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TWO-STAGE AND MODIFIED TWO-STAGE ESTIMATION IN THRESHOLD FIRST-ORDER AUTOREGRESSIVE PROCESS

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Abstract. In this paper, we discuss the two-stage and the modified two-stage procedures for the estimation of the threshold autoregressive parameter in a first-order threshold autoregressive model (TAR(1)). This is motivated by the problem of finding a final sample size when the sample size is unknown in advance. For this purpose, a two-stage stopping variable and a class of modified two-stage stopping variables are proposed. Afterward, we prove the significant properties of the procedures, including asymptotic efficiency and asymptotic risk efficiency for the point estimation based on least-squares estimators. To illustrate this theory, comprehensive Monte Carlo simulation studies is conducted to observe the significant properties of the procedures. Furthermore, the performance of procedures based on Yule-Walker estimators is investigated and the results are compared in practice that confirm our theoretical results. Finally, real-time-series data is studied to demonstrate the application of the procedures.

Keywords: Two-stage procedure, Modified two-stage procedure, Threshold autoregressive process, Point estimation, Monte Carlo simulation.
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1. Introduction

Time series data is used and deployed in large amount of ways. These contexts are to show changes in metrics over time, or to predict future values of a metric. Three different aspects of the time series data are used in this analysis. Time series forecasting is used by organizations to predict the probability of upcoming events. A common way to estimate the parameter is using an optimal fixed size procedure. When the sample size is not fixed and known in advance there will be difficulty in analyzing the time series data. Sequential procedures are an appropriate approach to tackle this problem.

When significant results are observed the sequential procedure will stop based on a predefined stopping rule. The performance of these procedures is evaluated in terms of asymptotic properties. The best way for stopping the sampling procedure is the relevant stopping rule strategy of these sequential

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procedures. The accuracy of these procedures is shown by analyzing the final sample size of these procedures as well as the sample size that is just large enough for ensuring optimality. [1], [25] and [6], the first people who introduced, an alternative inferential method for point and interval estimation of an unknown population mean.

The proposed method provides a stopping strategy that the required sample size is determined by sequentially stopping rules. The difference between procedures is the stopping strategy and how to implement it. Purely sequential, two-stage, and modified two-stage procedures are the most widely used. Many researchers have published studies on a broad variety of sequential procedures such as purely sequential, two-stage, and modified two-stage. These studies have given an better understanding of how sequential procedures can be used to tackle the unsolved problems in point and interval estimation that we give a glimpse of these in the following.

A two-stage procedure was first introduced by [31, 32]. He proposed a two-stage procedure to construct a fixed-width confidence interval and hypotheses testing for the mean in a normal population. [20] studied a modified two-stage procedure to construct a confidence interval for the mean in a normal population. [21] extended the results based on their previous research a year later. In addition, [19] discussed a two-stage procedure for the mean estimation in a normal population under the condition of unknown variance for constructing a confidence interval. The autoregressive parameter and the mean of a first-order autoregressive model are investigated via a purely sequential sampling scheme by [27,28]. [2] described a sequential sampling procedure to estimate the autoregressive parameter in a first-order autoregressive process with Weibull errors. [7] analyzed a purely sequential procedure to estimate the mean vector parameter in a multivariate linear process.

Also, a purely sequential procedure is considered for the mean vector estimation in a p-independent first-order autoregressive model by [22]. [14] researched the point and confidence interval estimation of the autoregressive parameter in a pth order auto regressive model via a purely sequential procedure. Sequential estimation of autoregressive parameters is considered in a multiple pth order autoregressive model by [3]. An estimation problem involving a threshold AR(1) model is discussed via a purely sequential procedure by [16]. Moreover, the fixed-size confidence region is investigated in single and multiple first-order threshold autoregressive models through a purely sequential procedure by [29].A purely sequential procedure in a stochastic regression model is studied by [15]. The first-order RCA model is considered by [12] that suggested a strongly consistent sequential estimator of coefficients of a univariate p order RCA model. [8] carry out studying to construct a confidence interval with time series observations through a sequential procedure. Parameters estimation is investigated through a two-stage procedure and purely sequential procedure under a modified Linex loss function in a normal distribution by [23]. A sequential parameters estimation is suggested in a single and multivariate random coefficient autoregressive model by [10]. A two-stage procedure is investigated to estimate a parameter in a stress-strength model by [17]. Furthermore, [13] have worked on a two-stage procedure to construct a confidence interval for a parameter in an exponential distribution. Recently, [30] reconsidered a purely sequential procedure to estimate a parameter in an AR(1) model that was previously studied by [28]. An innovative and general class of modified two-stage sampling schemes proposed by assuming the squared error loss by [9]. Also, a two-stage procedure for point and interval estimation in the AR(1) model is discussed by [26]. Furthermore, the performance of this procedure with the purely sequential procedure is compared.

As mentioned, many authors investigated the purely sequential and two-stage procedures in time series models. The advantages of the modified two-stage and two-stage procedure include the simplicity of implementation and lower cost of this procedure compared to the purely sequential procedure. A modified two-stage procedure is presented inspired by the two-stage procedure that determines the pilot sample size by providing a strategy. In the situation where we can provide a strategy for determining the initial sample size in the two-stage procedure, the procedure is proposed. This procedure in many cases prevents overestimation of the final sample size. Also, the modified two-stage reduced the weakness of the two-stage procedure in estimating. We are also interested in investigating the performance of the two-stage procedure and modified two-stage procedure because of the widely used and operational savings of these procedures. As we know, nonlinear time series modeling drew much attention in the 1970s, due to the nonlinear time series models compared to the linear models providing a much wider spectrum of possible dynamics for economic and financial time series data. In this regard, [34] introduced a threshold autoregressive model that not only provides a better fit than linear models but also exhibits strictly nonlinear behavior. We can show that this model captures the dynamic behavior of time series by switching the regimes. Among the features of this class of models, items limit cycles, amplitude-dependent frequencies, and jump phenomena can be mentioned which linear models fail to capture. These models are generally agreed to be useful in modeling discrete time series that exhibit piece-wise linearity.

The advantages of procedures and using time series data extensively encourage us to examine the performance of point estimation by proposing different stopping rules. The purpose of this paper is to exhibit the performance of the two-stage stopping variable and the modified two-stage variable with further conditions on it. To this end, the problem of point estimation via the two-stage procedure and the modified two-stage is investigated in the threshold autoregressive model. The results are presented in Theorems 2.2, 3.2 and 3.4. The point estimation is studied based on the least-squares estimator as the reciprocal of the cost per observation tends to infinity and theorems demonstrate asymptotic properties of the procedures including asymptotic risk efficiency and asymptotic efficiency. Several methods are usually used to estimate the autoregressive parameters, such as least squares, Yule-Walker, and Burg's method. For a large sample, these estimation techniques should lead to approximately the same estimates. In some special cases, the least-squares estimation method leads to poor parameter estimates and we decide to review the results based on the Yule-Walker estimators. Simulation studies are conducted to investigate the performance of both procedures based on both estimators. Results in terms of the stopping variables, the ratio of the average stopping variable to the optimal fixed sample size, the root of mean square error (RMSE) of the parameter, and the ratio of risk efficiency functions are reported. Furthermore, the asymptotic properties and applicability of the procedure are investigated via real application implementations.

The paper is structured as follows: In Section 2, we discuss the point estimation through a two-stage procedure. In Section 3, the class of modified two-stage procedure is suggested and is reviewed for point estimation. In Section 4, we proceed with the study of comprehensive simulations. Finally, Section 5 is devoted to illustrating the applicability of the two-stage and modified two-stage procedures with real-time-series data.

