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REPRODUCING KERNEL METHOD FOR SOLVING PARTIAL TWO-DIMENSIONAL NONLINEAR FRACTIONAL VOLTERRA INTEGRAL EQUATION

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Abstract. This article discusses the replicating kernel interpolation collocation method related to Jacobi polynomials to solve a class of fractional system of equations. The reproducing kernel function that is executed as an (RKM) was first created in the form of Jacobi polynomials. To prevent Schmidt orthogonalization, researchers compare the numerical solutions achieved by varying the parameter value. Through various numerical examples, it is demonstrated that this technique is practical and precise.

Keywords: Reproducing, Shifted Jacobi Polynomials, Two-dimensional fractional integral equation, Fractional derivatives.
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1. Introduction

Hilbert established the kernel theory in 1904, describing a kernel (collection of interrelated processes). Numerous integral equations were derived from his attempt, including the Fredholm integral equation and eventually Schmidt. Introducing Hilbert space was accomplished through such a method. Further research into quantum mechanics' Hilbert space led to the development of a valuable tool for formulation. Mercer improved Hilbert's work and presented Mercer's Theorem in 1909 [16], because of Thomas Hilbert and Schmidt's discoveries in Hilbert space Frchet, Banach, Hahn, and Eduard Helly developed a new notion known as Banach space during 1920 to 1922. The Banach space is a Hilbert space subset. As a specific example of Hilbert space, the kernel Hilbert space (RKHS) possesses certain features. This term is a neologism for a Hilbert space of functions defined by replicating kernels. RKHS was initially proposed by [9].

In [16], many enhancements to RKHS concepts were implemented. The RKHS continued in pure mathematics, [18], after introducing Kernel (SVM). Additionally, eigenfunctions have been created to apply the eigenvalue issue to operating and functioning [8] and their usage in machine learning and physics

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[19, 22]. Since it implements weighted inner products in Hilbert space [13], RKHS is a Hilbert space of functions with a replicating kernel, there is a correlation between this and RKHS. In the field of science and math, shifted Jacobi polynomials proved to be an valuable tool for discovering a broad range of issues. There have been many applications of the stochastic operational matrix, such as the solution of stochastic Ito-Volterra integral equations [1], the solution of systems of nonlinear integral equations [2], and the solution of space-time integral equations [4]. M. H. Heydari et al. utilized Hat funactions (HFs) to solve nonlinear stochastic Ito integral equations [1, 3]. HFs were also invented by M. P. Tripathi et al. to solve fractional differential equations [6]. In [17] the Triangular functions (TFs) have been used to find the solution of nonlinear 2D Volterra-Fredholm integro-differential equations. For the solution of two-dimensional Fredholm integral equations, researchers modified Hat functions [25]. In [5] the nonlinear mixed Volterra-Fredholm integro-differential equations have been solved by two-dimensional block-pulse functions. In [6] developers created a Hat fractional integration-operating matrix. The fractional-order integro-differential equation may be used to represent a broad variety of scientific phenomena. Numerous applications are found in economics, viscoelasticity, and signal processing, electromagnetics, see [7, 10] for more information. Many academics have utilized numerical analysis to discover approximations of solutions to extensive diversity of differential equations. To specify the integration, the operational matrix must be correctly defined by some orthogonal function, such as a triangular function. Saeedi has solved the nonlinear fractional Volterra integral equations by using Triangular functions [14], Saeedi et al. found the numerical solution of the weakly singular Volterra integral equations by the a spectral Chelyshkov wavelet method [15]. Alahviranllo et. al have been solved fractional delay differential equations by reproducing kernel method [23]. The 2D nonlinear mixed Volterra-Fredholm integro-differential equations have been solved by black pulse functions [11]. Recently, the 2D nonlinear partial mixed Volterra-Fredholm integral equations have been solved by TFs [12].

