^*(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \int_a^b F^*|$$

$$= \max|d(\sum_{i=1}^{n} F^{*(-)}(t_{i_n})\nu(U_{i_n})), I_0^-) < \varepsilon, d(\sum_{i=1}^{n} F^{*(+)}(t_{i_n})\nu(U_{i_n})), I_0^+) < \varepsilon|.$$

Hence, The interval Sequential Henstock is equivalent to interval Henstock integral. $(ISH)\int_a^b F^* = (ITH)\int_\Omega F^*$ on $X^* \subset \mathbb{R}$, when the sets in $\triangle$ are $\Omega = [a, b] \subset \mathbb{R}$. Thus by Proposition 1.12, Definition 1.9 implies Definition 1.8. □

Corollary 2.3. *A function $F^* : [a, b] \to \mathbb{R}$ with interval values is Sequential Henstock integrable on $[a, b]$ if and only if it is Topological Henstock integrable there. .*

Proof. The proof follows easily from Theorems 2.1 and 2.2. This completes the proof. □

Example 2.4. *Suppose that $[a, b] = [0, 1]$. If $Q \subset [0, 1]$ and $F^* : [0, 1] \to I_{\mathbb{R}}$ be defined by*

$$F^*(x) = \begin{cases} [-1, 0], & if\ x \in Q \\ [1, 2], & if\ x \in [0, 1]\backslash Q. \end{cases}$$

Suppose we define our gauge

$$\delta_n(x) = \begin{cases} \frac{1}{n2^n}, & \text{if } x \in [0,1] \\ 1, & \text{if } x \notin [0,1]. \end{cases}$$

So, we have our

$$\begin{aligned} U(F^*, P_n) &= \sum_{i\in\Pi\cup\Pi'} F^*(t_{i_n})(u_{i_n} - u_{(i-1)_n}) \\ &= [-1,0]\sum_{i\in\Pi} 0.\frac{1}{i2^i} + [1,2]\sum_{i\in\Pi'} 1.1 \\ &= [1,2] \end{aligned}$$

Then

$$(I\text{-}SH[a,b])\int_0^1 F^* = [(SH[a,b])\int_0^1 F^{*(-)}] + (SH[a,b]\int_0^1 F^{*(+)}) = [1,2].$$

Clearly, F^* is Interval Sequential Henstock integrable on $[0,1]$ and that $\int_0^1 F^*(x)dx = [1,2]$. So by Theorems 2.1 and 2.2. it follows that $F^*(x)$ is also Interval Sequential Topological Henstock integrable on $[0,1]$.

3. Application of ITH integral

The study of fuzzy-valued functions, interval optimization, and interval-valued differential equations may benefit from our newly introduced kind of integral for interval-valued functions(see[3,4,10]).
Holzmann et al.[2], Lang[9] as well as Kramer and Wedner[7] have successfully applied the techniques of interval analysis for approximate continuous functions to adaptive Gaussian quadrature(see[10]).
One other good application for consideration in the study of Sequential The theory of trigonometric series and trigonometric integrals includes the Henstock integral; in which the principle concerning trigonometric series is the concept of recovering every convergent trigonometric series' coefficients from the sum(see[4,10]).

4. Conclusion

In this article, we provide a novel equivalence results for the Sequential Henstock and Topological Henstock Integrals For interval valued functions. This concept could easily link functional analysis and approximately differentiable interval valued functions to the Henstock integrals. By directly defining it alternatively based on sequences, one gains more insight into all sorts of equivalences of Henstock-type integrals. The sequential approach provide an alternative to the usual definitions of the Henstock integrals.

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7. Conflicts of Interest.

The authors have no conflicts of interest.

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