2. **Two-stage procedure**

A threshold first-order autoregressive model (TAR(1)) is given by,

$$X_i = \theta_1 X_{i-1}^{+} + \theta_2 X_{i-1}^{-} + \varepsilon_i,\ \ i = 1, 2, ..., n,$$

where $x^+ = \max(x, 0), x^- = \min(x, 0)$ are defined for a real number x and $\boldsymbol{\theta} = (\theta_1, \theta_2)$ are assumed to be real parameters not neccessarily equal. Also, we suppose that $\{\varepsilon_i, i \geq 1\}$ are iid random variable with unknown distribution F and $\mathbb{E}(\varepsilon_1) = 0 < \mathbb{E}(\varepsilon_1^2) = \sigma^2 \in (0, \infty)$.

It is well known that the process $\{X_i; i \geq 1\}$ is ergodic if and only if

$$\boldsymbol{\theta} \in \Theta = \{\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} : \theta_1 < 1, \theta_2 < 1, \theta_1\theta_2 < 1\}. \tag{1}$$

Due to equation (1), the existence of an invariant probability distribution of process $\{X_i; i \geq 1\}$ is confirmed that is proven by [24]. The initial random variable X_0 has distribution $\pi(.)$ the invariant probability distribution of the Markov chain $\{X_i; i \geq 1\}$ that strictly stationary of $\{X_i; i \geq 1\}$ is implied. [4] proposed sufficient condition for stationarity and ergodicity for TAR(P)

$$\max_j \sum\nolimits_{i=1}^{p} |\theta_i| < 1.$$

It is noteworthy that $\mathbb{E}[|X_1|^k] < \infty$ for some integer $k \geq 1$ and $\mathbb{E}[|X_0|^k] < \infty$ for each $\boldsymbol{\theta} \in \Theta$, which is noted in [5]. The least-squares estimators of θ_1 and θ_2 are given by

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{n} X_i X_{i-1}^{+}}{\sum_{i=1}^{n} X_{i-1}^{+2}},\ \hat{\theta}_2 = \frac{\sum_{i=1}^{n} X_i X_{i-1}^{-}}{\sum_{i=1}^{n} X_{i-1}^{-2}}.$$

The corresponding loss function is defined as follows:

$$L_n(\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}, \boldsymbol{\theta}) = An^{-1}\left(\hat{\boldsymbol{\theta}}_{\boldsymbol{n}} - \boldsymbol{\theta}\right)'\Gamma_n\left(\hat{\boldsymbol{\theta}}_{\boldsymbol{n}} - \boldsymbol{\theta}\right) + n,$$

where $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}} = \binom{\hat{\theta}_1}{\hat{\theta}_2}$ and $\Gamma_n = diag(\sum_{i=1}^{n} X_{i-1}^{+2}, \sum_{i=1}^{n} X_{i-1}^{-2})$ is a diagonal matrix. We assume $A(>0)$ known weight that reflects the reciprocal of the cost per observation. It follows that

$$\sigma^{-2}\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right)'\Gamma_n\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right) \xrightarrow{D} \chi_2^2,\ as\ n \to \infty, \tag{2}$$

where it follows from Theorem 3.2 [24]. Here, χ_2^2 is a chi-square random variable with two degrees of freedom. From Proposition 2.1 in [16] it is established that $\{Q_n = (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta})'\Gamma_n(\hat{\boldsymbol{\theta}}_{\boldsymbol{n}} - \boldsymbol{\theta}), n \geq 1\}$ is uniformly integrable, under certain regularity conditions. Consider the risk function due to (2) and the uniformly integrable property

$$R_n = \mathbb{E}\big[L_n(\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}, \boldsymbol{\theta})\big] = 2n^{-1}A\sigma^2 + n + o(n^{-1}).$$

By ignoring the term $o(n^{-1})$, the optimal sample size is approximately obtained $n_A \simeq (2A)^{1/2}\sigma$. We have the corresponding minimum risk function

$$R_{n_A} \simeq 2(2A)^{1/2}\sigma.$$

We cannot find an appropriate sample size to minimize the risk function in practice when σ^2 is unknown. It should be noted that σ^2 is defined

$$\hat{\sigma}_n^2 = n^{-1}\sum\nolimits_{i=1}^{n}\left(X_i - \hat{\theta}_1 X_{i-1}^{+} - \hat{\theta}_2 X_{i-1}^{-}\right)^2 = n^{-1}\sum\nolimits_{i=1}^{n}\varepsilon_i^2 - n^{-1}Q_n$$

for any n. To solve the problem, we determine the final sample size by a two-stage procedure due to the initial sample size m. The two-stage stopping rule is defined analogy with n_A,

$$N_m = \max\{m, \lfloor(2A)^{1/2}\hat{\sigma}_m\rfloor + 1\},$$

where $\hat{\sigma}_m$ is the least squares estimator based on the initial sample size m and $\lfloor x\rfloor$ denotes the largest integer smaller than x. We investigate the performance of the two-stage procedure based on the proposed stopping rule relative to the optimal fixed sample procedure. The result of this section is Theorem 2.2 that shows the two-stage procedure as much as the optimal-fixed sample procedure efficient as $A \to \infty$. Also, the asymptotically efficient and the asymptotically risk efficient properties are established as a consequence of the following theorem. To achieve these properties, we shall present Lemma 2.1.

Lemma 2.1. *Assume for $s \geq 1$ that $\mathbb{E}|\varepsilon_1|^{4s} < \infty$ and $\mathbb{E}|X_0|^{4s} < \infty$. In addition if $A^{1/2(1+\eta)} \leqslant m = o(A^{1/2})$ for some $\eta > 0$, then for any $0 < \theta < 1$,*

$$P\left(N_m < (1-\theta)(2A)^{1/2}\sigma\right) = O\left(A^{-s/2(1+\eta)}\right), \tag{3}$$

$$P\left(N_m > \left[(1+\theta)(2A)^{1/2}\sigma\right] + 1\right) = O\left(A^{-s/2(1+\eta)}\right). \tag{4}$$

Proof. The proof is similar to Lemma 1 [26], which is omitted. □

Theorem 2.2. *Suppose for $s > 1$ that $\mathbb{E}|\varepsilon_1|^{4s} < \infty$, $\mathbb{E}|X_0|^{4s} < \infty$ and $A^{1/2(1+\eta)} \leqslant m = o(A^{1/2})$ for some $\eta \in (0, (s+1)/2 - 1)$. Then as $A \to \infty$,*

$$\frac{N_m}{n_A} \xrightarrow{a.s} 1, \tag{5}$$

$$\mathbb{E}\left[\frac{N_m}{n_A}\right] \to 1, \text{ (asymptotically efficient)} \tag{6}$$

$$\frac{R_{N_m}}{R_{n_A}} \to 1, \text{ (asymptotically risk efficient)}. \tag{7}$$

Proof. In view of N_m, write

$$\hat{\sigma}_m (2A)^{1/2} \leq N_m \leq \hat{\sigma}_m (2A)^{1/2} + m.$$

$\sigma_m^2 \xrightarrow{a.s} \sigma^2$ follows as $A \to \infty$ since $\sigma_n^2 \xrightarrow{a.s} \sigma^2$ [24]. Then, by dividing and taking limit, we obtain equation (5) as $A \to \infty$. Also, by taking expectation of equation (5), equation(6) is yielded. It remain to prove equation (7), set

$$\begin{aligned} R_{N_m}/R_{n_A} &= \{\mathbb{E}[L_{N_m}(\hat{\boldsymbol{\theta}}_{\boldsymbol{N_m}}, \boldsymbol{\theta})]\}/R_{n_A} \\ &= A\mathbb{E}\left[N_m^{-1}\left(\hat{\boldsymbol{\theta}}_{N_m} - \boldsymbol{\theta}\right)\Gamma_{N_m}\left(\hat{\boldsymbol{\theta}}_{N_m} - \boldsymbol{\theta}\right)'\right]/R_{n_A} + \mathbb{E}[N_m]/R_{n_A}. \end{aligned} \tag{8}$$