In this paper, we consider the following system of two-dimensional fractional Volterra partial integro-differential equations (S2DFVPIDEs)

$$\begin{aligned} D_x^{\theta} u_1(x,t) &+ \int_0^x \int_0^t k_{11}(x,t,s,y)u_1(s,y)) + k_{12}(x,t,s,y)u_2(s,y))dyds = g_1(x,t), \\ D_x^{\theta} u_2(x,t) &+ \int_0^x \int_0^t k_{11}(x,t,s,y)u_1(s,y)) + k_{12}(x,t,s,y)u_2(s,y))dyds = g_2(x,t), \end{aligned} \tag{1}$$

with the initial conditionals

$$\frac{\partial^j}{\partial x^j} u_i(0,t) = \delta_{i,j}(t), \quad t \in \Omega \quad j = 0, 1, \cdots, \rho - 1, i = 0, \cdots, n, \tag{2}$$

where D_x^θ is a Riemann-Liouville fractional differential operator with $\rho - 1 < \theta \leq \rho$, the function $g_1(x,t), g_2(x,t)$ and $k_{11}(x,t,s,y)$ and $k_{12}(x,t,s,y)$ are known functions and $u_1(x,t)$ and $u_1(x,t)$ are unknown function which should be approximated. Also ρ is positive integer number and $\Omega = [0,a] \times [0,b]$. This paper is organized as follows: In Section 2, the basic concepts of fractional calculus are presented. Some necessary properties of the Shifted Jacobi polynomials are discussed in Section 3. In general, we describe reproducing kernel space method in this section. Afterwards, the reproducing kernel interpolation collocation method are discussed in Section 5. Section 6 shows the efficiency and accuracy of the proposed scheme by solving some numerical examples. Finally, Section 7 contains the concluding remarks.

2. Preliminaries

In this section, we review some of the properties of fractional differential equations. Some definitions and formulas containing these lines of representation may be cited throughout the paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order θ_1 is defined as

$$I_{x_0}^{\theta_1} u(x) = \frac{1}{\Gamma(\theta_1)} \int_{x_0}^{x} (x-t)^{\theta_1 - 1} u(t)dt, \quad \theta_1 > 0, \quad x > 0. \tag{3}$$

The properties of the operator I^{θ_1} can be found in [26]. Riemann-Liouville and Caputo fractional derivatives of order θ_1, are defined in the following equations, respectively:

$$D_{x_0}^{\theta_1} u(x) = \frac{d^n}{dx^n}[I_{x_0}^{n-\theta_1} u(x)], \tag{4}$$

$$D_{*x_0}^{\theta_1} u(x) = I_{x_0}^{n-\theta_1}[\frac{d^n}{dx^n} u(x)], \tag{5}$$

where $n - 1 \leq \theta_1 < n$ and $m \in \mathbb{N}$. From (3) and (4), we have,

$$D_{x_0}^{\theta_1} u(x) = \frac{1}{\Gamma(n-\theta_1)} \frac{d^n}{dx^n} \int_{x_0}^{x} (x-t)^{n-\theta_1-1} u(t)dt, \quad x > x_0. \tag{6}$$

Lemma 2.2. *[26]. If $n - 1 < \theta_1 \leq n$, $n \in \mathbb{N}$, then $D_x^{\theta_1} I^{\theta_1} u(x,t) = u(x,t)$, and:*

$$I^{\theta_1} D_x^{\theta_1} \mathrm{u}(x,t) = \mathrm{u}(x,t) - \sum_{k=0}^{n-1} \frac{\partial^k u(0^+,t)}{\partial x^k} \frac{x^k}{k!}, \quad x > 0.$$

Definition 2.3. [26]. Let $(\theta_1, \theta_2) \in (0,\infty) \times (0,\infty), \theta = (0,0), \Omega := [0,a] \times [0,b]$, and $u \in L^1(\Omega)$. The left-sided mixed Riemann-Liouille integral of order

(θ_1, θ_2) of u is defined by

(7) $$(I_\theta^{(\theta_1,\theta_2)}u)(x,t) = \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^x \int_0^t (x-s)^{(\theta_1-1)}(t-y)^{(\theta_2-1)}u(s,y)dyds.$$

