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GENERALIZATIONS OF BANACH'S CONTRACTION PRINCIPLE AND KANNAN AND CHATTERJEA'S THEOREMS FOR CYCLIC AND NONCYCLIC MAPPINGS

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Abstract. Two interesting extensions of Banach contraction principle to mappings that do not to be continuous, are Kannan and Chatterjea's theorems. Before this, in the cyclical form, extensions of these two theorems and Banach contraction principle were produced. But so far, these theorems have not been studied in the noncyclical form. In this paper, we answer the question whether there are versions of these theorems for noncyclic mappings, also we give generalizations of existing results. For this purpose, in the setting of metric spaces we introduce the notions of cyclic and noncyclic contraction of Fisher type. We establish the existence of fixed points for these mappings and iterative algorithms are furnished to determine such fixed points. As a result of our results we give new theorems for cyclic orbital contractions.

Keywords: Fixed point, Cyclic and noncyclic contractions of Fisher-type, Kannan and Chatterjea mappings, Cyclic orbital contraction.
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1. **Introduction**

One of the most important result in fixed point theory is the Banach Contraction Mapping Principle which basically shows that any contraction on a complete metric space (X, d), that is, any mapping $T : X \to X$ satisfying

(I) $$d(Tx, Ty) \leq cd(x, y) \ for\ all\ x, y \in X,$$

where $c \in (0, 1)$ is a constant, has a unique fixed point. Notice that any contraction is continuous on X. It is natural to ask if there exist contractive conditions which do not imply the continuity of T all over the whole space X. Kannan [10] in 1968, answered the question positively, he proved a fixed point theorem, which extends Banach's contraction principle to mappings that need not to be continuous, by considering instead of (I) this condition: there exists $c \in [0, \frac{1}{2})$ such that

(II) $$d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)] \ for\ all\ x, y \in X.$$

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Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T; see [1, 4, 7–9, 15, 17, 18] and refrences therien. One of them, due to Chatterjea [2], is based on a condition similar to (II): there exists $c \in [0, \frac{1}{2})$ such that

(III) $$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \; for \; all \; x, y \in X.$$

On the other hand, in [16] Kirk, Srinivasan and Veeramani obtained an extension of Banach's fixed point theorem by considering a cyclical contractive condition. For nonempty subsets A and B of a metric space X, a self mapping $T : A \cup B \to A \cup B$ is said to be cyclic provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. They proved the following theorem.

Theorem 1.1. *[16] Let A and B be two nonempty closed subsets of a complete metric space (X, d) and suppose $T : A \cup B \to A \cup B$ is a cyclic map satisfies the following condition*

$$d(Tx, Ty) \leq cd(x, y) \; for \; all \; x \in A, \; y \in B,$$

where $c \in (0, 1)$. Then T has a unique fixed point in $A \cap B$.

Later, many authors interested to obtaining fixed point theorems for cyclic mappings; see [3, 9, 11, 12, 19, 20] and refrences therien. In the cyclical form, in [19], the contractive condition due to Kannan [10], was introduced as a cyclic Kannan contraction, and in [11], the contractive condition due to Chatterjea [2], was introduced as a cyclic Chatterjea contraction. For two sets A and B we have the following special results.

Theorem 1.2. *[19] Let A and B be two nonempty closed subsets of a complete metric space (X, d) and suppose $T : A \cup B \to A \cup B$ is a cyclic map such that*

$$d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)] \; for \; all \; x \in A, \; y \in B,$$

where $c \in [0, \frac{1}{2})$. Then T has a unique fixed point x^* in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in A \cup B$.

Theorem 1.3. *[11] Let A and B be nonempty and closed subsets of a complete metric space (X, d). Let T be a cyclic mapping on $A \cup B$ such that*

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \; for \; all \; x \in A, \; y \in B,$$

where $c \in [0, \frac{1}{2})$. Then T has a unique fixed point x^* in $A \cap B$.

For nonempty subsets A and B of a metric space, a self mapping $T : A \cup B \to A \cup B$ is said to be noncyclic provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. Fernández-León and Gabeleh in [6] proved that:

Theorem 1.4. *[6] Let A and B be nonempty and closed subsets of a complete metric space (X, d). Let T be a noncyclic mapping on $A \cup B$ such that*

$$d(Tx, Ty) \leq cd(x, y) \; for \; all \; x, y \in X,$$

where $c \in [0,1)$. Then $d(A,B)=0$. Moreover, the mapping T has a fixed point if and only if $A\cap B\neq\emptyset$.

We show that in previous theorem T has a unique fixed point in $A\cap B$, so Theorem 1.1 holds for noncyclic maps. This is natural to ask, do the Theorems 1.2 and 1.3 hold for noncyclic maps, too? Our answer is negative for Theorem 1.2 and positive for Theorem 1.3. In this article, in the setting of metric spaces we first, introduce the notion of cyclic contraction of Fisher-type as a generalization of cyclic Kannan contraction. Then, we prove the existence of fixed point for such mappings. Also, uniqueness and iterative algorithms for finding the fixed points of such mappings are given. Our results in this section extend Theorems 1.1 and 1.2. In the next section, we introduce the notion of noncyclic contraction of Fisher-type as a generalization of Chatterjea contraction. Then, we prove the existence of fixed point for such mappings. Also, we give iterative algorithms for finding the fixed points of such mappings. Our results in this section extend Chetterjea's theorem. As a result we give generalizations for Theorem 2.2 and Corollary 2.3 in [13] for cyclic orbital contractions.