From equation (5) and $R_{n_A} \approx 2(2A)^{1/2}\sigma$, it is sufficient to show that the first term (8) tends to 1/2 as $A \to \infty$. For this purpose, suppose that for $\phi \in (0,1)$, $n' = [(1-\phi)(2A)^{1/2}\sigma]$, $n'' = [(1+\phi)(2A)^{1/2}\sigma] + 1$, $B = \{n' \leq N_m \leq n''\}$. By Cauchy Schwartz inequality, equation (2.6) of [16], and (3) result follows similarily equation (2.22) of [16]

$$(2A)^{1/2}\sigma\mathbb{E}N_m^{-1}Q_{N_m}I_{N_m<n'} \longrightarrow 0, \ \textit{as}\ A \to \infty. \tag{9}$$

by utlizing (4) and similar argument (9), we have

$$(2A)^{1/2}\sigma\mathbb{E}N_m^{-1}Q_{N_m}I_{N_m>n''} \longrightarrow 0, \ \textit{as}\ A \to \infty.$$

From [16], $\{Q_n, n \geq 1\}$ is u.c.i.p. Then from equations (2), (5) and the Anscombe's theorem follows that

$$(2A)^{1/2}\sigma N_m^{-1} Q_{N_m} I_B \xrightarrow{D} \chi_2^2\sigma^2, \ as \ A \to \infty.$$

Moreover, by equation (2.6) of [16], for any $\beta > 1$ follows,

$$\mathbb{E}\left[(2A)^{1/2}\sigma N_m^{-1} Q_{N_m} I_B\right]^{\beta} \leq (2A)^{\beta/2}(n')^{-\beta}\sigma\beta\mathbb{E}\left[\max_{n'\leq n\leq n''}(Q_n)^{\beta}\right] = O(1).$$

Therefore, $\{(2A)^{1/2}\sigma N_m^{-1} Q_{N_m} I_B; A \geq 1\}$ is u.i and hence using previous result,

$$(2A)^{1/2}\sigma\mathbb{E}N_m^{-1} Q_{N_m} I_B \to 2\sigma^2, \ as \ A \to \infty.$$

and the proof is completed. □

3. Modified Two-stage procedure

This section aims to provide a brief discussion for the class of the modified two-stage procedure for the proposed model. It is divided into discussions regarding the modified two-stage procedure and the modified two-stage procedure under condition, respectively. As mentioned, the modified two-stage procedure is reviewed. The first strategy is determining the pilot sample size. The stopping rule for $\gamma \in (1/2, \infty)$ and $m_0 \geq 2$ is given by

$$m = \max\{m_0, [A^{1/2(1+\gamma)}] + 1\}.$$

In the spirit of a two-stage procedure, we define a final stopping rule

$$N'_m = \max\{m, \lfloor (2A)^{1/2}\hat{\sigma}_m \rfloor + 1\}.$$

In the following, the asymptotic properties of the modified two-stage procedure are examined. To prove the properties, before presenting the theorem such as the two-stage procedure, we need to state the following lemma.

Lemma 3.1. *Assume that $\mathbb{E}|\varepsilon_1|^{4s} < \infty$ and $\mathbb{E}|X_0|^{4s} < \infty$ for $s \geq 1$. Also, $m_0 = o(A^{1/2})$ and for any $0 < \theta < 1$,*

$$P\left(N'_m < (1-\theta)(2A)^{1/2}\sigma\right) = O\left(A^{-s/2(1+\gamma)}\right),$$

$$P\left(N'_m > \left[(1+\theta)(2A)^{1/2}\sigma\right] + 1\right) = O\left(A^{-s/2(1+\gamma)}\right).$$

Proof. The proof is similar to Lemma 2.1, which refuse to repeat. □

The main theorem of this section is now addressed.

Theorem 3.2. *Suppose for $s > 1$ that $\mathbb{E}|\varepsilon_1|^{4s} < \infty$, $\mathbb{E}|X_0|^{4s} < \infty$ and $m_0 = o(A^{1/2})$. Then as $A \to \infty$,*

$$\frac{N'_m}{n_A} \xrightarrow{a.s} 1, \tag{10}$$

$$\mathbb{E}\left[\frac{N'_m}{n_A}\right] \to 1, \ \text{(asymptotically efficient)}$$

$$\frac{R_{N'_m}}{R_{n_A}} \to 1, \text{ (asymptotically risk efficient).}$$

Proof. According to the definition of N'_m, we have

$$\hat{\sigma}_m(2A)^{1/2} \leq N'_m \leq \hat{\sigma}_m(2A)^{1/2} + mI_{(N'_m=m)}.$$

The properties are proved similar to Theorem 2.2, which are not mentioned. □

In the following, we intend to provide the stopping rules that a decision rule satisfies the condition in which a lower bound of variance is known. To this end, we suppose $\sigma > \sigma_l > 0$ while σ_l is known. We first determine the pilot sample size due to σ_l as follows:

$$m_1 = \max\{m_0, [\sigma_l A^{1/2}] + 1\}.$$

Then, the desired stopping rule based on the optimal fixed sample size is given by

$$N_{m_1} = \max\{m_1, \lfloor A^{1/2}\hat{\sigma}_{m_1}\rfloor + 1\}. \tag{11}$$

As before, the asymptotic properties are examined as $A \to \infty$ that Theorem 3.4 and Lemma 3.3 as a practical and important results are expressed.

Lemma 3.3. *Assume that* $\mathbb{E}|\varepsilon_1|^{4s} < \infty$ *and* $\mathbb{E}|X_0|^{4s} < \infty$ *for* $s \geq 1$*. In addition if* $A^{1/2(1+\eta)} \leqslant m_1$ *for some* $\eta > 0$*, then for any* $0 < \theta < 1$*,*

$$P\left(N_{m_1} < (1-\theta)(2A)^{1/2}\sigma\right) = O\left(A^{-s/2(1+\eta)}\right),$$

$$P\left(N_{m_1} > \left[(1+\theta)(2A)^{1/2}\sigma\right] + 1\right) = O\left(A^{-s/2(1+\eta)}\right).$$

Proof. The proof is similar to Lemma 2.1, which we refuse to repeat. □

In the following, we present the results of the performance and efficiency of the procedure.

Theorem 3.4. *Suppose for* $s > 1$ *that* $\mathbb{E}|\varepsilon_1|^{4s} < \infty$, $\mathbb{E}|X_0|^{4s} < \infty$ *and* $A^{1/2(1+\eta)} \leqslant m_1$ *for some* $\eta \in (0, (s+1)/2 - 1)$*. Then as* $A \to \infty$*,*

$$\frac{N_{m_1}}{n_A} \xrightarrow{p} 1, \tag{12}$$

$$\mathbb{E}\left[\frac{N_{m_1}}{n_A}\right] \to 1, \text{ (asymptotically efficient)} \tag{13}$$

$$\frac{R_{N_{m_1}}}{R_{n_A}} \to 1, \text{ (asymptotically risk efficient).} \tag{14}$$

Proof. Note that

$$\hat{\sigma}_{m_1} A^{1/2} \leq N_{m_1} \leq \hat{\sigma}_{m_1} A^{1/2} + m_1 I_{(N_{m_1}=m_1)}.$$

Clearly, $m_1/n_A \to \sigma_l/\sigma$ as $A \to \infty$, divide throughout above equation by n_A. In view of Lemma 3.3, as $A \to \infty$, we have

$$m_1 I_{(N_{m_1}=m_1)} \xrightarrow{\text{p}} 0.$$

Equation (12) follows and immediately achieve (13). The assertion of (14) follows similar to Theorem 2.2. The proof is complete. □

As can be seen, the performance of the modified two-stage procedure is close to the results of the two-stage procedure. Also, the asymptotic properties of the proposed procedures are obtained the same as the properties of the purely sequential procedure ([16]). As mentioned, the two-stage and the modified two-stage procedures are preferable to the purely sequential procedure in terms of simplicity of implementation. Based on the results, the proposed procedures are suitable replacement procedures for determining the sample size. In an appropriate situation, the modified two-stage procedure can be used as a proposed alternative to the two-stage procedure, which is more accurate than the two-stage procedure. In other words, the two-stage and the modified two-stage procedures share almost the same asymptotic properties. Also, the two-stage procedure is operationally much more convenient because of sampling at most two batches but the modified two-stage procedure prevents overestimation final sample size. In the following, we examine the accuracy and performance of the execution procedures. Before checking, the Yule-Walker estimators of the model are given by,

$$\hat{\theta}_1 = \frac{\sum_{i=2}^{n} X_i X_{i-1}^{+}}{\sum_{i=2}^{n+1} X_{i-1}^{+2}}, \ \hat{\theta}_2 = \frac{\sum_{i=2}^{n} X_i X_{i-1}^{-}}{\sum_{i=2}^{n+1} X_{i-1}^{-2}}.$$

As we know, the Yule-Walker estimators are asymptotically identical to the least-squares estimators. All previous studies are based on the least-squares estimators, and we are curious to examine the performance of stopping rules based on these estimators. A similar argument in [16], can be shown that $\{Q_n, n \geq 1\}$ based on the Yule-Walker estimators is uniformly integrable, under certain regularity conditions. The properties of procedures discussed in the previous sections are proved also based on the Yule-Walker estimators and we refrain from repeating these theorems. In the next section, the performance of the procedures is examined based on both estimators.