In particular

1. $(I_\theta^{(\theta_1,\theta_2)}u)(x,t) = u(x,t)$,
2. $(I_\theta^{\upsilon}u)(x,t) = \int_0^x \int_0^t u(s,y)dyds, (x,t) \in \Omega, \upsilon = (1,1)$,
3. $(I_\theta^{(\theta_1,\theta_2)}u)(x,0) = (I_\theta^{(\theta_1,\theta_2)})(0,t) = 0, x \in [0,a], t \in [0,b]$,
4. $I_\theta^{(\theta_1,\theta_2)} x^\lambda t^\nu = \frac{\Gamma(1+\lambda)\times\Gamma(1+\nu)}{\Gamma(1+\lambda+\theta_1)\times\Gamma(1+\nu+\theta_2)} x^{\lambda+\theta_1}t^{\nu+\theta_2}, (x,t) \in \nu, \lambda, \nu \in (-1,\infty)$.

3. Shifted Jacobi Polynomials

The method is developed by using well-known polynomials. These polynomials are known as shifted Jacobi polynomials and we will describe their properties here. In the current section, you can find some descriptions and properties. The shifted Jacobi polynomials are defined on the interval [0,1] as:

$$P_{l,i}^{\theta_1,\theta_2}(x) = \sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(i+\theta_2+1)\Gamma(i+k+1+\theta_1+\theta_2)}{\Gamma(k+1+\theta_2)\Gamma(i+\theta_1+\theta_2+1)(i-k)!k!}x^k$$

where

(8) $$P_{l,i}^{\theta_1,\theta_2}(0) = (-1)^i\frac{\Gamma(i+\theta_2+1)}{\Gamma(\theta_2+1)i!}$$

(9) $$P_{l,i}^{\theta_1,\theta_2}(1) = (-1)^i\frac{\Gamma(i+\theta_2+1)}{\Gamma(\theta_1+1)i!}.$$

The shifted Jacobi polynomials on the interval $[0,1]$ are orthogonal

(10) $$\int_0^1 P_{1,n}^{\theta_1,\theta_2}(x)P_{1,m}^{\theta_1,\theta_2}(x)\omega^{\theta_1,\theta_2}(x)dx = L_k$$

where $\omega(x) = x(1-x)^{\theta_1}$ is a weight function, and

$$L_k = \begin{cases} \frac{\Gamma(k+\theta_2+1)\Gamma(k+\theta_2+1)}{(2k+\theta_1+\theta_2+1)k!\Gamma(k+\theta_1+\theta_2+1)(i-k)!k!}, & n=m \\ 0, & n \neq m, \end{cases}$$

3.1. **Two-dimensional shifted Jacobi polynomials.** We introduce the definition of two-dimensional shifted Jacobi polynomials (SJPs) briefly in this section. The two-variable SJPs are defined on the domain $D = [0 \times l_1] \times [0 \times l_1]$

(11) $$P_{i,j}^{\theta_1,\theta_2}(x,t) = P_{l_1,1}^{\theta_1,\theta_2}(x)P_{l_2,j}^{\theta_1,\theta_2}(t),$$

for $i,j = 0,1,2,...,n$, with the following orthogonality property on the domain X:

(12) $$\int_0^{l_1}\int_0^{l_2} P_{i,j}^{\theta_1,\theta_2}(x,t)P_{i_1,j_2}^{\theta_1,\theta_2}(x,t)\omega^{\theta_1,\theta_2}(x,t)dxdt = L_{l_1}L_{l_2}$$

where $\omega(x,t)=\omega_1^{\theta_1}\omega_2^{\theta_2}$ is the weight function.

4. Reproducing Kernel Space

In this section, we represent the inner product and norm of of shifted Jacobi in reproducing kernel space. To do so, let

$$H_{n[0,1]\times[0,1]}=\left\{P_{i,j}^{\theta_1,\theta_2}(x,t)\Big|\int_0^1\int_0^1\omega(x,t)|P_{i,n}^{\theta_1,\theta_2}(x,t)|^2dxdt<\infty\right\}. \tag{13}$$

where $i,j=0,1,...,n.$ be the weighted inner product space of the SJPs on $[0\times l_1]\times[0\times l_1]$. The inner product and norm are defined as