2. Cyclic contraction of Fisher-type

Ćirić [4] defined quasi-contraction mappings and proved that if (X,d) is a complete metric space and $T: X\to X$ is a quasi-contraction mapping, then T has a unique fixed point. Fisher [7] extended the definition of a quasi-contraction map. By motivation of the notion of quasi-contraction of Fisher-type in [7], we introduce the concept of cyclic contraction of Fisher-type. Let A and B be nonempty subsets of a metric space (X,d). Suppose T be a cyclic mapping on $A\cup B$. Throughout this section and for each $x\in A,\ y\in B$ and $n,m\in\mathbb{N}$, let

$$A^{x,y}_{n,m} := \big\{T^{2i}x, T^{2j+1}y :\ 0\le i\le \lfloor\frac{n}{2}\rfloor,\ 0\le j\le\lfloor\frac{m-1}{2}\rfloor\big\},$$

and

$$B^{x,y}_{n,m} := \big\{T^{2j+1}x, T^{2i}y :\ 0\le j\le \lfloor\frac{n-1}{2}\rfloor,\ 0\le i\le\lfloor\frac{m}{2}\rfloor\big\}.$$

Also let $\delta[C,D] := \sup\ \{d(x,y):\ x\in C, y\in D\}$. To establish our main results in this section, we introduce the following class of cyclic contraction mappings.

Definition 2.1. Let A and B be nonempty subsets of a metric space (X,d) and let T be a cyclic mapping on $A\cup B$. Then T is said to be a contraction of Fisher-type, if there exist $p,q\in\mathbb{N}$ such that $p-q$ is even and

$$d(T^px, T^qy)\le c\delta\big[A^{x,y}_{p,q}, B^{x,y}_{p,q}\big],$$

for all $x\in A$ and $y\in B$, where $c\in[0,1)$.

We begin with the following lemmas which will be used later.

Lemma 2.2. *Let A and B be nonempty subsets of a metric space (X,d) and let T be a cyclic map on $A\cup B$ such that for some $q\in\mathbb{N}$ satisfying*

$$d(T^q x, T^q y)\le c\delta\big[A^{x,y}_{q,q}, B^{x,y}_{q,q}\big], \tag{1}$$

for all $x\in A$ and $y\in B$, where $c\in[0,1)$. For $x_0\in A$, define $x_{n+1}:=Tx_n$ for each $n\ge 0$. Then there exists $M_{x_0}\in\mathbb{R}^+$ such that

$$d(x_n,x_{n+1})\le c^{\lfloor \frac{2n}{q}\rfloor}\ M_{x_0}.$$

Proof. For simplicity, we assume that q is even. For each $x\in A$ and $n\in\mathbb{N}$, let

$$\mathfrak{A}^x_{2n}:=\{x,T^2x,T^4x,\dots,T^{2n}x\},\ \text{ and }\mathfrak{B}^x_{2n}:=\{Tx,T^3x,T^5x,\dots,T^{2n+1}x\}.$$

Now we show that for each $n\in\mathbb{N}$, we have

$$\delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}]=d(T^{2k}x_0,T^{2l+1}x_0),\ \text{ where either } 2k<q \text{ or } 2l<q. \tag{2}$$

We may assume that $\delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}]=d(T^{2i}x_0,T^{2j+1}x_0)$, where $q\le 2i,2j\le 2n$. Since T satisfies in (1), then we have

$$\begin{aligned} d(T^{2i}x_0,T^{2j+1}x_0)&=d(T^qT^{2i-q}x_0,T^qT^{2j-q+1}x_0)\\ &\le c\delta\big[A^{T^{2i-q}x_0,T^{2j-q+1}x_0}_{q,q},B^{T^{2i-q}x_0,T^{2j-q+1}x_0}_{q,q}\big]\\ &\le c\delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}]. \end{aligned} \tag{3}$$

Thus, we get $\delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}]=0$, then $\delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}]=d(x_0,Tx_0)$, and so (2) holds. Now we show that for each $n\in\mathbb{N}$,

$$\delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}]\le M_{x_0}, \tag{4}$$

where

$$M_{x_0}=\frac{1}{1-c}\max\big\{d(T^ix_0,T^jx_0):\ 0\le i,j\le q+1\big\}.$$

To prove the claim note that from (2) we have, $\delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}]=d(T^{2k}x_0,T^{2l+1}x_0)$, where either $2k<q$ or $2l<q$. If $2k,2l<q$, then (4) trivially holds. If $2k<q$ and $q\le 2l\le 2n$. Then similar (3), we get

$$\begin{aligned} \delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}]&=d(T^{2k}x_0,T^{2l+1}x_0)\le d(T^{2k}x_0,T^qx_0)+d(T^qx_0,T^{2l+1}x_0)\\ &\le d(T^{2k}x_0,T^qx_0)+c\delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}], \end{aligned}$$

and so (4) holds. Similarly if $q\le 2k\le 2n$ and $2l<q$, we get

$$\delta[\mathfrak{A}^{x_0}_{2n},\mathfrak{B}^{x_0}_{2n}]\le\frac{1}{1-c}d(T^{q+1}x_0,T^{2l+1}x_0)\le M_{x_0}.$$