4. **Simulation study**

In this section, Monte Carlo simulation studies are conducted to evaluate the performance of point estimation for $\boldsymbol{\theta} = (\theta_1, \theta_2)$ when $\varepsilon_i \sim N(0,1)$. The performance of the two-stage procedure and modified two-stage procedure to the optimal fixed sample procedure are compared. To assess, the ratio of the average two-stage stopping variable to the optimal fixed sample size, the root of mean square error (RMSE) of $\hat{\boldsymbol{\theta}}_n$, the ratio of the risk function to the optimal fixed sample size risk function are investigated. Also, the pairs of initial sample size (m, m_0), the reciprocal of the cost per observation (A) are chosen $(10, 800), (15, 1000), (30, 9000)$ that values of n_A are obtained 40, 44.7213 and 134.1641, respectively. Tables 1-3 report the simulation results of estimators and the root of mean square error (RMSE) of $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}$ based on the least-squares and the Yule-Walker estimators (within parentheses), respectively. Also, the ratios of the average two-stage stopping variable to the optimal fixed sample size and the ratios of the risk function to the optimal fixed sample size risk function 1-6 are drawn. It should be noted that all the computations using R software by $10,000$ replications.

From Figures 1 and 2, the ratios of the average two-stage stopping variable (N_m) and modified two-stage variables (N'_m) to the optimal fixed sample size variable are close to 1 as A increasing, as we expected. From Figure 3, the ratios of the average modified two-stage stopping variables (N_{m_1}) to the optimal fixed sample size variable are obtained around 1 for small pair (m_0, A). The performance of stopping variable N_{m_1} is better than the other variables for small pairs (m_0, A). For larger values of pair initial sample size and A, all three stopping variables are achieved similar. Moreover, from Tables 1-3, RMSEs of variables decrease as A increases and the results confirm the accuracy of both procedures in estimation, as we expected. From Figures 4-6, the ratios of the risk function to the optimal fixed sample size risk function are geted values close to 1 for different A and m or m_0 that results confirm both procedures have the same performance. In the end, the results demonstrate the close performance of the variables, which, of course, N_{m_1} is recommended for small values A and m_0 if the conditions are met. The results of both procedures based on both estimators are very close, which show the good performance of both processes.

5. **Data analysis**

we aim to briefly describe the proposed and threshold (TAR) models in the annual record of the numbers of Canadian lynx trapped in the Mackenzie River district of northwest Canada and investigate the performance of the proposed sequential point estimation in these data. The periodic fluctuation exhibited in this series has profoundly influenced ecological theory. It has also started a benchmark series to investigate a new statistical methodology for time series analysis. The first time series model built for this particular data set was probably that of [18]. [18] was one of the first people to propose a model for this particular data set. He fitted the linear $AR(2)$ model to the lynx data with $\varepsilon \sim N(0, 0.24^2)$ after taking a log_{10} transformation. After further examination of the model, [18] immediately realized the limitation of the linear fitting, as he pointed out in the same paper a ”curious feature”.

Table 1. Point estimation of the two-stage procedure (N_m).

$(m, A, \theta_1, \theta_2)$	$\hat{\theta}_{1N_m}$	$\hat{\theta}_{2N_m}$	RMSE $\hat{\theta}_{1N_m}$	RMSE $\hat{\theta}_{2N_m}$
$(10, 800, 0.2, 0.5)$	0.1493(0.1610)	0.3797(0.3873)	0.6731(0.5654)	0.6162(0.5445)
$(10, 800, 0.1, 0.7)$	0.0658 (0.0799)	0.5597(0.5702)	0.6896(0.4585)	0.1493(0.0870)
$(10, 800, 0.2, 0.1)$	0.1356(0.1386)	0.0639(0.0705)	0.0967(0.0901)	0.1700(0.1116)
$(10, 800, 0.1, 0.3)$	0.0825(0.0863)	0.2077(0.2160)	0.1090 (0.0990)	0.1236(0.0922)
$(10, 800, 0.2, 0.8)$	0.0038(0.0150)	0.6483(0.6629)	1.4254(1.6650)	0.3173(0.3711)
$(10, 800, -0.2, -0.3)$	-0.2209(-0.2210)	-0.2613(-0.2609)	0.1403(0.1349)	0.0890(0.0813)
$(10, 800, -0.2, -0.6)$	-0.2519(-0.2530)	-0.4449(-0.4467)	0.0686 (0.0622)	0.1094(0.0994)
$(10, 800, -0.7, -0.8)$	-0.6423(-0.6450)	-0.7012(0.7050)	0.0471 (0.0429)	0.0584(0.0535)
$(15, 1000, 0.2, 0.5)$	0.1823(0.1773)	0.4092(0.4094)	0.0824(0.1182)	0.0654 (0.0668)
$(15, 1000, 0.1, 0.7)$	0.1037 (0.1120)	0.5911(0.5956)	0.1843 (0.1533)	0.0565 (0.0506)
$(15, 1000, 0.2, 0.1)$	0.1530(0.1515)	0.0823(0.0844)	0.0638(0.0607)	0.0681(0.0651)
$(15, 1000, 0.1, 0.3)$	0.0953(0.0953)	0.2337 (0.2302)	0.0706(0.0683)	0.0632 (0.07115)
$(15, 1000, 0.2, 0.8)$	0.1478(0.1454)	0.6880(0.6974)	0.3205(0.3923)	0.0531(0.0439)
$(15, 1000, -0.2, -0.3)$	-0.2128(-0.2056)	-0.2585(-0.2637)	0.0567(0.0539)	0.0612(0.0586)
$(15, 1000, -0.2, -0.6)$	-0.2650(-0.2620)	-0.4420(-0.4486)	0.0515(0.0500)	0.0857(0.0764)
$(15, 1000, -0.7, -0.8)$	-0.6546(-0.6568)	-0.7137 (-0.7164)	0.0334(0.0316)	0.0432 (0.0415)
$(30, 9000, 0.2, 0.5)$	0.2187(0.2191)	0.4644(0.4613)	0.0184(0.0179)	0.0140(0.0140)
$(30, 9000, 0.1, 0.7)$	0.1585 (0.1590)	0.6589(0.6596)	0.0258(0.0250)	0.0090(0.0101)
$(30, 9000, 0.2, 0.1)$	0.1783(0.1775)	0.1078(0.1069)	0.0168 (0.0166)	0.0167(0.0169)
$(30, 9000, 0.1, 0.3)$	0.1173(0.1200)	0.2673 (0.2656)	0.0176 (0.0178)	0.0158(0.0163)
$(30, 9000, 0.2, 0.8)$	0.2382(0.2386)	0.7630(0.7631)	0.0269(0.0261)	0.0074(0.0073)
$(30, 9000, -0.2, -0.3)$	-0.2179(-0.2210)	-0.2695(-0.2682)	0.0160 (0.0158)	0.0168(0.0164)
$(30, 9000, -0.2, -0.6)$	-0.2789(-0.2796)	-0.4619(-0.4608)	0.0194(0.0194)	0.0343(0.0341)
$(30, 9000, -0.7, -0.8)$	-0.6987(-0.6993)	-0.7609(-0.7610)	0.0071(0.0073)	0.0099(0.0099)

Table 2. Point estimation of the modified two-stage procedure (N'_m).