$$\langle P_{i,j}^{\theta_1,\theta_2}(x,t),P_{i,j}^{\theta_1,\theta_2}(x,t)\rangle=\int_0^{l_1}\int_0^{l_2}P_{i,j}^{\theta_1,\theta_2}(x,t)P_{i_1,j_2}^{\theta_1,\theta_2}(x,t)\omega(x,t)dxdt \tag{14}$$

$$\|P_{i,j}^{\theta_1,\theta_2}(x,t)\|=\sqrt{\langle P_{i,j}^{\theta_1,\theta_2}(x,t),P_{i,j}^{\theta_1,\theta_2}(x,t)\rangle} \tag{15}$$

where for any $P_{i,j}^{\theta_1,\theta_2}(x,t)\in H_{n[0,1]\times[0,1]}$.
Let

$$L^2[0,1]\times[0,1]=\{f(x,t)|\int_0^1\int_0^1\omega(x,t)|f(x,t)|^2dxdt<\infty\} \tag{16}$$

Its reproducing kernel is

$$R(x,t)=R_x(t)=\sum_{i=0}^{n}e_i(x)e_i(t), \tag{17}$$

where

$$e_i(x)=\sqrt{(2k+\theta_1+\theta_2+1)k!\Gamma(k+\theta_1+\theta_2+1)/\Gamma(k+\theta_1+1)\Gamma(k+\theta_2+1)}P_{i,j}^{\theta_1,\theta_2}(x),$$

$$e_i(t)=\sqrt{(2k+\theta_1+\theta_2+1)k!\Gamma(k+\theta_1+\theta_2+1)/\Gamma(k+\theta_1+1)\Gamma(k+\theta_2+1)}P_{i,j}^{\theta_1,\theta_2}(t).$$

5. The Reproducing Kernel Interpolation Collocation Method

The aim of the section is to introduce reproducing kernel interpolation collocation method

Definition 5.1. Assume that

$$\bar{H}_n[0,1]=\{u|u\in H_n[0,1],u(0)=0\}. \tag{18}$$

As a result, its norm is the same as the norm of $\bar{H}_{n,[0,1]}$.

It can easily be shown that $\bar{H}_{n[0,1]}$ is a reproducing kernel Hilbert space. According to [24], the reproducing kernel of $\bar{H}_{n[0,1]}$ is

$$K(x,t)=K_x(t)=R(x,t)-\frac{R(0,x)R(t,0)}{\|R(0,0)\|^2}. \tag{19}$$

Definition 5.2. The inner product space is defined as

$\bar{H}_{n,[0,1]\times[0,1]} \oplus \bar{H}_{n,[0,1]\times[0,1]} = \{U(x,t) = [u_1(x,t), u_2(x,t)]^T | u_1(x,t), u_2(x,t) \in \bar{H}_{n,[0,1]\times[0,1]}\}$.

Its corresponding inner product and norm defined as

$$\langle U(x), V(x)\rangle = \sum_{i=1}^{2}\sum_{j=1}^{2} \langle u_i(x), v_j(t)\rangle_{\bar{H}_n([0,1]\otimes[0,1])\oplus\bar{H}_n([0,1]\otimes[0,1])}, \tag{20}$$

$$\| U(x,t) \|^2 = \sum_{i=1}^{2}\sum_{i=1}^{2} \| u_i(x,t) \|_{\bar{H}_n(([0,1]\otimes[0,1]))\oplus\bar{H}_n([0,1]\otimes[0,1])}. \tag{21}$$

It is easy to verify that $\bar{H}_n(([0,1]\otimes[0,1])\oplus\bar{H}_n([0,1]\otimes[0,1])$ is a Hilbert space with the definition of inner product (21). Similarly, $L^2(([0,1]\otimes[0,1])\oplus L^2([0,1]\otimes[0,1])$ is also a Hilbert space. To solve equation (1), let

$$\begin{aligned}
P_{11}u_1 &= D^\theta u_1(x,t) + \int_0^s\int_0^y k_{11}(x,t,s,y)u_1(x,t)dsdy,\\
P_{12}u_2 &= \int_0^s\int_0^y k_{12}(x,t,s,y)u_2(x,t)dsdy,\\
P_{21}u_1 &= \int_0^s\int_0^y k_{21}(x,t,s,y)u_1(x,t)dsdy,\\
P_{22}u_2 &= D^\theta u_2(x,t) + \int_0^s\int_0^y k_{22}(x,t,s,y)u_2(x,t)dsdy.
\end{aligned}$$

So, equation (1) can be turned into

$$PU(x,t) = G(x,t), \tag{22}$$

where

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}.$$

The operator $L : \bar{H}_{n[0,1]\times[0,1]} \oplus \bar{H}_{n[0,1]\times[0,1]} \to L^2_{n[0,1]\times[0,1]} \oplus L^2_{n[0,1]\times[0,1]}$ is a bounded linear operator.