Now we prove that for every $r,s\in\mathbb{N}$ with $2r\ge q$, we have

$$\delta[\mathfrak{A}^{x_{2r}}_{2s},\mathfrak{B}^{x_{2r}}_{2s}]\le c\delta\big[\mathfrak{A}^{x_{2r-q}}_{2s+q},\mathfrak{B}^{x_{2r-q}}_{2s+q}\big]. \tag{5}$$

Since T satisfies in (1), then for some $k, l \leq s$, we have

$$\begin{aligned}\delta[\mathfrak{A}_{2s}^{x_{2r}}, \mathfrak{B}_{2s}^{x_{2r}}] &= d(T^{2k}x_{2r}, T^{2l+1}x_{2r}) = d(T^q x_{2r+2k-q}, T^q x_{2r+2l-q+1})\\ &\leq c\ \delta\big[A_{q,q}^{x_{2r+2k-q}, x_{2r+2l-q+1}}, B_{q,q}^{x_{2r+2k-q}, x_{2r+2l-q+1}}\big]\\ &= c\delta\Big[\big\{T^{2i}x_{2r+2k-q}, T^{2j+1}x_{2r+2l-q+1} :\ 0 \leq i \leq \frac{q}{2},\ 0 \leq j \leq \frac{q}{2} - 1\big\},\\ &\quad \big\{T^{2j+1}x_{2r+2k-q}, T^{2i}x_{2r+2l-q+1} :\ 0 \leq j \leq \frac{q}{2} - 1,\ 0 \leq i \leq \frac{q}{2}\big\}\Big]\\ &\leq c\delta\big[\mathfrak{A}_{2s+q}^{x_{2r-q}}, \mathfrak{B}_{2s+q}^{x_{2r-q}}\big].\end{aligned}$$

Now we prove that for every $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) \leq c^{\lfloor \frac{2n}{q} \rfloor}\ M_{x_0}.$$

Since T satisfies in (1), then for every $n \in \mathbb{N}$ with $n \geq \frac{q}{2}$, we have

$$\begin{aligned}d(x_{2n}, x_{2n+1}) =& d(T^q x_{2n-q}, T^q x_{2n-q+1})\\ \leq& c\delta\big[A_{q,q}^{x_{2n-q}, x_{2n-q+1}}, B_{q,q}^{x_{2n-q}, x_{2n-q+1}}\big]\\ =& c\delta\Big[\big\{T^{2i}x_{2n-q}, T^{2j+1}x_{2n-q+1} :\ 0 \leq i \leq \frac{q}{2},\ 0 \leq j \leq \frac{q}{2} - 1\big\},\\ & \big\{T^{2j+1}x_{2n-q}, T^{2i}x_{2n-q+1} :\ 0 \leq j \leq \frac{q}{2} - 1,\ 0 \leq i \leq \frac{q}{2}\big\}\Big]\\ \leq& c\delta\big[\mathfrak{A}_q^{x_{2n-q}}, \mathfrak{B}_q^{x_{2n-q}}\big],\end{aligned}$$

then from (5) for $n \geq q$ we have

$$d(x_{2n}, x_{2n+1}) \leq c^2\big[\mathfrak{A}_{2q}^{x_{2n-2q}}, \mathfrak{B}_{2q}^{x_{2n-2q}}\big].$$

By continuing this process and using (4), we obtain

$$\begin{aligned}d(x_{2n}, x_{2n+1}) &\leq c^{\lfloor \frac{2n}{q} \rfloor}\ \delta\big[\mathfrak{A}_{\lfloor \frac{2n}{q} \rfloor q}^{x_{2n-\lfloor \frac{2n}{q} \rfloor q}}, \mathfrak{B}_{\lfloor \frac{2n}{q} \rfloor q}^{x_{2n-\lfloor \frac{2n}{q} \rfloor q}}\big]\\ &\leq c^{\lfloor \frac{2n}{q} \rfloor}\ \delta[\mathfrak{A}_{2n}^{x_0}, \mathfrak{B}_{2n}^{x_0}].\end{aligned}$$

Similarly we can prove that $d(x_{2n+2}, x_{2n+1}) \leq c^{\lfloor \frac{2n}{q} \rfloor}\ \delta[\mathfrak{A}_{2n+2}^{x_0}, \mathfrak{B}_{2n+2}^{x_0}]$. Therefore

$$d(x_n, x_{n+1}) \leq c^{\lfloor \frac{2n}{q} \rfloor}\ M_{x_0}.$$

□

Lemma 2.3. *Let A and B be nonempty subsets of a complete metric space (X, d). Assume that T is a cyclic contraction of Fisher-type on $A \cup B$. Then the Picard iteration $\{T^n x_0\}$ is Cauchy for any starting point $x_0 \in A \cup B$.*

Proof. Let $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Without loss of generality we may assume that $q \geq p$ then T satisfy in condition (1) so from Lemma 2.2 we get

$$d(x_n, x_{n+1}) \leq c^{\lfloor \frac{2n}{q} \rfloor} M_{x_0}.$$

Therefore

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq M_{x_0} \sum_{n=1}^{\infty} c^{\lfloor \frac{2n}{q} \rfloor} \leq M_{x_0} \sum_{n=1}^{\infty} c^{\lfloor \frac{n}{q} \rfloor} = qM_{x_0} \sum_{n=0}^{\infty} c^n < \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

Now we are ready to state our first fixed point result in this section.