$(m_0, A, \theta_1, \theta_2)$	$\hat{\theta}_{1N'_m}$	$\hat{\theta}_{2N'_m}$	RMSE $\hat{\theta}_{1N'_m}$	RMSE $\hat{\theta}_{2N'_m}$
$(10, 800, 0.2, 0.5)$	0.1493(0.1610)	0.3797(0.3873)	0.3318(0.1654)	0.1578(0.1445)
$(10, 800, 0.1, 0.7)$	0.0658 (0.0799)	0.5597 (0.5702)	0.6896(0.4585)	0.1493 (0.0870)
$(10, 800, 0.2, 0.1)$	0.1356(0.1386)	0.0639(0.0705)	0.0967 (0.0901)	0.1700(0.1116)
$(10, 800, 0.1, 0.3)$	0.0825(0.0863)	0.2077(0.2160)	0.1090(0.0990)	0.1236 (0.0922)
$(10, 800, 0.2, 0.8)$	0.0038(0.0150)	0.6483(0.6629)	1.4296 (1.3642)	0.3173(0.2998)
$(10, 800, -0.2, -0.3)$	- 0.2209(-0.2210)	-0.2613(-0.2609)	0.1403(0.1349)	0.0890(0.0813)
$(10, 800, -0.2, -0.6)$	-0.2519 (-0.2530)	-0.4449(-0.4467)	0.0686(0.0622)	0.1094(0.0994)
$(10, 800, -0.7, -0.8)$	-0.6388(-0.6439)	-0.6988(0.7009)	0.0458(0.0418)	0.0588 (0.0550)
$(15, 1000, 0.2, 0.5)$	0.1775 (0.1733)	0.4089 (0.4116)	0.0839 (0.0782)	0.0705 (0.0677)
$(15, 1000, 0.1, 0.7)$	0.1078 (0.0998)	0.5894 (0.5984)	0.1619 (0.4002)	0.0658 (0.0511)
$(15, 1000, 0.2, 0.1)$	0.1480 (0.1525)	0.0829 (0.0820)	0.0672 (0.0695)	0.0698 (0.0690)
$(15, 1000, 0.1, 0.3)$	0.0888 (0.0951)	0.2374 (0.2323)	0.0714 (0.0715)	0.0625 (0.0628)
$(15, 1000, 0.2, 0.8)$	0.1609 (0.1454)	0.6880 (0.6974)	0.25733 (0.3923)	0.0520 (0.0439)
$(15, 1000, -0.2, -0.3)$	-0.2106 (-0.2056)	-0.2638 (-0.2637)	0.0579 (0.0539)	0.0588 (0.0586)
$(15, 1000, -0.2, -0.6)$	-0.2607 (-0.2620)	-0.4417 (-0.4486)	0.0502 (0.0500)	0.0789 (0.0764)
$(15, 1000, -0.7, -0.8)$	-0.6545 (-0.6568)	-0.7140 (-0.7164)	0.03356 (0.0316)	0.0436 (0.0415)
$(30, 9000, 0.2, 0.5)$	0.2175 (0.2215)	0.4620 (0.4620)	0.0180 (0.0178)	0.0137 (0.0138)
$(30, 9000, 0.1, 0.7)$	0.1562 (0.1612)	0.6579 (0.6588)	0.0253 (0.0253)	0.0100 (0.0100)
$(30, 9000, 0.2, 0.1)$	0.1758 (0.1782)	0.1076 (0.1063)	0.0169 (0.0165)	0.0168 (0.0170)
$(30, 9000, 0.1, 0.3)$	0.1211 (0.1182)	0.2671 (0.2660)	0.0176 (0.0179)	0.0163 (0.0163)
$(30, 9000, 0.2, 0.8)$	0.2359 (0.2366)	0.7624 (0.7626)	0.0265 (0.0254)	0.0076 (0.0075)
$(30, 9000, -0.2, -0.3)$	-0.2201 (-0.2190)	-0.2685 (-0.2691)	0.0158 (0.0160)	0.0173 (0.0166)
$(30, 9000, -0.2, -0.6)$	-0.2806 (-0.2808)	-0.4600 (-0.4605)	0.0195 (0.0194)	0.0353 (0.0350)
$(30, 9000, -0.7, -0.8)$	-0.6989 (-0.6996)	-0.7638 (-0.7626)	0.0072 (0.00731)	0.0098 (0.0099)

For many biological populations, birth rates rely on population sizes for example, due to competition for the resources of habitat, the limitation of food, the predator-prey interaction, and other factors. Typically, an increasing phase occurs when the birth rate will increase in the early stage of a population cycle and it will decrease when the population is oversized in the latter stage, leading to a decreasing phase. A population decrease for one species will cause, in due course, a population decrease of its predators and a population increase of

TABLE 3. Point estimation of the modified two-stage procedure (N_{m_1}).

$(m_0, A, \theta_1, \theta_2)$	$\hat{\theta}_{1N_{m_1}}$	$\hat{\theta}_{2N_{m_1}}$	RMSE $\hat{\theta}_{1N_{m_1}}$	RMSE $\hat{\theta}_{2N_{m_1}}$
$(10, 800, 0.2, 0.5)$	0.1501(0.1488)	0.0845(0.0848)	0.3034(0.2898)	0.2573(0.2562)
$(10, 800, 0.1, 0.7)$	0.0965(0.0968)	0.5955(0.5924)	0.1943(0.2584)	0.0533(0.0532)
$(10, 800, 0.2, 0.1)$	0.1511(0.1493)	0.0859(0.0778)	0.0666(0.0669)	0.0661(0.0710)
$(10, 800, 0.1, 0.3)$	0.0935(0.0885)	0.2371(0.2258)	0.0711(0.0760)	0.0630(0.0695)
$(10, 800, 0.2, 0.8)$	0.1127(0.0896)	0.6827(0.6877)	1.4462(1.3426)	0.0576(0.0502)
$(10, 800, -0.2, -0.3)$	-0.2151(-0.2051)	-0.2552(-0.2572)	0.0579(0.0571)	0.0600(0.0623)
$(10, 800, -0.2, -0.6)$	-0.2644 (-0.2612)	-0.4458 (-0.4527)	0.0527(0.0500)	0.0831(0.0799)
$(10, 800, -0.7, -0.8)$	-0.6598 (-0.6659)	-0.7200 (-0.7259)	0.0340(0.0326)	0.0430(0.0430)
$(15, 1000, 0.2, 0.5)$	0.1855 (0.1847)	0.4167 (0.4187)	0.0687 (0.0730)	0.0562 (0.05399)
$(15, 1000, 0.1, 0.7)$	0.1185 (0.1215)	0.6051 (0.6057)	0.1152 (0.1950)	0.0464 (0.0417)
$(15, 1000, 0.2, 0.1)$	0.1531 (0.1539)	0.0884 (0.0874)	0.0577 (0.0569)	0.0597 (0.0552)
$(15, 1000, 0.1, 0.3)$	0.0961 (0.0979)	0.2395 (0.2367)	0.0628 (0.0587)	0.0577 (0.0569)
$(15, 1000, 0.2, 0.8)$	0.1024 (0.1664)	0.6966 (0.6971)	0.2234 (0.2560)	0.0428 (0.0426)
$(15, 1000, -0.2, -0.3)$	-0.2168 (-0.2088)	-0.2614 (-0.2616)	0.0515 (0.0500)	0.0537 (0.0521)
$(15, 1000, -0.2, -0.6)$	-0.2644 (-0.2612)	-0.4458 (-0.4527)	0.0463 (0.0454)	0.0743 (0.0716)
$(15, 1000, -0.7, -0.8)$	-0.6598 (-0.6659)	-0.7200 (-0.7259)	0.0293 (0.0267)	0.0371 (0.0342)
$(30, 9000, 0.2, 0.5)$	0.2198 (0.2227)	0.4629 (0.4624)	0.0175 (0.0172)	0.0132 (0.0134)
$(30, 9000, 0.1, 0.7)$	0.1605 (0.1617)	0.6600 (0.6601)	0.0246 (0.0242)	0.0092 (0.0091)
$(30, 9000, 0.2, 0.1)$	0.1800 (0.1792)	0.1068 (0.1071)	0.0151 (0.0154)	0.0160 (0.0157)
$(30, 9000, 0.1, 0.3)$	0.1208 (0.1182)	0.2671 (0.2658)	0.0168 (0.0167)	0.0151 (0.0156)
$(30, 9000, 0.2, 0.8)$	0.2362 (0.2381)	0.7634 (0.7647)	0.0245 (0.0247)	0.0071 (0.0069)
$(30, 9000, -0.2, -0.3)$	-0.2209 (-0.2208)	-0.2678 (-0.2703)	0.0149 (0.0150)	0.0159 (0.0155)
$(30, 9000, -0.2, -0.6)$	-0.2788 (0.2807-)	-0.4611 (-0.4601)	0.0184 (0.0192)	0.0341 (0.0339)
$(30, 9000, -0.7, -0.8)$	-0.6993 (-0.7002)	-0.7631 (-0.7633)	0.0067 (0.0067)	0.0094 (0.0092)