Theorem 5.3. *Assuming that $\{x_i\}_{i=1}^\infty$ is dense in the interval $[0,1]$, put $\phi_{ijkh} = l^*_{ij}k_{x_{kh}}(x,t)$, in which l^*_{ij} is the adjoint operator of l_{ij}, we have*

$$\phi_{ijkh}(x,t) = l_{ij}K_x(x_k,t_h), \quad i,j,k,h = 1,2,.... \tag{23}$$

Let us put

$$\begin{aligned}
\Phi_{i,j}(x,t) &= (\psi_{11,ij}(x,t), \psi_{12,ij}(x,t))^T \qquad (24)\\
\Phi_{i,j}(x,t) &= (\psi_{21,ij}(x,t), \psi_{22,ij}(x,t))^T, \quad i,j = 1,2,... \qquad (25)
\end{aligned}$$

For each fixed n, $\{\Phi_{ijkh}\}_{(1,1)}^{(n,2)}$ is linearly independent in $\bar{H}_{n,[0,1]\times[0,1]}\oplus\bar{H}_{n,[0,1]\times[0,1]}$.

Proof. Letting

$$0 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(c_{ij,1}\Phi_{ij,1}(x,t) + c_{ij,2}\Phi_{ij,2}(x,t)), \tag{26}$$

$$U_{kh}(x,t) = [u_{kh,1}(x,t), u_{kh,2}(x,t)]^T, \tag{27}$$

where $U_{kh}(x,t) \in L^2[0,1] \times [0,1]$, when $x = x_1, x_2, ..., x_{k-1}, x_{k-1}, ..., x_n$, $t = t_1, t_2, ..., t_{k-1}, t_{k-1}, ..., t_n$. So,

$$\begin{aligned} 0 &= \langle U_{kh}, \sum_{i=1}^{n}\sum_{i=1}^{m}(c_{ij}\Phi_{ij,1}(x,t) + c_{ij,2}\Phi_{ij,2}(x,t)\rangle) \\ &= \sum_{i=1}^{n}\sum_{i=1}^{m}(c_{ij}\langle U_{kh}\Phi_{ij,1}\rangle + c_{ij,2}\langle U_{kh}, \Phi_{ij,2}\rangle \\ &= \sum_{i=1}^{n}\sum_{i=1}^{m}(c_{11,ij}(\langle u_{kh,1}\phi_{ij,11}\rangle + \langle u_{kh,2}\phi_{ij,12}\rangle) \\ &\quad + c_{ij,2}(\langle u_{kh,1}, \phi_{ij,21}\rangle + \langle u_{kh,2}, \phi_{ij,22}\rangle)) \\ &= \sum_{i=1}^{n}\sum_{i=1}^{m}(c_{ij,1}(p_{11}u_{kh,1}(x_i,t_j) + p_{12}u_{kh,2}(x_i,t_j)) \\ &\quad + c_{ij,2}(p_{21}u_{kh,1}(x_i,t_j)) + p_{22}u_{kh,2},(x_i,t_j))) \\ &= \sum_{i=1}^{n}\sum_{i=1}^{m}(c_{ij,1}u_{kh,1}(x_i,t_j) + c_{ij,2}(p_{21}u_{kh,1}(x_i,t_j)) = c_{kh,1}u_{kh,1}(x_i,t_j). \end{aligned}$$

□

So, $c_{kh} = 0, \quad k,h = 1,2,...,n$. Similarly, we have $c_{kh,2} = 0$.