Theorem 2.4. *Let A and B be nonempty and closed subsets of a complete metric space (X, d). Assume that T is a cyclic contraction of Fisher-type on $A \cup B$ for which the restriction of T to A (or B) is continuous. Then T has a unique fixed point x^* in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in A \cup B$.*

Proof. Let $x_0 \in A$, define $x_{n+1} := Tx_n$ from Lemma 2.3 the sequence $\{x_n\}$ is Cauchy and thus there exist a $x^* \in A \cup B$ such that $\{x_n\}$ converges to it. Now $\{x_{2n}\}$ is a sequence in A and $\{x_{2n+1}\}$ is a sequence in B and both converges to x^*. Since A and B are closed, $x^* \in A \cap B$. Since $T_{|_A}$ is continuous, it follows that $Tx^* = x^*$. To prove the uniqueness, suppose $\overline{x}$ be another fixed point of T in $A \cup B$. Because T is cyclic map clearly $\overline{x} \in A \cap B$, so we have

$$d(x^*, \overline{x}) = d(T^p x^*, T^q \overline{x}) \leq c\delta\big[A_{p,q}^{x^*,\overline{x}}, B_{p,q}^{x^*,\overline{x}}\big] = cd(x^*, \overline{x}).$$

so $\overline{x} = x^*$. □

The following example shows that the continuity condition of T in the above theorem is not superfluous.

Example 2.5. *Let $\mathbb{R}$ with the usual metric and let $A = B = \{0\} \cup \{\frac{1}{2^n}\}_{n=0}^{\infty}$. We define the self mapping $T : A \to A$ by $T(0) = \frac{1}{2}$ and $T(x) = \frac{1}{2}x$ if $x \neq 0$. It is straightforward to show that for each $x, y \in A$*

$$|T^2x - T^2y| \leq \frac{1}{2}|Tx - Ty| \leq \frac{1}{2}\delta\big[\{x, T^2x, Ty\}, \{Tx, y, T^2y\}\big] = \frac{1}{2}\delta\big[A_{2,2}^{x,y}, B_{2,2}^{x,y}\big].$$

Thus T is a cyclic contraction of Fisher-type on $A \cup B$ but T has not any fixed point. Note that T is not continuous at $x = 0 \in A$.

The next example shows that Theorem 2.4 is stronger than Theorem 1.1.

Example 2.6. *Let $\mathbb{R}$ with the usual metric and let $A = [0, 1]$ and $B = [-1, 0]$. We define the cyclic mapping $T : A \cup B \to A \cup B$ by $T(x) = -x$ when $x \in A$ and $T(y) = -\frac{1}{2}y$ if $y \in B$. It is easy to show that for each $x \in A$ and $y \in B$*

$$|T^2x - T^2y| \leq \frac{1}{2}|x - y| \leq \frac{1}{2}\delta\big[\{x, T^2x, Ty\}, \{Tx, y, T^2y\}\big] = \frac{1}{2}\delta\big[A_{2,2}^{x,y}, B_{2,2}^{x,y}\big].$$

Thus T is a cyclic contraction of Fisher-type on $A\cup B$ and T has a unique fixed point $x^*=0\in A\cap B$. Note that in this example Theorem 1.1 is not useful and Theorem 1.4 only requires that x^* be a fixed point of T^2.

In the next theorem, we relax the continuity condition of T in Theorem 2.4 in the case $p=1$ or $q=1$.

Theorem 2.7. *Let A and B be nonempty and closed subsets of a complete metric space (X,d). Assume that T is a cyclic contraction of Fisher-type on $A\cup B$ for which $p=1$. Then T has a unique fixed point x^* in $A\cap B$ and the Picard iteration $\{T^nx_0\}$ converges to x^* for any starting point $x_0\in A\cup B$.*

Proof. The proof is quite similar to the proof of the preceding theorem. Just enough to show that x^* is a fixed point ot T. Since T is a contraction of Fisher-type, we have (note that q is odd)

$$\begin{aligned}
d(x^*,Tx^*) &= \lim_{n\to\infty} d(Tx^*,T^{2n}x_0)=\lim_{n\to\infty} d(Tx^*,T^qT^{2n-q}x_0)\\
&\le \lim_{n\to\infty} c\delta\big[A_{1,q}^{x^*,T^{2n-q}x_0},B_{1,q}^{x^*,T^{2n-q}x_0}\big]\\
&= \lim_{n\to\infty} c\delta\Bigg[\big\{x^*,T^{2j+1}T^{2n-q}x_0,0\le j\le \frac{q-1}{2}\big\}\\
&\qquad ,\big\{Tx^*,T^{2i}T^{2n-q}x_0,0\le i\le \frac{q-1}{2}\big\}\Bigg]\\
&= cd(x^*,Tx^*).
\end{aligned}$$

Thus we get $Tx^*=x^*$. □

As a corollary, we derive the following result that is generalization of Theorems 1.1 and 1.2 and special case of Theorem 2 in [20] and Remark 3.20 in [5] without using property UC and WUC.