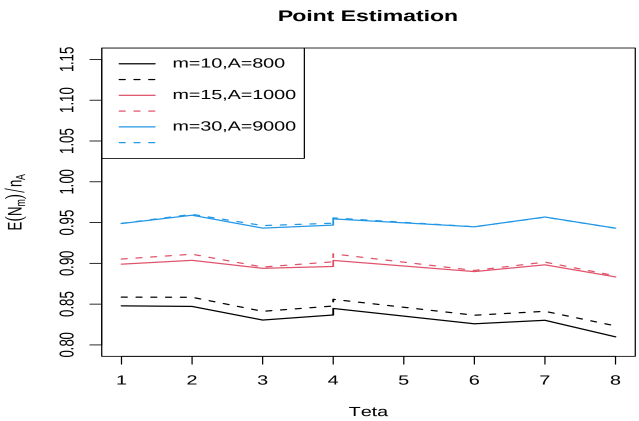

FIGURE 1. Ratio of the two-stage stopping variable to the optimal fixed sample size

its prey and also the abundance of resources. This in turn will cause to start to a new increasing phase. Therefore, it looks very alluring to model population dynamics in terms of a threshold model in which different regimes would demonstrate different phases or stages in population cycles. The difference of the coefficients in increasing and decreasing phases depicts the so-called phase-dependence and density-dependence in ecology, which can only be reflected in a nonlinear model. The phase-dependence means that the both lynx and the hare behave differently (in hunting or escaping) when the lynx population increases or decreases. The density-dependence implies that the reproduction

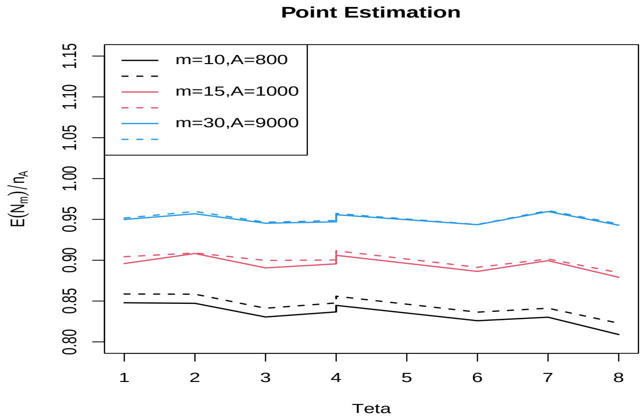

FIGURE 2. Ratio of the modified two-stage stopping variable (N'_m) to the optimal fixed sample size

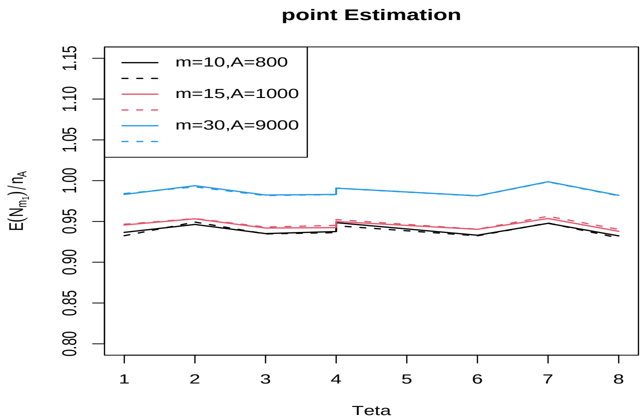

FIGURE 3. Ratio of the modified two-stage stopping variable (N_{m_1}) to the optimal fixed sample size

rates of animals as well as their behavior depending on the abundance of the population.

The lynx time series plots including time plot and reversed-time plot show that the lynx population exhibits a periodic-like fluctuation with most cycles around nine or ten years. It is also obvious that there exists some characteristic in this series that is not time-reversible. Having incorporated the biological evidence, [35] fitted the TAR model with two regimes with delay variable $d = 2$. For further discussion on the biological meaning of TAR fitting for the lynx data, we refer the reader to [33].

According to the reviews, both models are examined in this paper. To this end, the parameters are estimated for different m, m_0, and A using the proposed

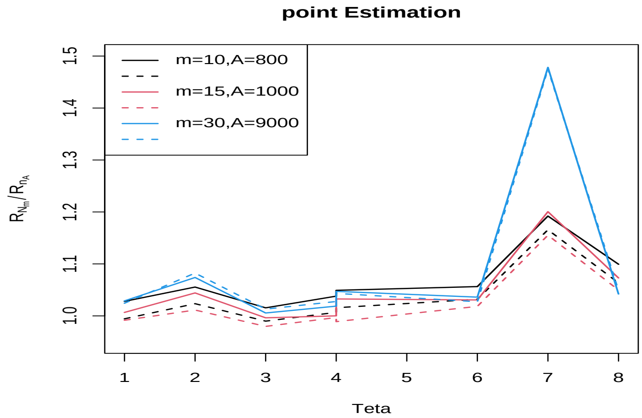

FIGURE 4. Ratio of the two-stage risk function to the optimal fixed sample size risk function

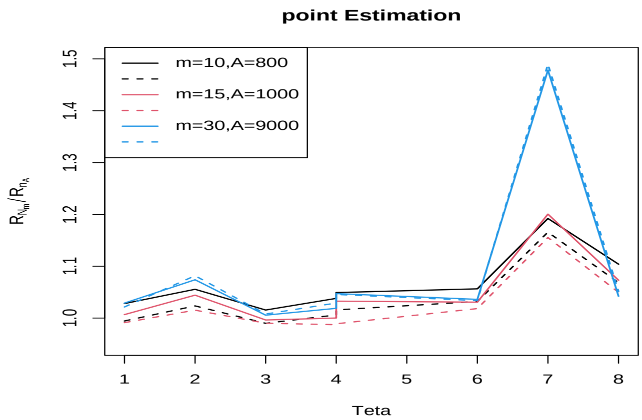

FIGURE 5. Ratio of the modified two-stage risk function ($R_{N'_m}$)to the optimal fixed sample size risk function

stopping variables. As mentioned before, sequential procedures are examined to determine the final sample size. So, if the initial sample size might not be enough then the difference of the final sample size and the initial sample size is generated at the second stage. Also, to make a better comparison, the procedures have been compared with the widely used purely sequential procedure. The purely sequential stopping rule analogy with n_A is

$$N_p = \inf \left\{ n \geq m \,\middle|\, n \geq (2Ap)^{1/2} \hat{\sigma}_n \right\}.$$