Theorem 5.4. *Assume that* $\{\Phi_{i,j}\}_{(1,1)}^{\infty,2}$ *is complete in space* $\bar{H}_{n,[0,1]\times[0,1]} \oplus \bar{H}_{n,[0,1]\times[0,1]}$.

Proof. For each

$$U(x,t) = [u_1(x,t), u_2(x,t)]^T \in \bar{H}_{n,[0,1]\times[0,1]} \oplus \bar{H}_{n,[0,1]\times[0,1]}, \tag{28}$$

it follows that $\langle U(x,t), \Phi_{ij}(x,t)\rangle = 0$, for $i,j = 1,2,....$ Thus

$$\begin{aligned} 0 &= \langle U(x,t), \Phi_{ij,1}(x,t)\rangle_{\bar{H}_{n,[0,1]\times[0,1]}\oplus\bar{H}_{n,[0,1]\times[0,1]}} \\ &= \langle u_1(x,t), p_{11}K_{x_{ij}}(x,t)\rangle_{\bar{H}_{n,[0,1]\times[0,1]}} + \langle u_2(x,t), p_{12}K_{x_{ij}}(x,t)\rangle_{\bar{H}_{n,[0,1]\times[0,1]}} \\ &= p_{11}u_1(x,t) + p_{12}u_2(x,t), \\ 0 &= \langle U(x,t), \Phi_{ij,2}\rangle_{\bar{H}_{n,m}[0,1]} = \langle u_1(x,t), p_{21}K_{x_{ij}}(x,t)\rangle_{\bar{H}_{n,m}[0,1]} \\ &\quad + \langle u_2(x,t), p_{22}K_{x_{ij}}(x,t)\rangle_{\bar{H}_{n,m}[0,1]} \\ &= p_{21}u_1(x,t) + p_{22}u_2(x,t). \end{aligned}$$

Hence equation (1) has a unique solution, therefore $U(x,t) = 0$. □

The exact solution of Eq (1) can be expressed as

$$U(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{2}\sum_{h=1}^{2} c_{ijkh}\Phi_{ijkh}(x,t), \tag{29}$$

and we obtain the approximate solution of Eq (1) by truncating the infinite series of the analytical solution.

$$U_m(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{2}\sum_{h=1}^{2} c_{ijkh}\Phi_{ijkh}(x,t), \tag{30}$$

Theorem 5.5. *Let $U \in \bar{H}_{n,[0,1]\times[0,1]} \oplus \bar{H}_{n,[0,1]\times[0,1]}$ be the exact solution of Eq (1), U_m be the approximate solution of U. Then U_m converges uniformly to U.*

Proof. In order to prove the theorem, we construct the following proposition:

$$\begin{aligned} |u_1(x,t) - u_{1,m}(x,t)| &= |\langle u_1 - u_{1,m}, K_x\rangle| \\ &\leq \|u_1 - u_{1,m}\|_{\bar{H}_{n,[0,1]\times[0,1]}} \|K_x\|_{\bar{H}_{n,[0,1]\times[0,1]}} \\ &\leq M\|u_1 - u_{1,m}\|_{\bar{H}_{n,[0,1]\times[0,1]}} \end{aligned}$$

Similarly,

$$|u_2(x,t) - u_{2,m}(x,t)| \leq M\|u_2 - u_{2,m}\|_{\bar{H}_{n,[0,1]\times[0,1]}} \tag{31}$$

□

As long as the coefficients of $\Phi_{ij}(x,t)$ can be obtained, we can also obtain the approximate solution $U_m(x,t)$. Using $\Phi_{ij}(x,t)$ to do the inner products with both sides of Eq (30), we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\langle\Phi_{ij,1}, \Phi_{ij,1}\rangle \quad + \quad \sum_{k=1}^{m}\sum_{h=1}^{m}\langle c_{ij}\langle\Phi_{ij,1}, \Phi_{ij,1}\rangle = f_1(x,t) \tag{32}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\langle\Phi_{ij,1}, \Phi_{ij,1}\rangle \quad + \quad \sum_{k=1}^{m}\sum_{h=1}^{m}\langle c_{ij}\langle\Phi_{ij,1}, \Phi_{ij,1}\rangle = f_2(x,t). \tag{33}$$