Theorem 2.8. *Let A and B be nonempty and closed subsets of a complete metric space (X,d). Let T be a cyclic mapping on $A\cup B$ such that*

$$d(Tx,Ty)\le c\max\big\{d(x,y),d(x,Tx),d(y,Ty)\big\}, \tag{6}$$

for all $x\in A$ and $y\in B$ where $c\in[0,1)$. Then T has a unique fixed point x^* in $A\cap B$ and the Picard iteration $\{T^nx_0\}$ converges to x^* for any starting point $x_0\in A\cup B$.

Example 2.9. *Let $\mathbb{R}$ with the usual metric and let $A=\{0,2.25,2.5\}$ and $B=\{0,1,4\}$. We define the cyclic mapping $T:A\cup B\to A\cup B$ by*

$$T(2.25)=1,T(2.5)=0,T(0)=0,T(4)=2.25,T(1)=0.$$

With $c=0.9$, we can check that relation (6) is true for all $x\in A$ and $y\in B$ and $x^*=0$ is unique fixed point of T in $A\cap B$. Also it is not difficult to check that Theorems 1.1 and 1.2 are not useful in this example.

From previous theorem, we immediately obtain the following common fixed point result.

Corollary 2.10. *Let (X, d) be a complete metric space and let $T : X \to X$ and $S : X \to X$ are mappings satisfying*

$$d(Tx, Sy) \leq c \max\big\{d(x, y), d(x, Tx), d(y, Sy)\big\},$$

for each $x, y \in X$ where $c \in [0, 1)$. Then T and S have a unique common fixed point in X.

In [13], S. Karpagam and S. Agrawal introduced the notions of the cyclic orbital contraction as follows.

Definition 2.11. [13] Let A and B be nonempty subsets of a metric space X and $T : A \cup B \to A \cup B$ be a cyclic map such that for some $x \in A$, there exists a $k_x \in (0, 1)$ such that

$$d(T^{2n}x, Ty) \leq k_x\ d(T^{2n-1}x, y),\ n \in \mathbb{N} \text{ and } y \in A, \tag{7}$$

then T is called a cyclic orbital contraction.

They proved that if A and B be nonempty closed subsets of a complete metric space X and $T : A \cup B \to A \cup B$ be a cyclic orbital contraction. Then $A \cap B$ is nonempty and T has a unique fixed point. Moreover, for self-map T on a complete metric space X such that for some $x \in X$, there exists a k_x, $0 < k_x < 1$, such that

$$d(T^{2n}x, Ty) \leq k_x d(T^{2n-1}x, y),\ n \in \mathbb{N} \text{ and } y \in A,$$

they showed that T has a unique fixed point.
In [3, 12–14] authors study cyclic orbital contraction types. In the following we obtain Theorem 2.2 and corollary 2.3 in [13] as a results of Lemma 2.3 and Theorem 2.8.

Theorem 2.12. *Let A and B be nonempty closed subsets of a complete metric space X and $T : A \cup B \to A \cup B$ be a cyclic map such that for some $x \in A$, there exists a $k_x \in [0, 1)$ such that*

$$\begin{aligned} d(T^{2n}x, Ty) \leq & k_x \max\big\{d(T^{2n-1}x, y), d(T^{2n-1}x, T^{2n}x), d(y, Ty)\big\}, \\ & \forall n \in \mathbb{N} \text{ and } \forall y \in A. \end{aligned} \tag{8}$$

Then $A \cap B$ is nonempty and T has a unique fixed point.

Proof. Let $\mathfrak{A} := \{x, T^2x, T^4x, \ldots\}$ and $\mathfrak{B} := \{Tx, T^3x, T^5x, \ldots\}$. It is obvious that $T(\mathfrak{A}) \subseteq \mathfrak{B}$ and $T(\mathfrak{B}) \subseteq \mathfrak{A}$. According to relation (8), we have

$$\begin{aligned} d(T(T^{2n-1}x), Ta) \leq & k_x\ \max\big\{d(T^{2n-1}x, a), d(T^{2n-1}x, T^{2n}x), d(a, Ta)\big\}, \\ & \forall n \in \mathbb{N} \text{ and } \forall a \in \mathfrak{A}, \end{aligned}$$

hence

$$d(Tb, Ta) \leq k_x \max\big\{d(b, a), d(b, Tb), d(a, Ta)\big\},\ \forall b \in \mathfrak{B} \text{ and } \forall a \in \mathfrak{A}.$$

Now by Lemma 2.3 the sequence $\{T^n x\}$ is Cauchy, and thus there exist a $x^* \in A \cup B$ such that $\{T^n x\}$ converges to it. Now $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n+1} x\}$ is a sequence in B and both converges to x^*. Since A and B are closed, $x^* \in A \cap B$. Just enough to show x^* is a unique fixed point ot T. From (8), we have

$$\begin{aligned} d(Tx^*, x^*) &= \lim_{n\to\infty} d(T^{2n}x, Tx^*) \\ &\leq \lim_{n\to\infty} k_x \max\left\{d(T^{2n-1}x, x^*), d(T^{2n-1}x, T^{2n}x), d(x^*, Tx^*)\right\} \\ &= k_x \; d(x^*, Tx^*). \end{aligned}$$