Tables 4, 5, 6 and 7 report point estimation including the two-stage stopping variable (N_m), the modified two-stage stopping variable (N'_m), the modified two-stage stopping variable (N_{m_1}), the purely stopping variable (N_p),

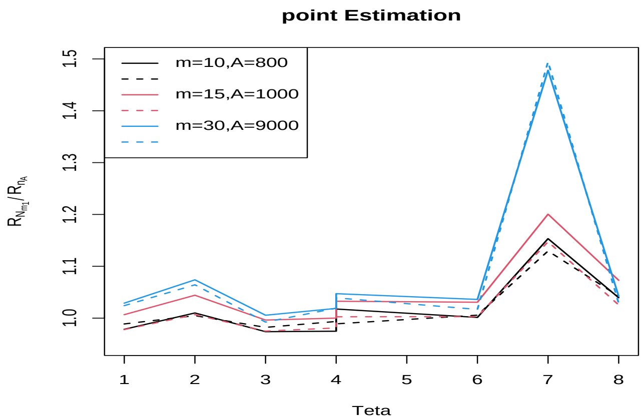

FIGURE 6. Ratio of the modified two-stage risk function ($R_{N_{m_1}}$) to the optimal fixed sample size risk function

the threshold autoregressive estimators and the variance of model based on the least-squares and the Yule-Walker estimators (within parentheses), respectively. Also, the pairs initial sample size and A for point estimation are assumed $(2,2),(4,6),(5,10),(20,25),(40,50),(80,100),(100,140)$.

Tables 4 and 5 report the results based on AR(2) model. The stopping variables resulting from the proposed procedures for values $(2,2)$ and $(4,6)$ have lower values than the purely sequential stopping variable. As we observed from the tables, the stopping variables of both procedures in terms of both estimators are close together by increasing A. Moreover, $\hat{\boldsymbol{\theta}}$ and $\widehat{\sigma}$ are not significant different which confirm the same performance of both procedures. The results of the proposed procedures are close to the purely sequential procedure. Therefore, the proposed stopping rules are a suitable alternative to the purely sequential procedure for determining the sample size.

Based on the achievements by [35], the model TAR(2) is fitted to the data with a threshold value 3.25. From Tables 6 and 7, the two-stage stopping variables and the modified two-stage stopping variables have lower values than the purely sequential procedure for small values of the initial sample size. The values of the estimators also do not differ much by increasing the value of A and different the initial sample size. Also, the results demonstrate the good performance of the proposed procedures compared to the purely sequential procedure. The results show the same performance in terms of both sampling schemes and estimators. As seen from the results, model TAR(2) has results close to model AR(2), which indicates the characteristics of the linear model, despite the features of the nonlinear model. Sequential procedures also perform well based on this model.

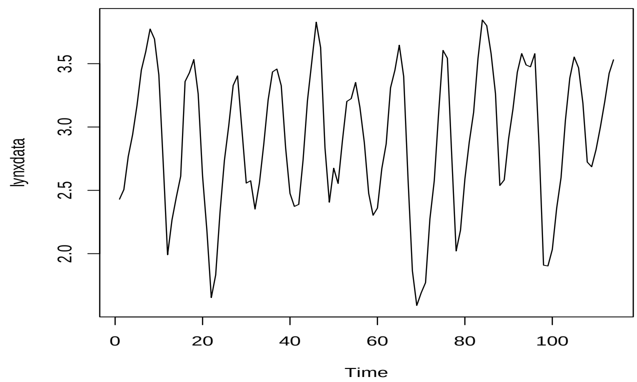

FIGURE 7. Time plot of lynx data

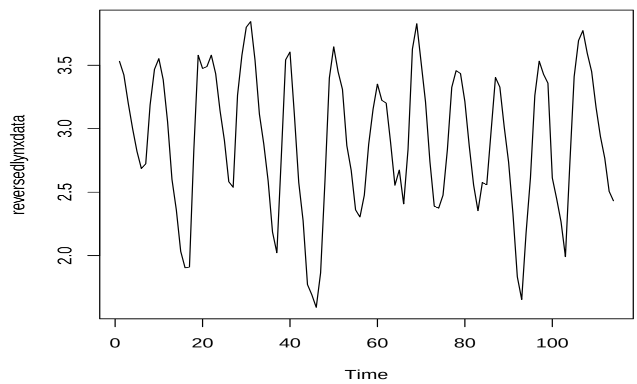

FIGURE 8. Reversed-time plot of lynx data

Conclusions

In this paper, we investigate the two-stage procedure and the modified two-stage procedure for the point estimation as a solution to determine the sample size in a TAR(1) model. The properties of the two-stage procedure and the modified two-stage procedure are established as the cost per observation tends to zero. These properties indicate the efficiency of the procedures compared to the optimal fixed sample procedure. In the following, these properties confirm via the Monte Carlo simulations studies. The performance of the two-stage procedure and the modified two-stage procedure based on the least-squares and the Yule-Walker estimators are compared. The results demonstrate the two-stage procedure (N_m) as efficient as the modified two-stage procedure (N'_m) for different the initial sample size and A. The stopping variable (N_{m_1}) indicates

TABLE 4. Point estimation of two-stage and modified two-stage procedures (AR(2)).

$(morm_0, A)$	N_m	N'_m	N_{m_1}	N_p
(2, 2)	7(8)	7(7)	7(7)	9(10)
(4, 6)	14(15)	14(13)	14(13)	15(15)
(5, 10)	18(21)	18(18)	18(18)	20(20)
(20, 25)	31(32)	31(31)	31(31)	30(30)
(40, 50)	42(43)	42(42)	42(42)	41(41)
(80, 100)	80(80)	80(80)	80(80)	80(80)
(100, 140)	100(100)	100(100)	100(100)	100(100)
$(morm_0, A)$	$\widehat{\theta}_{1N_m}$	$\widehat{\theta}_{1N'_m}$	$\widehat{\theta}_{1N_{m_1}}$	$\widehat{\theta}_{1N_p}$
(2, 2)	0.9398(0.9684)	0.9398(0.9515)	0.9398(0.9515)	0.9685(1.0045)
(4, 6)	0.9903(0.9735)	0.9903(0.9914)	0.9903(0.9914)	0.9728(0.9735)
(5, 10)	0.9800(0.9995)	0.9800(0.9906)	0.9800(0.9909)	0.9956(1.0013)
(20, 25)	0.9929(0.9886)	0.9929(0.9916)	0.9929(0.9916)	0.9907(0.9934)
(40, 50)	0.9908(0.9847)	0.9908(0.9872)	0.9908(0.9872)	0.9933(0.9911)
(80, 100)	0.9901(0.9893)	0.9901(0.9893)	0.9901(0.9893)	0.9901(0.9893)
(100, 140)	0.9922(0.9917)	0.9922(0.9917)	0.9922(0.9917)	0.9752(0.9738)

TABLE 5. Point estimation of two-stage and modified two-stage procedures (AR(2)).