Letting

$$P = \begin{pmatrix} \langle\Phi_{ij,1}, \Phi_{ij,1}\rangle & \cdots & \langle\Phi_{ij,1}, \Phi_{ij,1}\rangle \\ \cdots & \cdots & \cdots \\ \langle\Phi_{ij,1}, \Phi_{ij,1}\rangle & \cdots & \langle\Phi_{ij,1}, \Phi_{ij,1}\rangle \end{pmatrix}_{i,j,n=1,2,\cdots,m,}$$

and $F = (f_1(x,t), \cdots, f_1(x,t), f_2(x,t), \cdots, f_2(x,t))^T$.
It is obvious that the inverse of A_{2m} exists by Theorem 5.3. So, we have

$$(c_{11}, c_{12}, \cdots, c_{1m}, c_{21}, c_{22}, \cdots, c_{2m2})^T = P_{2m}^{-1}F. \tag{34}$$

6. Numerical examples

In this part, two examples are given to show the accuracy of the proposed method.

Example 6.1. *Consider the following S2DFVPIDEs*

$$
\begin{aligned}
\frac{\partial^{1/2}u_1(x,t)}{\partial x^{1/2}} &+ u_2(x,t) - \int_0^x\int_0^t s^2(u_1^2(s,y)u_2^2(s,y))dyds = g_1(x,t),\\
\frac{\partial^{2/3}u_1(x,t)}{\partial x^{2/3}} &+ u_1^2(x,t) - \int_0^x\int_0^t y^2(u_1(s,y)u_2(s,y))dyds = g_2(x,t),
\end{aligned} \tag{35}
$$

with

$$
\begin{aligned}
g_1(x,t) &= \frac{2e^t\sqrt{x}}{\sqrt{\pi}} - 2te^{-t} + \frac{1}{15}x^5t^3,\\
g_2(x,t) &= \frac{3}{2\pi}(e^t x^{\frac{1}{3}}\sqrt{3}\Gamma(\frac{2}{3})) + te^{-t} + x(e^t) - \frac{1}{8}t^4x^2,
\end{aligned}
$$

for $x,t\in[0,1]$ *and with supplementary conditions*

$$
u_1(0,t)=0,\quad u_2(0,t)=te^{-t}, \tag{36}
$$

which the exact solutions are $u_1(x,t)=xe^t$ *and* $u_2(x,t)=te^{-t}$. *In Table 1, the numerical results are presented.*

Example 6.2. *In this example, we present the following S2DFVPIDEs*

$$
\begin{aligned}
\frac{\partial^{1/3}u_1(x,t)}{\partial x^{1/3}} &+ \int_0^x\int_0^t s^2\cos(y)(u_1^2(s,y)+u_2^2(s,y))dyds = g_1(x,t),\\
\frac{\partial^{3/2}u_1(x,t)}{\partial x^{3/2}} &+ \int_0^x\int_0^t s\cos(y)(u_1(s,y)+\frac{\partial u_2(s,y)}{\partial y})dyds = g_2(x,t),
\end{aligned} \tag{37}
$$

where

$$
\begin{aligned}
g_1(x,t) &= \frac{3}{2\Gamma(\frac{3}{2})}\sin(t)x^{\frac{2}{3}} - x\sin(t) - \frac{1}{15}x^5\sin(t)(\cos^2(t)-\sin^2(t)+2),\\
g_2(x,t) &= -x\sin(t) + x\sin(t) - \frac{x^3}{3}(\cos(t)\sin(t)+t),
\end{aligned}
$$

for $x,t\in[0,1]$ *and with supplementary conditions*

$$
u_1(0,t)=0,\quad u_2(0,t)=0. \tag{38}
$$

The exact solutions of this example are $u_1(x,t)=x\cos(t)$ *and* $u_2(x,t)=x\sin(t)$. *Numerical results are presented in Table 2.*

Table 1. Numerical results and errors estimation for Example (6.1).