Thus we get $Tx^* = x^*$. To prove the uniqueness, suppose $\overline{x}$ be another fixed point of T in $A \cup B$, we have

$$\begin{aligned} d(x^*, \overline{x}) &= \lim_{n\to\infty} d(T^{2n}x, T\overline{x}) \\ &\leq \lim_{n\to\infty} k_x \max\left\{d(T^{2n-1}x, \overline{x}), d(T^{2n-1}x, T^{2n}x), d(\overline{x}, T\overline{x})\right\} \\ &= k_x \; d(x^*, \overline{x}), \end{aligned}$$

so $\overline{x} = x^*$. □

Corollary 2.13. *Let T be a self-map on a complete metric space X such that for some $x \in X$, there exists a $k_x \in [0, 1)$, such that*

$$d(T^{2n}x, Ty) \leq k_x \max\left\{d(T^{2n-1}x, y), d(T^{2n-1}x, T^{2n}x), d(y, Ty)\right\}, \quad \forall n \in \mathbb{N} \text{ and } \forall y \in X.$$

Then T has a unique fixed point.

The next example shows that Theorem 2.12 is stronger than Theorem 2.2 in [13].

Example 2.14. *Let $\mathbb{R}^2$ with the usual metric and let*

$$A = \{(0,0), (2.25,0), (2.5,0), (4,3)\} \text{ and } B = \{(0,0), (1,0), (4,0)\}.$$

We define the cyclic mapping $T : A \cup B \to A \cup B$ by

$$T(4,3) = (4,0), T(2.25,0) = (1,0), T(2.5,0) = (0,0),$$
$$T(0,0) = (0,0), T(4,0) = (2.25,0), T(1,0) = (0,0).$$

With $x = (4,3)$ and $k_x = 0.95$, we can check that relation 8 is true here and $x^* = (0,0)$ is unique fixed point of T in $A \cap B$. Also we can check that relation 7 is not correct, so Theorem 2.2 in [13] is not useful here.

3. Noncyclic contractions of Fisher-type

We begin this section by introducing the concept of noncyclic contraction of Fisher-type.

Definition 3.1. Let A and B be nonempty subsets of a metric space (X,d) and let T be a noncyclic mapping on $A\cup B$. Then, T is said to be a contraction of Fisher-type if for some $p,q\in\mathbb{N}$

$$d(T^px,T^qy)\leq c\delta[\mathcal{O}(x,p),\mathcal{O}(y,q)],$$

for all $x\in A$ and $y\in B$ where $c\in[0,1)$ and $\mathcal{O}(u,n):=\{u,Tu,T^2u,\ldots,T^nu\}$ for $u\in X$ and $n\in\mathbb{N}$.

The following lemma is essential to proving our main result in this section.

Lemma 3.2. *Let A and B be nonempty subsets of a metric space (X,d) and let T be a noncyclic contraction of Fisher-type on $A\cup B$. For $x_0\in A$ and $y_0\in B$, define $x_{n+1}:=Tx_n$ and $y_{n+1}:=Ty_n$ for each $n\geq 0$. Then*

$$\lim_{m,n\to\infty} d(x_n,y_m)=0.$$

Proof. We first show that for each $n,m\in\mathbb{N}$, we have

$$\delta[\mathcal{O}(x_0,n),\mathcal{O}(y_0,m)]=d(T^kx_0,T^ly_0),\ \text{where either } k<p \text{ or } l<q. \tag{9}$$

Suppose that $\delta[\mathcal{O}(x_0,n),\mathcal{O}(y_0,m)]=d(T^ix_0,T^jy_0)$, where $p\leq i\leq n$ and $q\leq j\leq m$. Since T is a noncyclic contraction of Fisher-type, we have

$$\begin{aligned} d(T^ix_0,T^jy_0)&=d(T^pT^{i-p}x_0,T^qT^{j-q}y_0)\\ &\leq c\delta[\mathcal{O}(x_{i-p},p),\mathcal{O}(y_{j-q},q)]\\ &\leq c\delta[\mathcal{O}(x_0,n),\mathcal{O}(y_0,m)]. \end{aligned}\tag{10}$$

Thus, we obtain $\delta[\mathcal{O}(x_0,n),\mathcal{O}(y_0,m)]=0$, then $\delta[\mathcal{O}(x_0,n),\mathcal{O}(y_0,m)]=d(x_0,y_0)$, and so (9) holds.
Now, we show that for each $m,n\in\mathbb{N}$

$$\delta[\mathcal{O}(x_0,n),\mathcal{O}(y_0,m)]\leq M_{x_0,y_0}, \tag{11}$$

where

$$\begin{aligned} M_{x_0,y_0}=\frac{1}{1-c}\max\big\{&d(T^ix_0,T^jy_0),d(T^ix_0,T^jx_0),d(T^iy_0,T^jy_0)\\ &:\ 0\leq i,j\leq\max\{p,q\}\big\}. \end{aligned}$$