$(morm_0, A)$	$\widehat{\theta}_{2N_m}$	$\widehat{\theta}_{2N'_m}$	$\widehat{\theta}_{2N_{m_1}}$	$\widehat{\theta}_{2N_p}$
(2, 2)	0.8941(0.9608)	0.8941(0.9253)	0.8941(0.9253)	0.9544(1.0031)
(4, 6)	0.9565(0.9446)	0.9565(0.9613)	0.9565(0.9613)	0.9405(0.9446)
(5, 10)	0.9643(0.9877)	0.9643(0.9830)	0.9643(0.9830)	0.9876(0.9962)
(20, 25)	0.9763(0.9675)	0.9763(0.9743)	0.9763(0.9743)	0.9754(0.9792)
(40, 50)	0.9701(0.9606)	0.9701(0.9655)	0.9701(0.9655)	0.9768(0.9721)
(80, 100)	0.9703(0.9684)	0.9703(0.9684)	0.9703(0.9684)	0.9703(0.9684)
(100, 140)	0.9752(0.9738)	0.9752(0.9738)	0.9752(0.9738)	0.9752(0.9738)
$(morm_0, A)$	$\widehat{\sigma}_{N_m}$	$\widehat{\sigma}_{N'_m}$	$\widehat{\sigma}_{N_{m_1}}$	$\widehat{\sigma}_{N_p}$
(2, 2)	9.0365(11.7114)	9.0365(9.9123)	9.0365(9.9123)	10.0503(10.8742)
(4, 6)	8.8141(9.9417)	8.8141(8.7320)	8.8141(8.7320)	8.6858(8.7720)
(5, 10)	9.1785(9.2092)	9.1785(9.7380)	9.1785(9.7380)	9.1054(9.3653)
(20, 25)	8.3430(8.8578)	8.3430(8.2895)	8.3430(8.2895)	8.4015(8.5108)
(40, 50)	8.3004(9.0322)	8.3004(8.1663)	8.3004(8.1663)	8.3676(8.2534)
(80, 100)	8.2002(8.8527)	8.2002(8.1575)	8.2002(8.1575)	8.2002(8.1575)
(100, 140)	8.4887(9.0701)	8.4887(8.4579)	8.4887(8.4579)	8.4887(8.4579)

TABLE 6. Point estimation of two-stage and modified two-stage procedures (TAR(2)).

$(morm_0, A)$	N_m	N'_m	N_{m_1}	N_p	$\widehat{\theta}^+_{1N_m}$	$\widehat{\theta}^-_{1N_m}$
(2, 2)	8(8)	8(8)	8(8)	10(10)	0.9644(0.9859)	0.9341(0.9978)
(4, 6)	14(15)	14(15)	14(15)	15(16)	0.9859(0.9597)	0.9949(0.9977)
(5, 10)	19(20)	19(20)	19(20)	20(20)	0.9716(0.9716)	1.0201(1.0417)
(20, 25)	31(31)	31(31)	31(31)	30(30)	0.9666(0.9666)	1.0199(1.0099)
(40, 50)	42(42)	42(42)	42(42)	42(42)	0.9711(0.9671)	1.0083(1.0013)
(80, 100)	80(80)	80(80)	80(80)	80(80)	0.9651(0.9616)	1.0095(1.0080)
(100, 140)	100(100)	100(100)	100(100)	100(100)	0.9703(0.9703)	1.0130(1.0086)
$(morm_0, A)$	$\widehat{\theta}^+_{1N'_m}$	$\widehat{\theta}^-_{1N'_m}$	$\widehat{\theta}^+_{1N_{m_1}}$	$\widehat{\theta}^-_{1N_{m_1}}$	$\widehat{\theta}^+_{1N_p}$	$\widehat{\theta}^-_{1N_p}$
(2, 2)	0.9644(0.9859)	0.9341(0.9978)	0.9644(0.9859)	0.9341(0.9978)	0.9859(0.9859)	0.9934(1.0176)
(4, 6)	0.9859(0.9597)	0.9949(0.9977)	0.9859(0.9597)	0.9949(0.9977)	0.9548(0.9716)	0.9949(0.9977)
(5, 10)	0.9716(0.9716)	1.0201(1.0417)	0.9716(0.9716)	1.0201(1.0417)	0.9716(0.9716)	1.0316(1.0417)
(20, 25)	0.9666(0.9666)	1.0199(1.0099)	0.9666(0.9666)	1.0199(1.0099)	0.9666(0.9666)	1.0166(1.0173)
(40, 50)	0.9711(0.9671)	1.0083(1.0013)	0.9711(0.9671)	1.0083(1.0013)	0.9711(0.9671)	1.0083(1.0013)
(80, 100)	0.9651(0.9616)	1.0095(1.0080)	0.9651(0.9616)	1.0095(1.0080)	0.9651(0.9616)	1.0095(1.0080)
(100, 140)	0.9703(0.9703)	1.0130(1.0086)	0.9703(0.9703)	1.0130(1.0086)	0.9703(0.9703)	1.0130(1.0086)

Table 7. Point estimation of two-stage and modified two-stage procedures (TAR(2)).

$(morm_0, A)$	$\widehat{\theta}^+_{2N_m}$	$\widehat{\theta}^-_{2N_m}$	$\widehat{\theta}^+_{2N'_m}$	$\widehat{\theta}^-_{2N'_m}$	$\widehat{\theta}^+_{2N_{m_1}}$	$\widehat{\theta}^-_{2N_{m_1}}$
$(2, 2)$	0.9859(0.9859)	0.9257(1.0698)	0.9859(0.9859)	0.9257(1.0698)	0.9859(0.9859)	0.9257(1.0698)
$(4, 6)$	0.9548(0.9716)	0.9977(1.0125)	0.9548(0.9716)	0.9977(1.0125)	0.9548(0.9716)	0.9977(1.0125)
$(5, 10)$	0.9716(0.9716)	1.0374(1.0369)	0.9716(0.9716)	1.0374(1.0369)	0.9716(0.9716)	1.0374(1.0369)
$(20, 25)$	0.9666(0.9666)	1.0173(1.0068)	0.9666(0.9666)	1.0173(1.0068)	0.9666(0.9666)	1.0173(1.0068)
$(40, 50)$	0.9711(0.9711)	1.0013(1.0053)	0.9711(0.9635)	1.0013(1.0053)	0.9711(0.9635)	1.0013(1.0053)
$(80, 100)$	0.9651(0.9597)	1.0080(1.0101)	0.9651(0.9597)	1.0080(1.0101)	0.9651(0.9597)	1.0080(1.0101)
$(100, 140)$	0.9703(0.9685)	1.0120(1.0104)	0.9703(0.9685)	1.0120(1.0104)	0.9703(0.9685)	1.0120(1.0104)
$(morm_0, A)$	$\widehat{\theta}^+_{2N_p}$	$\widehat{\theta}^-_{2N_p}$	$\widehat{\sigma}_{N_m}$	$\widehat{\sigma}_{N'_m}$	$\widehat{\sigma}_{N_{m_1}}$	$\widehat{\sigma}_{N_p}$
$(2, 2)$	0.9859(0.9859)	1.0364(1.0251)	10.6709(12.4990)	10.6709(12.4990)	10.6709(12.4990)	10.5713(10.6875)
$(4, 6)$	0.9581(0.9716)	0.9977(1.0418)	9.1115(9.6330)	9.1115(9.6330)	9.1115(9.6330)	9.2987(10.0678)
$(5, 10)$	0.9716(0.9716)	1.0484(1.0369)	9.5764(9.3477)	9.5764(9.3477)	9.5764(9.3477)	9.3506(9.3477)
$(20, 25)$	0.9666(0.9666)	1.0225(1.0160)	8.6039(8.4315)	8.6039(8.4315)	8.6039(8.4315)	8.7082(8.6621)
$(40, 50)$	0.9711(0.9635)	1.0013(1.0053)	8.5935(8.4724)	8.5935(8.4724)	8.5935(8.4724)	8.5935(8.4724)
$(80, 100)$	0.9651(0.9597)	1.0080(1.0101)	8.5202(8.4617)	8.5202(8.4617)	8.5202(8.4617)	8.5202(8.4617)
$(100, 140)$	0.9703(0.9685)	1.0120(1.0104)	8.7761(8.7078)	8.7761(8.7078)	8.7761(8.7078)	8.7761(8.7078)

better performance for small initial sample size compared to the other stopping variables. Of course, all three stopping variables have the good and same performance, as increasing A. Moreover, the real data set is indicated the application of the two-stage procedure and the modified two-stage procedure, in practice. The results of the real data set show the close performance of the two procedures. The simplicity and the operational savings of these procedures are an advantage compared to the most commonly used purely sequential procedure which encourages us to suggest these procedures in analysis time series models. The modified two-stage procedure also solves the weakness of the two-stage procedure in overestimation. If there are conditions for using the modified two-stage procedure, this procedure is preferable to the two-stage procedure and is more accurate.

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