	$m = 8$	$m = 16$	$m = 32$
$u_1(x,t)$			
$(0.1, 0.1)$	4.5775×10^{-4}	2.7410×10^{-6}	4.0104×10^{-8}
$(0.3, 0.3)$	7.1710×10^{-4}	2.1918×10^{-7}	4.4040×10^{-7}
$(0.5, 0.5)$	2.8747×10^{-3}	3.7011×10^{-6}	8.1012×10^{-8}
$(0.7, 0.7)$	8.9674×10^{-4}	5.7412×10^{-6}	6.5408×10^{-8}
$(0.9, 0.9)$	3.7417×10^{-3}	3.7047×10^{-6}	5.1650×10^{-8}
$u_2(x,t)$			
$(0.1, 0.1)$	3.2585×10^{-4}	1.0474×10^{-6}	6.6504×10^{-7}
$(0.3, 0.3)$	2.2582×10^{-4}	2.0216×10^{-5}	2.9658×10^{-7}
$(0.5, 0.5)$	8.9674×10^{-4}	4.8410×10^{-5}	8.6325×10^{-6}
$(0.7, 0.7)$	6.3250×10^{-3}	5.8501×10^{-5}	3.0604×10^{-6}
$(0.9, 0.9)$	0.2870×10^{-3}	8.5204×10^{-4}	5.0214×10^{-6}

Table 2. Numerical results and errors estimation for Example (6.2).

	$m = 8$	$m = 16$	$m = 32$
$u_1(x,t)$			
$(0.1, 0.1)$	3.1221×10^{-4}	9.0025×10^{-6}	3.2575×10^{-9}
$(0.3, 0.3)$	0.1242×10^{-4}	4.0578×10^{-6}	3.4217×10^{-8}
$(0.5, 0.5)$	1.4205×10^{-3}	3.4745×10^{-8}	0.1012×10^{-9}
$(0.7, 0.7)$	2.3224×10^{-4}	5.7412×10^{-8}	6.5408×10^{-11}
$(0.9, 0.9)$	3.6202×10^{-4}	3.1065×10^{-8}	5.1650×10^{-11}
$u_2(x,t)$			
$(0.1, 0.1)$	3.2585×10^{-6}	3.9854×10^{-8}	6.6504×10^{-12}
$(0.3, 0.3)$	3.6521×10^{-6}	0.5016×10^{-8}	8.1547×10^{-12}
$(0.5, 0.5)$	2.0210×10^{-6}	3.8651×10^{-8}	1.6148×10^{-10}
$(0.7, 0.7)$	4.8421×10^{-6}	7.7041×10^{-7}	6.5474×10^{-9}
$(0.9, 0.9)$	1.6968×10^{-6}	9.3004×10^{-7}	2.0853×10^{-13}

7. Conclusion

In this paper, for the first time, this article solves fractional-order linear integro-differential equations utilizing reproducing kernel interpolation collocation technique with a reproducing kernel function in the form of Jacobi polynomials. The approximate and precise answers are compared. We demonstrate the methods viability by varying the parameters μ, α, as well, as β. The algorithm is exact and practical, as shown by the tables and figures.

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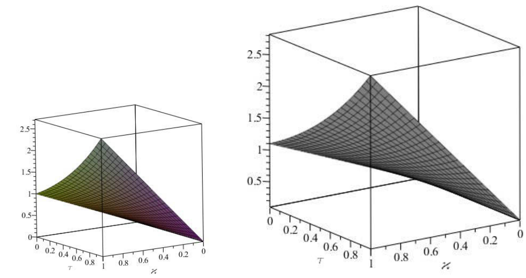

FIGURE 1. The numerical (left part) and analytical solutions (right part) test in example (6.1) for $u_2(x,t)$ with $m = 32$.

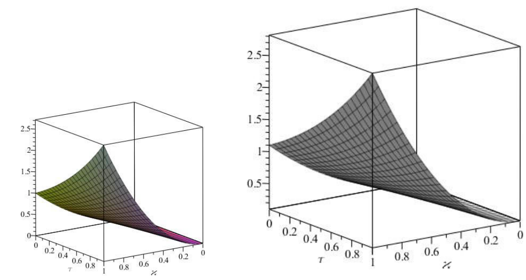

FIGURE 2. The numerical (left part) and analytical solutions (right part) test in example (6.2) for $u(x,t)$ with $m = 32$.

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