To prove the claim, note that from (9), we have

$$\delta[\mathcal{O}(x_0,n),\mathcal{O}(y_0,m)]=d(T^kx_0,T^ly_0),$$

where either $0\leq k<p$ or $0\leq l<q$. If $k<p$ and $l<q$, then (11) trivially holds. So, without loss of generality we may assume that $0\leq k<p$ and $q\leq l\leq m$. Then, from (10) we get

$$\begin{aligned} \delta[\mathcal{O}(x_0,n),\mathcal{O}(y_0,m)]&=d(T^kx_0,T^ly_0)\leq d(T^kx_0,T^px_0)+d(T^px_0,T^ly_0)\\ &\leq d(T^kx_0,T^px_0)+c\delta[\mathcal{O}(x_0,n),\mathcal{O}(y_0,m)], \end{aligned}$$

and so (11) holds.
Now, we prove that for each $r, s, m, n \in \mathbb{N} \cup \{0\}$ with $n, m \geq \max\{p, q\}$ we have

$$\delta[\mathcal{O}(x_n, r), \mathcal{O}(y_m, s)] \leq c\delta[\mathcal{O}(x_{n-p}, r+p), \mathcal{O}(y_{m-q}, s+q)]. \tag{12}$$

From (10), for some $0 \leq r' \leq r, 0 \leq s' \leq s$ we have

$$\begin{aligned}\delta[\mathcal{O}(x_n, r), \mathcal{O}(y_m, s)] &= d(T^{r'}x_n, T^{s'}y_m) = d(T^{p+r'}x_{n-p}, T^{q+s'}y_{m-q}) \\ &\leq c\ \delta[\mathcal{O}(x_{n-p}, r+p), \mathcal{O}(y_{m-q}, s+q)].\end{aligned}$$

Hence, (12) holds. Then, from (12) for $n, m \geq \max\{2p, 2q\}$ we have

$$\begin{aligned}d(x_n, y_m) = \delta[\mathcal{O}(x_n, 0), \mathcal{O}(y_m, 0)] &\leq c\delta[\mathcal{O}(x_{n-p}, p), \mathcal{O}(y_{m-q}, q)] \\ &\leq c^2\ \delta[\mathcal{O}(x_{n-2p}, 2p), \mathcal{O}(y_{m-2q}, 2q)].\end{aligned}$$

By continuing this process and using (11), we obtain

$$\begin{aligned}d(A, B) &\leq d(x_n, y_m) \\ &\leq c^{k_{n,m}}\ \delta[\mathcal{O}(x_{n-k_{n,m}p}, k_{n,m}p), \mathcal{O}(y_{m-k_{n,m}q}, k_{n,m}q)] \\ &\leq c^{k_{n,m}}\ \delta[\mathcal{O}(x_0, n), \mathcal{O}(y_0, m)] \\ &\leq c^{k_{n,m}}\ M_{x_0,y_0}, \qquad (13)\end{aligned}$$

where $k_{n,m} = \min\{\lfloor \frac{n}{p} \rfloor, \lfloor \frac{m}{q} \rfloor\}$. Therefore, $\lim_{m,n\to\infty} d(x_n, y_m) = 0$. □

Now we are ready to prove our main result in this section which is an extension of Theorem 2 in [7] for noncyclic mappings.

Theorem 3.3. *Let A and B be nonempty and closed subsets of a complete metric space (X, d). Assume that T is a noncyclic contraction of Fisher-type on $A \cup B$ for which the restriction of T to A (or B) is continuous. Then T has a unique fixed point x^* in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in A \cup B$.*

Proof. Let $x_0 \in A$ and $y_0 \in B$. From Lemma 3.2, $d(A, B) = 0$ and the sequences $\{T^n x_0\}$ and $\{T^n y_0\}$ are Cauchy sequences in A and B respectively. By the completeness of A and B and using again from Lemma 3.2 the sequences $\{T^n x_0\}$ and $\{T^n y_0\}$ converges to some $x^* \in A \cap B$. Since $T_{|_A}$ is continuous, it follows that $Tx^* = x^*$. To prove the uniqueness, suppose $\overline{x}$ be another fixed point of T in $A \cup B$. By Lemma 3.2, we have

$$d(\overline{x}, x^*) = \lim_{n\to\infty} d(T^n\overline{x}, T^n x^*) = 0,$$

so $\overline{x} = x^*$. □

Example 3.4. *Equip $\mathbb{R}$ with the usual metric and let $A = [0, 2]$ and $B = [-1, 0]$. We define the noncyclic mapping $T : A \cup B \to A \cup B$ by*

$$Tx = \frac{1}{2}x \ if \ x \in A \cap [0, 1], Tx = \frac{1}{x} \ if \ x \in A \cap (1, 2] \ and \ Ty = \frac{1}{2}y \ if \ y \in B.$$

It is straightforward to show that for each $x \in A$ and $y \in B$

$$|T^2x - Ty| \leq \frac{1}{2}|x-y| \leq \frac{1}{2}\delta\big[\{x, Tx, T^2x\}, \{y, Ty\}\big] = \frac{1}{2}\delta[\mathcal{O}(x,2), \mathcal{O}(y,1)].$$

Thus T is a noncyclic contraction of Fisher-type on $A \cup B$ and T has unique fixed point $x^* = 0 \in A \cap B$.

The self mapping T in Example 2.5 is a noncyclic contraction of Fisher-type, too. So this example shows that the continuity condition of T in the above theorem is not extra. In the following theorem which generalizes Theorem 3 in [7], we relax the continuity condition of $T_{|_A}$ (resp. $T_{|_B}$), in the case of $p = 1$ (resp. $q = 1$).

Theorem 3.5. *Let A and B be nonempty and closed subsets of a complete metric space (X, d). Assume that T is a noncyclic contraction of Fisher-type on $A \cup B$ for which $p = 1$. Then T has a unique fixed point x^* in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in A \cup B$.*

Proof. The proof is quite similar to the proof of the preceding theorem. Just enough to show x^* is a fixed point ot T. Let $y_0 \in B$. Since T is a contraction of Fisher-type, we have

$$d(Tx^*, T^n y_0) = d(Tx^*, T^q T^{n-q} y_0) \leq c\delta[\mathcal{O}(x^*, 1), \mathcal{O}(T^{n-q} y_0, q)],$$

we know that $\{T^n y_0\}$ converges to x^*, so

$$\limsup_{n\to\infty} d(Tx^*, T^n y_0) \leq c\ \max\big\{0, \limsup_{n\to\infty} d(Tx^*, T^n y_0)\big\}.$$

Hence,

$$d(Tx^*, x^*) = \lim_{n\to\infty} d(Tx^*, T^n y_0) = 0.$$

Thus we get $Tx^* = x^*$. □

From Theorem 3.5, in the case $p = q = 1$ we get the following result that is extension theorems 1.1 and 1.3 to noncyclic mappings.

Theorem 3.6. *Let A and B be nonempty and closed subsets of a complete metric space (X, d). Let T be a noncyclic mapping on $A \cup B$ satisfying*

$$d(Tx, Ty) \leq c \max\big\{d(x, y), d(x, Ty), d(y, Tx)\big\},$$

for each $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then T has a unique fixed point x^* in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in A \cup B$.

From Theorem 3.6, we immediately obtain the following common fixed point result.

Corollary 3.7. *Let (X, d) be a complete metric space and let $T : X \to X$ and $S : X \to X$ are two mappings satisfying*

$$d(Tx, Sy) \leq c \max\big\{d(x, y), d(x, Sy), d(y, Tx)\big\}$$

for all $x, y \in X$ where $c \in [0,1)$. Then S and T have a unique common fixed point in X.

We end this section with give a example which shows that the theorem 1.2 is not hold for noncyclic maps.

Example 3.8. *Let $\mathbb{R}^2$ with the usual metric. Let $A = \{(0,0),(1,1)\}$ and $B = \{(1,0),(0,1)\}$ be subsets of $\mathbb{R}^2$. We define the noncyclic mapping $T : A \cup B \to A \cup B$ by $T(0,0) = (1,1)$, $T(1,1) = (0,0)$, $T(1,0) = (0,1)$ and $T(0,1) = (1,0)$. It is straightforward to see that for each $x \in A$ and $y \in B$*

$$1 = d(Tx, Ty) \leq \frac{2}{5}[d(x, Tx) + d(y, Ty)] = \frac{4\sqrt{2}}{5}.$$

Thus T is a noncyclic contraction in the sence of Kannan, but $d(A,B) \neq 0$, and T has not fixed point.

4. Application to Complex Function Theory

Theorem 4.1. *Let A and B be nonempty compact subsets of a domain D of the complex plane. Let f and g be functions in D such that f is analytic in D. Suppose that*

(a) *$f(A) \subseteq B$ and $g(B) \subseteq A$,*
(b) *$|f'(z)| < 1$, for all $z \in A \cup B$,*
(a) *$|f(z) - g(z')| \leq |f(z) - f(z')|$, for all $z \in A$ and $z' \in B$.*

Then f and g have a unique common fixed point $z^* \in A \cap B$.

Proof. Since $|f'(z)| < 1$ is continuous on the compact set $A \cup B$, it attains its maximum at some point, say $u \in A \cup B$. Let $\lambda = |f'(u)|$, then $\lambda < 1$. Hence for all $z \in A$, we have $|f'(z)| \leq \lambda < 1$. Now for all $z \in A$ and $z' \in B$, we have

$$|f(z) - g(z')| \leq |f(z) - f(z')| = |\int_{z'}^{z} f'(\xi) d\xi| \leq \lambda |z - z'|.$$

Now if define cyclic map $T : A \cup B \to A \cup B$ with

$$T(z) = \begin{cases} f(z) & \text{if } z \in A \\ g(z) & \text{if } z \in B \end{cases},$$

then the result follows by invoking Theorem 2.8. □

Similarly, by using Theorem 3.6, the following theorem can be proved.

Theorem 4.2. *Let A and B be nonempty compact subsets of a domain D of the complex plane. Let f and g be functions in D such that f is analytic in D. Suppose that*

(a) *$f(A) \subseteq A$ and $g(B) \subseteq B$,*
(b) *$|f'(z)| < 1$, for all $z \in A \cup B$,*
(a) *$|f(z) - g(z')| \leq |f(z) - f(z')|$, for all $z \in A$ and $z' \in B$.*

Then f and g have a unique common fixed point $z^* \in A \cap B$.

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